Technical supplement to "Population Games and Deterministic Evolutionary Dynamics"

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Contents

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After 3.3: Ordinary differential equations

Every continuous vector field $V\colon \mathbb{R}^n \to \mathbb{R}^n$ defines an *ordinary differential equation* on $\mathbb{R}^n$:

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x_t = V(x_t).$$

Often we write $\dot{x}_t$ for $\frac{\mathrm{d}}{\mathrm{d}t} x_t$; we also express the previous equation as

$$\text{(D)} \qquad \dot{x} = V(x).$$

When the current state is x_t, the current velocity of state is $V(x_t)$. (Show picture.)

The trajectory $\{x_t\}_{t\in I}$ is a *solution* to (D) if $\dot{x}_t = V(x_t)$ at all times t in the interval I.

Example: $\dot{x} = ax$ has solutions $x_t = \xi \exp(at)$.

Fix an open set $O \subseteq \mathbb{R}^n$. We call the function $f\colon O \to \mathbb{R}^m$ *Lipschitz continuous* if there exists a scalar K such that

$$\left| f(x) - f(y) \right| \leq K \left| x - y \right| \text{ for all } x, y \in O.$$

Theorem 0.1 (The Picard-Lindelöf Theorem)**.** *Let $V\colon O \to \mathbb{R}^n$ be Lipschitz continuous. Then for each $\xi \in O$, there exists a scalar $T > 0$ and a unique trajectory $x\colon (-T, T) \to O$ with $x_0 = \xi$ such that $\{x_t\}$ is a solution to* (D)*.*

Theorem 0.2 ((Forward) invariance on compact convex sets)**.** *Let $C \subset \mathbb{R}^n$ be a compact convex set, and let $V\colon C \to \mathbb{R}^n$ be Lipschitz continuous.*

(*i*) *Suppose that $V(\hat{x}) \in TC(\hat{x})$ for all $\hat{x} \in C$. Then for each $\xi \in C$, there exists a unique $x\colon [0, \infty) \to C$ with $x_0 = \xi$ that solves* (D)*.*

(*ii*) *Suppose that $V(\hat{x}) \in TC(\hat{x}) \cap (-TC(\hat{x}))$ for all $\hat{x} \in C$. Then for each $\xi \in C$, there exists a unique $x\colon (-\infty, \infty) \to C$ with $x_0 = \xi$ that solves* (D)*.*

Assume the conditions of Theorem 0.2(i). The *semiflow* $\phi\colon [0, \infty) \times C \to C$ generated by (D) is defined by $\phi_t(\xi) = x_t$, where $\{x_t\}_{t\geq 0}$ is the solution to (D) with initial condition $x_0 = \xi$. If we fix $\xi \in C$ and vary t, then $\{\phi_t(\xi)\}_{t\in[0,\infty)}$ is the solution orbit of (D) through initial condition ξ; note also that ϕ satisfies the group property $\phi_t(\phi_s(\xi)) = \phi_{s+t}(\xi)$. If we instead fix t and vary ξ, then $\{\phi_t(\xi)\}_{\xi\in C'}$ describes the positions at time t of solutions to (D) with initial conditions in $C' \subseteq C$.

Theorem 0.3 (Continuity of solutions in initial conditions)**.** *Suppose that $V\colon C \to \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant K. Let ϕ be the semiflow of* (D)*, and fix $t \in [0, \infty)$.*

Then $\phi_t(\cdot)$ *is Lipschitz continuous with Lipschitz constant* $\mathrm{e}^{K|t|}$ *: for all* $\xi, \chi \in C$*, we have that* $\left|\phi_t(\xi) - \phi_t(\chi)\right| \leq |\xi - \chi|\,\mathrm{e}^{K|t|}$.

Before 5.1.3: Extinction and invariance under imitative dynamics

Imitative dynamics are of the form

$$\dot{x}_i = x_i \sum_{j \in S} x_j (r_{ji}(F(x), x) - r_{ij}(F(x), x)). \tag{1}$$

We require that the *conditional imitation rates* r_{ij} are Lipschitz continuous, and that *net conditional imitation rates are monotone*:

$$\pi_j \geq \pi_i \iff [r_{kj}(\pi, x) - r_{jk}(\pi, x) \geq r_{ki}(\pi, x) - r_{ik}(\pi, x) \text{ for all } i, j, k \in S]. \tag{2}$$

It follows from equation (1) that all imitative dynamics satisfy *extinction*: if a strategy is unused, its growth rate is zero.

$$\text{If } x_i = 0, \text{ then } V_i(x) = 0. \tag{3}$$

Extinction implies that the growth rate vectors $V(x)$ are always tangent to the boundaries of X, in the sense that $V(x) \in TX(x) \cap (-TX(x))$. This implies:

Proposition 0.4 (Forward and backward invariance)**.** *Let* $\dot{x} = V_F(x)$ *be an imitative dynamic. Then for each initial condition* $\xi \in X$*, this dynamic admits a unique solution trajectory in* $\mathcal{T}_{(-\infty,\infty)} = \{x\colon (-\infty, \infty) \to X \mid x \text{ is continuous}\}$*.*

With an argument that utilizes uniqueness of solutions, extinction also implies:

Theorem 0.5 (Support invariance)**.** *If* $\{x_t\}$ *is a solution trajectory of an imitative dynamic, then the sign of component* $(x_t)_i$ *is independent of* $t \in (-\infty, \infty)$*.*

With 5.1.3: Monotone percentage growth rates and positive correlation

All dynamics of form (1) can be expressed as

$$\dot{x}_i = V_i(x) = x_i G_i(x), \text{ where } G_i(x) = \sum_{k \in S} x_k \,(r_{ki}(F(x), x) - r_{ik}(F(x), x)). \tag{4}$$

If strategy $i \in S$ is in use, then $G_i(x) = V_i(x)/x_i$ represents the *percentage growth rate* of the number of agents using this strategy.

Since imitative dynamics have monotone net conditional imitation rates (2), strategies' percentage growth rates are ordered by their payoffs:

Observation 0.6. *All imitative dynamics exhibit* monotone percentage growth rates*:*

$$G_i(x) \geq G_j(x) \text{ if and only if } F_i(x) \geq F_j(x). \tag{5}$$

Condition (5) is a strong restriction on strategies' *percentage* growth rates. We now show that it implies our basic payoff monotonicity condition, which imposes a weak restriction on strategies' *absolute* growth rates.

Theorem 0.7. *All imitative dynamics satisfy positive correlation* (PC)*.*

Proof. Let x be a social state at which $V(x) \neq \mathbf{0}$; we need to show that $V(x)'F(x) > 0$. To do so, we define

$$S_+(x) = \{i \in S\colon V_i(x) > 0\} \quad \text{and} \quad S_-(x) = \{j \in S\colon V_j(x) < 0\}$$

to be the sets of strategies with positive and negative absolute growth rates, respectively. By extinction (3), these sets are contained in the support of x. It follows that

$$S_+(x) = \{i \in S\colon x_i > 0 \text{ and } \frac{V_i(x)}{x_i} > 0\} \text{ and } S_-(x) = \{j \in S\colon x_j > 0 \text{ and } \frac{V_j(x)}{x_j} < 0\}.$$

Since $V(x) \in TX$, we know that

$$\sum_{k \in S_+(x)} V_k(x) = -\sum_{k \in S_-(x)} V_k(x),$$

and since $V(x) \neq \mathbf{0}$, these expressions are positive. Thus by Observation 0.6,

$$\begin{aligned}
V(x)'F(x) &= \sum_{k \in S_+(x)} V_k(x)\, F_k(x) + \sum_{k \in S_-(x)} V_k(x)\, F_k(x) \\
&\geq \min_{i \in S_+(x)} F_i(x) \sum_{k \in S_+(x)} V_k(x) + \max_{j \in S_-(x)} F_j(x) \sum_{k \in S_-(x)} V_k(x) \\
&= \left(\min_{i \in S_+(x)} F_i(x) - \max_{j \in S_-(x)} F_j(x) \right) \sum_{k \in S_+(x)} V_k(x) > 0. \blacksquare
\end{aligned}$$

With 5.1.3: Rest points and restricted equilibria

First recall the definition of Nash equilibrium:

$$NE(F) = \{x \in X\colon x_i > 0 \Rightarrow F_i(x) = \max_{j \in S} F_j(x)\}.$$

Define the set of *restricted equilibria* of F by

$$RE(F) = \{x \in X\colon x_i > 0 \Rightarrow F_i(x) = \max_{j \in S: x_j > 0} F_j(x)\}.$$

In words, x is a restricted equilibrium of F if it is a Nash equilibrium of a restricted version of F in which only strategies in the support of x can be played.

Theorem 0.8. *If $\dot{x} = V^F(x)$ is an imitative dynamic, then $RP(V^F) = RE(F)$.*

Proof.

$$\begin{aligned} x \in RP(V) &\Leftrightarrow V_i(x) = 0 \text{ for all } i \in S \\ &\Leftrightarrow \frac{V_i(x)}{x_i} = 0 \text{ whenever } x_i > 0 \qquad \text{(by (3))} \\ &\Leftrightarrow F_i(x) = c \text{ whenever } x_i > 0 \qquad \text{(by (5))} \\ &\Leftrightarrow x \in RE(F). \; \blacksquare \end{aligned}$$

Theorem 0.9. *Let V^F be an imitative dynamic for population game F, and let $\hat{x}$ be a non-Nash rest point of V^F. Then $\hat{x}$ is not Lyapunov stable under V^F, and no interior solution trajectory of V^F converges to $\hat{x}$.*

With 5.2.2: Construction of solutions of the best response dynamic

If during $[0, T]$ strategy i is the unique best response, the solution to the best response dynamic satisfies the affine ODE $\dot{x} = e_i - x$. Thus over this interval,

$$x_t = (1 - \mathrm{e}^{-t})\, e_i + \mathrm{e}^{-t}\, x_0.$$

That is, the solution moves toward vertex e_i, slowing down during the approach.

Example: Standard RPS (picture)

Example: 123 Coordination (picture)

Example: Zeeman's game (picture)

Example: The Golman-Page game (pictures)

With 5.2.3: More on perturbed best response dynamics

Perturbed optimization: a representation theorem

We can derive $\tilde{M}$ using two sorts of payoff perturbations.

Stochastic perturbations to payoffs of pure strategies:

$$\tilde{M}_i(\pi) = \mathbb{P}\left(i = \operatorname*{argmax}_{j \in S} \pi_j + \varepsilon_j \right). \tag{6}$$

We require the random vector ε to be an *admissible stochastic perturbation*: it must admit a positive density on $\mathbb{R}^n$, and this density must be smooth enough that the function $\tilde{M}$ is continuously differentiable.

Deterministic perturbations of payoffs of mixed strategies:

$$\tilde{M}(\pi) = \operatorname*{argmax}_{y \in \operatorname{int}(X)} \left(y'\pi - v(y) \right). \tag{7}$$

We require the function $v\colon \operatorname{int}(X) \to \mathbb{R}$ to be an *admissible deterministic perturbation*: it must be *differentiably strictly convex* (the second derivative at y, $\nabla^2 v(y) \in L^2_s(\mathbb{R}^n_0, \mathbb{R})$, must be positive definite for all $y \in \operatorname{int}(X)$) and *steep near* $\operatorname{bd}(X)$ ($|\nabla v(y)|$ must approach infinity whenever y approaches $\operatorname{bd}(X)$).

Formulation (6) is more appealing from an economic point of view, but formulation (7) is clearly more convenient for analysis. What is the relationship between them?

Since $\tilde{M}(\pi) = \tilde{M}(\Phi\pi)$ for all $\pi \in \mathbb{R}^n$, we can focus on $\bar{M}\colon \mathbb{R}^n_0 \to \operatorname{int}(X)$, the restriction of $\tilde{M}$ to $\mathbb{R}^n_0$.

Theorem 0.10. *Let $\tilde{M}$ be a perturbed maximizer function defined in terms of an admissible stochastic perturbation ε via equation* (6). *Then $\tilde{M}$ satisfies equation* (7) *for some admissible deterministic perturbation v. In fact, $\bar{M} = \tilde{M}|_{\mathbb{R}^n_0}$ and ∇v are invertible, and $\bar{M} = (\nabla v)^{-1}$.*

Taking as given the initial statements in the theorem, it is easy to verify the last one. Suppose that $\tilde{M}$ (and hence $\bar{M}$) can be derived from the admissible deterministic perturbation v, that $\nabla v : \operatorname{int}(X) \to \mathbb{R}^n_0$ is invertible, and that $\pi \in \mathbb{R}^n_0$. Then $y^* = \bar{M}(\pi)$ satisfies

$$y^* = \operatorname*{argmax}_{y \in \operatorname{int}(X)} \left(y'\pi - v(y) \right).$$

Taking the first order condition with respect to directions in $\mathbb{R}^n_0$ yields

$$\Phi(\pi - \nabla v(y^*)) = \mathbf{0}.$$

Since π and $\nabla v(y^*)$ are already in $\mathbb{R}^n_0$, the projection Φ does nothing, so

$$\bar{M}(\pi) = y^* = (\nabla v)^{-1}(\pi).$$

To prove Theorem 0.10, one must show that a function v with the desired properties exists. This is accomplished using Legendre transforms—see *PGED*, Sec. 6.B and 6.C.

Logit choice and the logit dynamic

By far the most common perturbed best response function is the *logit choice function*:

$$\tilde{M}_i(\pi) = \frac{\exp(\eta^{-1}\pi_i)}{\sum_{j\in S}\exp(\eta^{-1}\pi_j)}.$$

Its mean dynamic the *logit dynamic* with *noise level* η:

$$\text{(L)} \qquad \dot{x}_i = \frac{\exp(\eta^{-1}F_i(x))}{\sum_{j\in S}\exp(\eta^{-1}F_j(x))} - x_i\ .$$

Rest points of logit dynamics are called *logit equilibria*.

Example: 123 Coordination (pictures)

Stochastic derivation: ε_i are i.i.d. double exponential ($\mathbb{P}(\varepsilon_i \le c) = \exp(-\exp(-\eta^{-1}c - \gamma))$.

Deterministic derivation: v is (negated) entropy ($v(y) = \eta\sum_{j\in S} y_j \log y_j$).

See *PGED* Exercise 6.2.4 for more results on logit choice.

Perturbed incentive properties via virtual payoffs

By Theorem 0.10 it is enough to consider deterministic perturbations.

Define the set of *perturbed equilibria* of the pair (F, v) by

$$PE(F, v) = \{x \in X : x = \tilde{M}(F(x))\}.$$

Observation 0.11. *All perturbed best response dynamics satisfy* perturbed stationarity*:*

$$\text{(8)} \qquad V(x) = \mathbf{0} \textit{ if and only if } x \in PE(F, v).$$

Define the *virtual payoffs* $\tilde{F} : \text{int}(X) \to \mathbb{R}^n$ for the pair (F, v) by

$$\tilde{F}(x) = F(x) - \nabla v(x).$$

Intuitively, strategies that very few agents use have high virtual payoffs. For example, when x_i is the only component of x that is close to zero, then for each alternate strategy $j \neq i$, moving "inward" in direction $e_i - e_j$ sharply decreases the value of v; thus, the directional derivative $\frac{\partial v}{\partial(e_i - e_j)}(x)$ is large in absolute value and negative.

Theorem 0.12. *Let $x \in X$ be a social state. Then $x \in PE(F, v)$ if and only if $\Phi\tilde{F}(x) = \mathbf{0}$.*

Proof.
$$\begin{aligned} x \in PE(F, v) &\Leftrightarrow V(x) = \mathbf{0} \\ &\Leftrightarrow \tilde{M}(F(x)) = x \\ &\Leftrightarrow \bar{M}(\Phi F(x)) = x \\ &\Leftrightarrow \Phi F(x) = \nabla v(x) \\ &\Leftrightarrow \Phi\tilde{F}(x) = \mathbf{0}. \quad \blacksquare \end{aligned}$$

For disequilibrium dynamics, we define *virtual positive correlation*:

(9) $\qquad V(x) \neq \mathbf{0}$ implies that $V(x)'\tilde{F}(x) > 0.$

Theorem 0.13. *All perturbed best response dynamics satisfy virtual positive correlation* (9).

Proof. Let $x \in X$ be a social state at which $V(x) \neq \mathbf{0}$. Then

(10) $\qquad y \equiv \tilde{M}(F(x)) = \bar{M}(\Phi F(x)) \neq x.$

Since $\nabla v = (\bar{M})^{-1}$, we can rewrite the equality in expression (10) as $\nabla v(y) = \Phi F(x)$. So since $V(x) \in TX$, we find that

$$\begin{aligned} V(x)'\tilde{F}(x) &= (\tilde{M}(F(x)) - x)'\Phi\tilde{F}(x) \\ &= \left(\bar{M}(\Phi F(x)) - x\right)' (\Phi F(x) - \nabla v(x)) \\ &= (y - x)'(\nabla v(y) - \nabla v(x)) > 0, \end{aligned}$$

where the final inequality holds because $y \neq x$ and v is strictly convex. $\blacksquare$

With 5.3.1: Characterization of Nash equilibrium via excess payoffs

The excess payoff vector $\hat{F}(x)$ cannot lie in the interior of the negative orthant $\mathbb{R}^n_-$: for this to happen, every strategy would have to earn a below average payoff. We can thus let the domain of the function τ be the set $\mathbb{R}^n_* = \mathbb{R}^n \setminus \text{int}(\mathbb{R}^n_-)$. Then $\text{int}(\mathbb{R}^n_*) = \mathbb{R}^n \setminus \mathbb{R}^n_-$ is the set of excess payoff vectors under which at least one strategy earns an above average payoff, while $\text{bd}(\mathbb{R}^n_*) = \text{bd}(\mathbb{R}^n_-)$ is the set of excess payoff vectors under which no strategy earns an above average payoff.

This definition leads to a new characterization of Nash equilibrium in terms of excess payoff vectors.

Lemma 0.14. *$\hat{F}(x) \in \text{bd}(\mathbb{R}^n_*)$ if and only if $x \in NE(F)$.*

Proof.
$$\begin{aligned}
\hat{F}(x) \in \text{bd}(\mathbb{R}^n_*) &\Leftrightarrow F_i(x) \le \textstyle\sum_{k\in S} x_k F_k(x) \text{ for all } i \in S \\
&\Leftrightarrow \text{ there exists a } c \in \mathbb{R} \text{ such that } F_i(x) \le c \text{ for all } i \in S, \\
&\qquad \text{with } F_j(x) = c \text{ whenever } x_j > 0 \\
&\Leftrightarrow F_j(x) = \max_{k\in S} F_k(x) \text{ whenever } x_j > 0 \\
&\Leftrightarrow x \in NE(F). \quad \blacksquare
\end{aligned}$$

With 5.3.2: Analysis of pairwise comparison dynamics

Pairwise comparison dynamics are those obtained from revision protocols ρ that are Lipschitz continuous and *sign preserving*:

(11) $$\text{sgn}(\rho_{ij}(\pi)) = \text{sgn}([\pi_j - \pi_i]_+) \quad \text{for all } i, j \in S.$$

Theorem 0.15. *Every pairwise comparison dynamic satisfies* (PC) *and* (NS).

The theorem follows from the following three lemmas.

Lemma 0.16. $x \in NE(F) \Leftrightarrow$ *For all $i \in S$, $x_i = 0$ or $\sum_{j\in S} [F_j(x) - F_i(x)]_+ = 0$.*

Proof. Both statements say that each strategy in use at x is optimal. ■

Lemma 0.17. $V(x) = \mathbf{0} \Leftrightarrow$ *For all $i \in S$, $x_i = 0$ or $\sum_{j\in S} \rho_{ij}(F(x)) = 0$.*

Proof. ($\Leftarrow$) Immediate.

($\Rightarrow$) Suppose that $V(x) = \mathbf{0}$. If j is an optimal strategy at x, then sign preservation implies that $\rho_{jk}(F(x)) = 0$ for all $k \in S$, and so that there is no "outflow" from strategy j:

$$x_j \sum_{i\in S} \rho_{ji}(F(x)) = 0.$$

Since $V_j(x) = 0$, there can be no "inflow" into strategy j either:

$$\sum_{i\in S} x_i \rho_{ij}(F(x)) = 0.$$

We can express this condition equivalently as

$$\text{For all } i \in S, \text{ either } x_i = 0 \text{ or } \rho_{ij}(F(x)) = 0.$$

If all strategies in S earn the same payoff at state x, the proof is complete. Otherwise, let i be a "second best" strategy—that is, a strategy whose payoff $F_i(x)$ is second highest among the payoffs available from strategies in S at x. The last observation in the previous paragraph and sign preservation tell us that there is no outflow from i. But since $V_i(x) = 0$, there is also no inflow into i:

$$\text{For all } k \in S, \text{ either } x_k = 0 \text{ or } \rho_{ki}(F(x)) = 0.$$

Iterating this argument for strategies with lower payoffs establishes the result. ■

Lemma 0.18. *(i)* $V(x)'F(x) \geq 0.$
(ii) $V(x)'F(x) = 0 \Leftrightarrow$ *For all* $i \in S, x_i = 0$ *or* $\sum_{j\in S} \rho_{ij}(F(x))[F_j(x) - F_i(x)]_+ = 0.$

Proof. We compute the inner product as follows:

$$\begin{aligned}
V(x)'F(x) &= \sum_{j\in S} \left(\sum_{i\in S} x_i \rho_{ij}(F(x)) - x_j \sum_{i\in S} \rho_{ji}(F(x)) \right) F_j(x) \\
&= \sum_{j\in S} \sum_{i\in S} \left(x_i \rho_{ij}(F(x)) F_j(x) - x_j \rho_{ji}(F(x)) F_j(x) \right) \\
&= \sum_{j\in S} \sum_{i\in S} x_i \rho_{ij}(F(x)) \left(F_j(x) - F_i(x) \right) \\
&= \sum_{i\in S} \left(x_i \sum_{j\in S} \rho_{ij}(F(x)) [F_j(x) - F_i(x)]_+ \right),
\end{aligned}$$

where the last equality follows from sign-preservation. Both claims directly follow. ■

With 5.3.2: Multiple revision protocols and hybrid dynamics

Recall:

Proposition 0.19. *If V^F satisfies* (PC)*, then $x \in NE(F)$ implies that $V^F(x) = \mathbf{0}$.*

Proof. If $x \in NE(F)$, then $F(x) \in NX(x)$. But $V^F(x) \in TX(x)$ (since it is a feasible direction of motion from x). Thus $V^F(x)'F(x) \le 0$, so (PC) implies that $V^F(x) = \mathbf{0}$. ■

If an agent uses the revision protocol ρ^V at intensity a and the revision protocol ρ^W at intensity b, then his behavior is described by the new revision protocol $\rho^C = a\rho^V + b\rho^W$. Since mean dynamics are linear in conditional switch rates, the mean dynamic for the combined protocol is a linear combination of the two original mean dynamics: $C^F = aV^F + bW^F$.

Theorem 0.20. *Suppose that V^F satisfies* (PC)*, that W^F satisfies* (PC) *and* (NS)*, and that $a, b > 0$. Then $C^F = aV^F + bW^F$ also satisfies* (PC) *and* (NS).

Thus imitation and Nash stationarity are not incompatible: if we combine an imitative dynamic V^F with any small amount of a pairwise comparison dynamic W^F, we obtain a combined dynamic C^F that satisfies both conditions.

Proof. To show that C^F satisfies (PC), suppose that $C^F(x) \neq \mathbf{0}$. Then either $V^F(x)$, $W^F(x)$, or both are not $\mathbf{0}$. Since V^F and W^F satisfy (PC), it follows that $V^F(x)'F(x) \geq 0$, that $W^F(x)'F(x) \geq 0$, and that at least one of these inequalities is strict. Consequently, $C^F(x)'F(x) > 0$, and so C^F satisfies (PC).

The proof that C^F satisfies (NS) is divided into three cases. First, if x is a Nash equilibrium of F, then it is a rest point of both V^F and W^F by Proposition 0.19, and hence a rest point of C^F as well. Second, if x is a non-Nash rest point of V^F, then it is not a rest point of W^F. Since $V^F(x) = \mathbf{0}$ and $W^F(x) \neq \mathbf{0}$, it follows that $C^F(x) = bW^F(x) \neq \mathbf{0}$, so x is not a rest point of C^F. Finally, suppose that x is not a rest point of V^F. Then by Proposition 0.19, x is not a Nash equilibrium, and so x is not a rest point of W^F either. Since V^F and W^F satisfy condition (PC), we know that $V^F(x)'F(x) > 0$ and that $W^F(x)'F(x) > 0$. Consequently, $C^F(x)'F(x) > 0$, implying that x is not a rest point of C^F. Thus, C^F satisfies (NS). ■

With 6.3: Efficiency in homogeneous full potential games

Definition. *We call a full potential game F* homogeneous of degree k *if each of its payoff functions $F_i : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous function of degree k (that is, if $F_i(tx) = t^k F_i(x)$ for all*

$x \in \mathbb{R}^n$ *and* $t > 0$)*, where* $k \neq -1$.

Example: Matching in normal form games with common interests.

Example: Isoelastic congestion games.

$$F_i(x) = -\sum_{\ell \in \mathcal{L}_i} c_\ell(u_\ell(x)), \quad \text{where } c_\ell(u) = a_\ell u^\eta \ \ (\text{so that } \tfrac{u c_\ell'(u)}{c_\ell(u)} \equiv \eta).$$

Theorem 0.21. *The full potential game F is homogeneous of degree* $k \neq -1$ *if and only if the normalized aggregate payoff function* $\frac{1}{k+1}\bar{F}(x)$ *is a full potential function for F and is homogeneous of degree* $k+1 \neq 0$.

Proof. If the potential game F is homogeneous of degree $k \neq -1$, then $\frac{1}{k+1}\bar{F}(x) = \frac{1}{k+1}\sum_{j \in S} x_j F_j(x)$ is clearly homogeneous of degree $k+1$. Thus full externality symmetry and Euler's Theorem imply that

$$\begin{aligned}
\frac{\partial}{\partial x_i}\left(\frac{1}{k+1}\bar{F}(x)\right) &= \frac{1}{k+1}\left(\sum_{j \in S} x_j \frac{\partial F_j}{\partial x_i}(x) + F_i(x)\right) \\
&= \frac{1}{k+1}\left(\sum_{j \in S} x_j \frac{\partial F_i}{\partial x_j}(x) + F_i(x)\right) \\
&= \frac{1}{k+1}\left(kF_i(x) + F_i(x)\right) \\
&= F_i(x),
\end{aligned}$$

so $\frac{1}{k+1}\bar{F}$ is a full potential function for F.

And if $\frac{1}{k+1}\bar{F}$ is homogeneous of degree $k+1 \neq 0$ and is a full potential function for F, then each payoff function $F_i = \frac{\partial}{\partial x_i}(\frac{1}{k+1}\bar{F})$ is homogeneous of degree k. ■

Intuition: the payoff an agent receives from choosing a strategy is directly proportional to the social impact of his choice.

After 6.4: Inefficiency and inefficiency bounds in congestion games

Inefficiency and inefficiency bounds (***"price of anarchy"***) ***in congestion games***

Braess's paradox: adding a link to a network can increase equilibrium travel times.

A congestion game's *inefficiency ratio* (or "*price of anarchy*") is the ratio between the game's equilibrium social cost and its minimal feasible social cost $\bar{C}(x) = -\bar{F}(x)$.

Example 0.22. Two parallel links, $c_1(u) = 1$ and $c_2(u) = u$. In the unique Nash equilibrium, all drivers travel on route 2, creating a social cost of 1. The efficient state, which minimizes $\bar{C}(x) = x_1 + (x_2)^2$, is $x_{\min} = (\frac{1}{2}, \frac{1}{2})$; it generates a social cost of $\bar{C}(x_{\min}) = \frac{3}{4}$. Thus, the inefficiency ratio in this game is $\frac{4}{3}$.

Remarkably, this example is the worst case for any network with affine costs.

Theorem 0.23. *Let C be a congestion game whose cost functions c_ℓ are nonnegative, nondecreasing, and affine: $c_\ell(u) = a_\ell + b_\ell u$ with $a_\ell, b_\ell \geq 0$. If $x^* \in NE(C)$ and $x \in X$, then $\bar{C}(x^*) \leq \frac{4}{3}\bar{C}(x)$.*

Proof. Fix $x^* \in NE(C)$ and $x \in X$, and write $v_\ell^* = u_\ell(x^*)$ and $v_\ell = u_\ell(x)$. Let $\underline{\mathcal{L}} = \{\ell \in \mathcal{L} : v_\ell < v_\ell^*\}$ be the set of facilities that are underutilized at x relative to x^*. Since x^* is a Nash equilibrium, and by our assumptions on c_ℓ, we have that

$$\begin{aligned}
\bar{C}(x^*) &= \sum_{\ell \in \mathcal{L}} c_\ell(v_\ell^*)\, v_\ell^* \\
&\leq \sum_{\ell \in \mathcal{L}} c_\ell(v_\ell^*)\, v_\ell \\
&= \sum_{\ell \in \mathcal{L}} c_\ell(v_\ell)\, v_\ell + \sum_{\ell \in \mathcal{L}} \big(c_\ell(v_\ell^*) - c_\ell(v_\ell)\big)\, v_\ell \\
&\leq \bar{C}(x) + \sum_{\ell \in \underline{\mathcal{L}}} \big(c_\ell(v_\ell^*) - c_\ell(v_\ell)\big)\, v_\ell \\
&\leq \bar{C}(x) + \tfrac{1}{4} \sum_{\ell \in \underline{\mathcal{L}}} v_\ell^* \cdot c_\ell(v_\ell^*) \\
&= \bar{C}(x) + \tfrac{1}{4}\bar{C}(x^*),
\end{aligned}$$

where the last inequality is easily verified by drawing a picture, using the functional form of c_ℓ. Rearranging yields $\bar{C}(x^*) \leq \frac{4}{3}\bar{C}(x)$. ■

Before 6.5: Stability and recurrence for flows

Let $X \subset \mathbb{R}^n$ be compact, and let $V : X \to \mathbb{R}^n$ be Lipschitz continuous with $V(x) \in TX(x)$ for all $x \in X$. Then

(D) $\qquad \dot{x} = V(x)$

has a unique solution from every initial condition in X, and these solutions exist on $[0, \infty)$ (see Theorems 0.1 and 0.2).

Let $\xi \in X$, and let $\{x_t\}_{[0,\infty}$ be the solution to (D) with $x_0 = \xi$. The *ω-limit* $\omega(\xi)$ is the set of all points that the solution from ξ approaches arbitrarily closely infinitely often:

$$\omega(\xi) = \left\{ y \in X\colon \text{ there exists } \{t_k\}_{k=1}^{\infty} \text{ with } \lim_{k\to\infty} t_k = \infty \text{ such that } \lim_{k\to\infty} x_{t_k} = y \right\}.$$

Proposition 0.24. (*i*) *$\omega(\xi)$ is non-empty and connected.*

(*ii*) *$\omega(\xi)$ is closed. In fact, $\omega(\xi) = \bigcap_{t\geq 0} \mathrm{cl}(\{x_s\colon s \geq t\})$, where $\{x_t\}$ solves* (D) *with $x_0 = \xi$.*

(*iii*) *$\omega(\xi)$ is invariant under* (D).

(The set $Y \subseteq X$ is *forward invariant* under (D) if for each solution $\{x_t\}$ of (D), $x_0 \in Y$ implies that $x_t \in Y$ for all $t > 0$. Y is *backward invariant* if $x_0 \in Y$ implies that x_t exists and is in Y for all $t < 0$. Y is *invariant* it is both forward and backward invariant.)

We write $\Omega = \bigcup_{\xi \in X} \omega(\xi)$ and $\bar{\Omega} = \mathrm{cl}(\Omega)$.

Let $A \subseteq X$ be a closed set, and call $O \subseteq X$ a *neighborhood* of A if it is open relative to X and contains A. We say that A is *Lyapunov stable* under (D) if for every neighborhood O of A there exists a neighborhood O' of A such that every solution $\{x_t\}$ that starts in O' is contained in O: that is, $x_0 \in O'$ implies that $x_t \in O$ for all $t \geq 0$. A is *attracting* if there is a neighborhood Y of A such that every solution that starts in Y converges to A: that is, $x_0 \in Y$ implies that $\omega(x_0) \subseteq A$. A is *globally attracting* if it is attracting with $Y = X$. Finally, the set A is *asymptotically stable* if it is Lyapunov stable and attracting, and it is *globally asymptotically stable* if it is Lyapunov stable and globally attracting.

Example: Consider a flow on a circle that proceeds clockwise except at a single rest point. The rest point is attracting, but not Lyapunov stable, and hence not asymptotically stable.

Example: An asymptotically stable set need not be backward invariant. Consider a flow on the real line that always moves toward the unique rest point at the origin. Then any closed interval containing the origin is asymptotically stable.

Before 6.5: Lyapunov functions

In general, a *Lyapunov function* is a function whose value changes monotonically along solution trajectories of (D). If monotonicity is strict whenever (D) is not at rest, the term *strict Lyapunov function* is often used.

Lemma 0.25. *Suppose that the function $L : Y \to \mathbb{R}$ and the trajectory $\{x_t\}_{t\geq 0}$ are Lipschitz continuous.*

(*i*) *If $\dot{L}(x_t) \leq 0$ for almost all $t \geq 0$, then the map $t \mapsto L(x_t)$ is nonincreasing.*

(*ii*) *If in addition* $\dot{L}(x_s) < 0$*, then* $L(x_t) < L(x_s)$ *for all* $t > s$.

Proof. The composition $t \mapsto L(x_t)$ is Lipschitz continuous, and therefore absolutely continuous. By the fundamental theorem of calculus, when $t > s$ we have

$$L(x_t) - L(x_s) = \int_s^t \dot{L}(x_u)\,\mathrm{d}u \leq 0,$$

where the inequality is strict if $\dot{L}(x_s) < 0$. ■

Call the (relatively) open set $Y \subset X$ *inescapable* if for each solution trajectory $\{x_t\}_{t\geq 0}$ with $x_0 \in Y$, we have that $\mathrm{cl}(\{x_t\}) \cap \mathrm{bd}(Y) = \varnothing$.

Theorem 0.26. *Let* $Y \subset X$ *be relatively open and inescapable under* (D)*. Let* $L\colon Y \to \mathbb{R}$ *be* C^1*, and suppose that* $\dot{L}(x) \equiv \nabla L(x)' V(x) \leq 0$ *for all* $x \in Y$*. Then* $\omega(\xi) \subseteq \{x \in Y\colon \dot{L}(x) = 0\}$ *for all* $\xi \in Y$*. Thus, if* $\dot{L}(x) = 0$ *implies that* $V(x) = \mathbf{0}$*, then* $\omega(\xi) \subseteq RP(V) \cap Y$*.*

Proof. Let $\{x_t\}$ be the solution to (D) with initial condition $x_0 = \xi \in Y$, let $\chi \in \omega(\xi)$, and let $\{y_t\}$ be the solution to (D) with $y_0 = \chi$. Since Y is inescapable, the closures of trajectories $\{x_t\}$ and $\{y_t\}$ are contained in Y.

Suppose by way of contradiction that $\dot{L}(\chi) \neq 0$. Since $\chi \in \omega(\xi)$, we can find a divergent sequence of times $\{t_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x_{t_k} = \chi = y_0$. Since solutions to (D) are unique, and hence continuous in their initial conditions, we have that

$$(12) \qquad \lim_{k\to\infty} x_{t_k+1} = y_1, \text{ and hence that } \lim_{k\to\infty} L(x_{t_k+1}) = L(y_1).$$

But since $y_0 = \chi \in \omega(\xi)$ and $\dot{L}(\chi) \neq 0$, applying Lemma 0.25 to both $\{x_t\}$ and $\{y_t\}$ yields

$$L(x_t) \geq L(\chi) > L(y_1)$$

for all $t \geq 0$, contradicting the second limit in (12). This proves the first claim of the theorem, and the second claim follows immediately from the first. ■

Theorem 0.27. *Let* $A \subseteq X$ *be closed, and let* $Y \subseteq X$ *be a neighborhood of* A*. Let* $L\colon Y \to \mathbb{R}_+$ *be* C^1 *with* $L^{-1}(0) = A$*.*

(*i*) *If* $\dot{L}(x) \equiv \nabla L(x)' V(x) \leq 0$ *for all* $x \in Y - A$*, then* A *is Lyapunov stable under* (D).
(*ii*) *If* $\dot{L}(x) < 0$ *for all* $x \in Y - A$*, then* A *is asymptotically stable under* (D).
(*iii*) *If in* (*ii*) $Y = X$*, then* A *is globally asymptotically stable under* (D).

Proof. For part (i), let O be a neighborhood of A such that $\mathrm{cl}(O) \subset Y$. Let $c = \min_{x \in \mathrm{bd}(O)} L(x)$, so that $c > 0$. Finally, let $O' = \{x \in O \colon L(x) < c\}$. Lemma 0.25 implies that solution trajectories that start in O' do not leave O, and hence that A is Lyapunov stable.

Parts (ii) and (iii) follow from part (i) and Theorem 0.26. ■

With 6.5: Global convergence and local stability in potential games

Global convergence in potential games

We saw earlier that if an evolutionary dynamic satisfies positive correlation (PC), then in any potential game, the potential function f serves as a strict Lyapunov function:

$$\dot{f}(x_t) = \nabla f(x_t)'\dot{x}_t = F(x_t)'V^F(x_t) \geq 0, \text{with equality only when } V^F(x) = \mathbf{0}.$$

This fact and Theorem 0.26 yield:

Theorem 0.28. *Let F be a potential game, and let $\dot{x} = V^F(x)$ be an Lipschitz continuous evolutionary dynamic for F that satisfies* (PC). *Then $\Omega(V^F) = RP(V^F)$. In particular,*

(*i*) *If V^F is an imitative dynamic, then $\Omega(V^F) = RE(F)$.*
(*ii*) *If V^F is an excess payoff dynamic or a pairwise comparison dynamic, then $\Omega(V^F) = NE(F)$.*

An analogous result holds for the best response dynamic.

For perturbed best response dynamics we have

Theorem 0.29. *Let F be a potential game with potential function f, and let $\dot{x} = V^{F,v}(x)$ be the perturbed best response dynamic for F generated by the admissible deterministic perturbation v. Define the* perturbed potential function $\tilde{f} \colon \mathrm{int}(X) \to \mathbb{R}$ *by*

$$\tilde{f}(x) = f(x) - v(x).$$

Then $\tilde{f}$ is a strict Lyapunov function for $V^{F,v}$, and so $\Omega(V^{F,v}) = PE(F, v)$.

Proof. Since $\nabla \tilde{f}(x) = F(x) - \nabla v(x) \equiv \tilde{F}(x)$, we have $\dot{\tilde{f}}(x) = \nabla \tilde{f}(x)'\dot{x} = \tilde{F}(x)'V^{F,v}(x)$, so virtual positive correlation (9) implies that $\tilde{f}$ is a strict Lyapunov function for $V^{F,v}$. Since $PE(F, v) \equiv RP(V^{F,v})$, that $\Omega(V^{F,v}) = PE(F, v)$ follows from Theorem 0.26. ■

Local stability in potential games

Theorem 0.30. *Let F be a potential game with potential function f, let V^F be an evolutionary dynamic for F.*

(*i*) *If $A \subseteq NE(F)$ is a local maximizer set of f, and V^F satisfies positive correlation* (PC), *then A is Lyapunov stable. If in addition V^F satisfies Nash stationarity* (NS) *and A is isolated in $NE(F)$, then A is asymptotically stable.*

(*ii*) *Conversely, if V^F satisfies* (PC) *and* (NS) *and $A \subseteq NE(F)$ is a smoothly connected asymptotically stable set, then A is a local maximizer set of f and is isolated in $NE(F)$.*

With 6.6: Local stability of strict equilibrium

We say that a dynamic V^F for game F satisfies *strong positive correlation* in $Y \subseteq X$ if

(SPC) There exists a $c > 0$ such that for all $x \in Y$,

$$V^F(x) \neq \mathbf{0} \text{ implies that } \operatorname{Corr}(V^F(x), F(x)) = \frac{V^F(x)'\Phi F(x)}{|V^F(x)|\,|\Phi F(x)|} \geq c.$$

That is, the correlation between strategies' growth rates and payoffs, or equivalently the cosine of the angle between the growth rate and excess payoff vectors, must be bounded away from zero on Y.

Theorem 0.31. *Let e_k be a strict equilibrium of F, and suppose that the dynamic* (D) *satisfies strong positive correlation* (SPC) *in some neighborhood of e_k in X. Define the function $L\colon X \to \mathbb{R}$ by*

$$L(x) = (e_k - x)'F(e_k).$$

Then $L(x) \geq 0$, with equality only when $x = e_k$, and there is a neighborhood of e_k on which $\dot{L}(x) \leq 0$, with equality only when $V^F(x) = \mathbf{0}$. Thus e_k is Lyapunov stable under (D), *and if e_k is an isolated rest point of* (D), *e_k is asymptotically stable under* (D).

Explain the construction in a picture.

Proof. To prove the first claim about L, observe that $L(x)$ is the difference between the payoffs to pure strategy k and mixed strategy x at pure state e_k. Thus, that $L(x) \geq 0$, with equality only when $x = e_k$, is immediate from the fact that e_k is a strict equilibrium.

To prove the second claim, note first that since e_k is a strict equilibrium, $F(e_k)$ is not a constant vector, and so $\Phi F(e_k) \neq \mathbf{0}$. Thus, since F is continuous, there is a neighborhood of

e_k in X on which $\Phi F(x)/|\Phi F(x)|$ is continuous, which implies that there is a neighborhood O of e_k such that

$$\sum_{i\in S}\left|\frac{\Phi F_i(x)}{|\Phi F(x)|}-\frac{\Phi F_i(e_k)}{|\Phi F(e_k)|}\right|<\frac{c}{2}\ \text{ for all } x\in O. \tag{13}$$

We can suppose that O is contained in the neighborhood where the implication in condition (SPC) holds. Hence if $V^F(x)\neq \mathbf{0}$ and $x\in O$, inequality (13) and condition (SPC) imply that

$$\begin{aligned}\frac{V^F(x)'\Phi F(e_k)}{|V^F(x)|\,|\Phi F(e_k)|} &= \sum_{i\in S}\frac{V_i^F(x)}{|V^F(x)|}\;\frac{\Phi F_i(e_k)}{|\Phi F(e_k)|}\\ &= \sum_{i\in S}\frac{V_i^F(x)}{|V^F(x)|}\;\frac{\Phi F_i(x)}{|\Phi F(x)|}+\sum_{i\in S}\frac{V_i^F(x)}{|V^F(x)|}\left(\frac{\Phi F_i(e_k)}{|\Phi F(e_k)|}-\frac{\Phi F_i(x)}{|\Phi F(x)|}\right)\\ &> c-\frac{c}{2}\\ &= \frac{c}{2}.\end{aligned} \tag{14}$$

Now, the time derivative of L under (D) is

$$\dot{L}(x)=\nabla L(x)'\dot{x}=-F(e_k)'V^F(x)=-V^F(x)'\Phi F(e_k). \tag{15}$$

Thus $\dot{L}(x)=0$ if $V^F(x)=\mathbf{0}$, while if $V^F(x)\neq\mathbf{0}$ and $x\in O$, (15) and (14) imply that

$$\dot{L}(x)<-\frac{c}{2}\,|V^F(x)|\,|\Phi F(e_k)|<0. \quad \blacksquare$$

With 7.2: Differential characterization of contractive games

Definition. *The population game $F: X\to\mathbb{R}^n$ is a* contractive game *if*

$$(y-x)'(F(y)-F(x))\leq 0 \text{ for all } x,y\in X. \tag{16}$$

If the inequality in condition (16) *holds strictly whenever $x\neq y$, we call F a* strictly contractive game*, while if this inequality always binds, we call F a* null contractive game*.*

Theorem 0.32. *Suppose the population game F is C^1. Then F is a contractive game if and only if it satisfies* self-defeating externalities*:*

$$DF(x) \text{ is negative semidefinite with respect to } TX \text{ for all } x\in X. \tag{17}$$

Proof of Theorem 0.32: Suppose F is contractive. Fix $x \in X$ and $z \in TX$. Since F is C^1, it is enough to consider x in the interior of X. Let $y_\varepsilon = x + \varepsilon z$. Take a Taylor expansion:

$$F(y_\varepsilon) = F(x) + DF(x)(y_\varepsilon - x) + o(\left|y_\varepsilon - x\right|)$$
$$\Rightarrow \quad (y_\varepsilon - x)'(F(y_\varepsilon) - F(x)) = (y_\varepsilon - x)'DF(x)(y_\varepsilon - x) + o(\left|y_\varepsilon - x\right|^2).$$

Now suppose condition (17) holds. Let $\alpha(t) = ty + (1 - t)x$.

$$\begin{aligned}(y - x)'(F(y) - F(x)) &= (y - x)' \left(\int_0^1 DF(\alpha(t))(y - x)\,\mathrm{d}t\right)\\ &= \int_0^1 (y - x)'DF(\alpha(t))(y - x)\,\mathrm{d}t \le 0. \blacksquare\end{aligned}$$

After 7.4: Existence of Nash equilibrium in contractive games

In addition to its role in establishing that the set of Nash equilibria of a contractive game is convex, the GNSS concept enables us to carry out an important theoretical exercise: devising an elementary proof of existence of Nash equilibrium in contractive games—in other words, one that does not rely on an appeal to a fixed point theorem. The heart of the proof, Proposition 0.33, is a finite analogue of the result we seek.

Proposition 0.33. *Let F be a contractive game, and let Y be a finite subset of X. Then there exists a state $x^* \in \mathrm{conv}(Y)$ such that $(y - x^*)'F(y) \le 0$ for all $y \in Y$.*

Proof. Suppose that Y has m elements. Define a two player zero-sum game $U = (U^1, U^2) = (Z, -Z)$ with $n^1 = n^2 = m$ as follows:

$$Z_{xy} = (x - y)'F(y).$$

In this game, player 2 chooses a "status quo" state $y \in Y$, player 1 chooses an "invader" $x \in Y$, and the payoff Z_{xy} is the invader's "relative payoff" in F. Split Z into its symmetric and skew-symmetric parts:

$$Z^S = \tfrac{1}{2}(Z + Z') \text{ and } Z^{SS} = \tfrac{1}{2}(Z - Z').$$

Since F is contractive,

$$Z^S_{xy} = \tfrac{1}{2}\left((x - y)'F(y) + (y - x)'F(x)\right) = \tfrac{1}{2}(x - y)'(F(y) - F(x)) \ge 0$$

for all $x, y \in Y$.

The Minmax Theorem tells us that in any zero sum game, player 1 has a strategy that guarantees him the value of the game. In the skew-symmetric game $U^{SS} = (Z^{SS}, -Z^{SS}) = (Z^{SS}, (Z^{SS})')$, the player roles are interchangeable, so the game's value must be zero. Since $Z = Z^{SS} + Z^S$ and $Z^S \geq 0$, the value of $U = (Z, -Z)$ must be *at least* zero. In other words, if $\lambda \in \mathbb{R}^m$ is a maxmin strategy for player 1, then

$$\sum_{x \in Y} \sum_{y \in Y} \lambda_x Z_{xy} \mu_y \geq 0$$

for all mixed strategies μ of player 2. If we let

$$x^* = \sum_{x \in Y} \lambda_x x \in \text{conv}(Y)$$

and fix an arbitrary pure strategy $y \in Y$ for player 2, we find that

$$0 \leq \sum_{x \in Y} \lambda_x Z_{xy} = \sum_{x \in Y} \lambda_x (x - y)' F(y) = (x^* - y)' F(y). \blacksquare$$

With this result in hand, existence of Nash equilibrium in contractive games follows from a simple compactness argument. Since F is contractive,

$$NE(F) = GNSS(F) = \bigcap_{y \in X} \{x \in X : (y - x)' F(y) \leq 0\}.$$

Proposition 0.33 shows that if we take the intersection above over an arbitrary finite set $Y \subset X$ instead of over X itself, then the intersection is nonempty. Since X is compact, the finite intersection property allows us to conclude that $GNSS(F)$ is nonempty itself.

With 7.5: Linear differential equations

The simplest ordinary differential equations on $\mathbb{R}^n$ are *linear differential equations*:

(L) $\qquad \dot{x} = Ax, \quad A \in \mathbb{R}^{n \times n}.$

Example: If $n = 1$, so that $\dot{x} = ax$, we have $x_t = \xi \exp(at)$. The flow of (L) is an contraction if $a < 0$, and an expansion if $a > 0$.

Example: If $n = 2$, the nature of the dynamics depends on the eigenvalues and diagonalizability of A. If A is real diagonalizable, then generically the flow of (L) is a stable or unstable node (if the eigenvalues have the same sign) or a saddle (if not). If A has a pair of complex eigenvalues, (L), the flow of (L) is a stable or unstable spiral (if the real part of the eigenvalues is negative or positive) or a center (if it is zero). If A is not diagonalizable, then generically the flow of (L) is a stable or unstable improper node. (Show pictures.)

More generally we have:

Theorem 0.34. *Let $\{x_t\}_{t\in(-\infty,\infty)}$ be the solution to* (L) *from initial condition x_0. Then each coordinate of x_t is a linear combination of terms of the form $t^k e^{at}\cos(bt)$ and $t^k e^{at}\sin(bt)$, where $a + \mathrm{i}\, b \in \mathbb{C}$ is an eigenvalue of A and $k \in \mathbb{Z}_+$ is less than the algebraic multiplicity of this eigenvalue.*

Theorem 0.34 shows in generic cases, the stability of the origin under the linear equation (L) is determined by the eigenvalues $\{a_1 + \mathrm{i}\, b_1, \dots, a_n + \mathrm{i}\, b_n\}$ of A: more precisely, by the real parts a_i of these eigenvalues. If each a_i is negative, then all solutions to (L) converge to the origin; in this case, the origin is called a *sink*, and the flow of (L) is called a *contraction*. If instead each a_i is positive, then all solutions besides the stationary solution at the origin move away from the origin; in this case, the origin is called a *source*, and the flow of (L) is called an *expansion*.

When the origin is a sink, solutions to (L) converge to the origin at an exponential rate: for any $a > 0$ satisfying $a < |a_i|$ for all $i \in \{1, \dots n\}$, there is a $C = C(a) \geq 1$ such that

$$\text{(18)} \qquad \mathbf{0} \text{ is a sink } \Leftrightarrow \left|\phi_t(\xi)\right| \leq C e^{-at} |\xi| \quad \text{for all } t \geq 0 \text{ and all } \xi \in \mathbb{R}^n.$$

If the origin is the source, the analogous statement holds if time is run backward.

More generally, the flow of (L) may be contracting in some directions and expanding in others. In the generic case in which each real part a_i of an eigenvalue of A is nonzero, the differential equation $\dot{x} = Ax$, its rest point at the origin, and the flow of (L) are all said to be *hyperbolic*. Hyperbolic linear flows come in three varieties: contractions (if all a_i are negative, as in (18)), expansions (if all a_i are positive), and *saddles* (if there is at least one a_i of each sign). If a linear flow is hyperbolic, then the origin is globally asymptotically stable if it is a sink, and it is unstable otherwise.

If the flow of (L) is hyperbolic, then A has k eigenvalues with negative real part (counting algebraic multiplicities) and $n - k$ eigenvalues with positive real part. In this case, we can view $\mathbb{R}^n = E^s \oplus E^u$ as the direct sum of subspaces of dimensions $\dim(E^s) = k$ and $\dim(E^u) = n - k$, where the *stable subspace* E^s contains all solutions of (L) that converge to the origin at an exponential rate (as in (18)), while the *unstable subspace* E^u contains all solutions of (L) that converge to the origin at an exponential rate if time is run backward.

With 7.5: Linearization of nonlinear differential equations

Now consider the C^1 differential equation

(D) $\dot{x} = V(x)$

with rest point x^*. By the definition of the derivative, we can approximate the value of V in the neighborhood of x^* via

$$V(y) = \mathbf{0} + DV(x^*)(y - x^*) + o(|y - x^*|).$$

This suggests that the behavior of the dynamic (D) near x^* can be approximated by the behavior near the origin of the linear equation

(L) $\dot{y} = DV(x^*)y.$

We say that ϕ and ψ are *topologically conjugate* on X and Y if there is a homeomorphism $h : X \to Y$ such that $\phi_t(x_0) = h^{-1} \circ \psi_t \circ h\,(x_0)$ for all times $t \in I$. In other words, ϕ and ψ are topologically conjugate if there is a continuous map with continuous inverse that sends trajectories of ϕ to trajectories of ψ (and vice versa), preserving the rate of passage of time.

Theorem 0.35 (The Hartman-Grobman Theorem)**.** *Let ϕ and ψ be the flows of the C^1 equation* (D) *and the linear equation* (L)*, where x^* is a hyperbolic rest point of* (D)*. Then there exist neighborhoods O_{x^*} of x^* and $O_{\mathbf{0}}$ of the origin $\mathbf{0}$ on which ϕ and ψ are topologically conjugate.*

Corollary 0.36. *Let x^* be a hyperbolic rest point of* (D)*. Then x^* is asymptotically stable if all eigenvalues of $DV(x^*)$ have negative real parts, and x^* is unstable otherwise.*

Suppose that $DV(x^*)$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part, counting algebraic multiplicities. The *stable manifold theorem* tells us that within some neighborhood of x^*, there is k dimensional *local stable manifold* M^s_{loc} on which solutions converge to x^* at an exponential rate, and an $n - k$ dimensional *local unstable manifold* M^u_{loc} on which solutions converge to x^* at an exponential rate if time is run backward. Both of these manifolds can be extended globally: the k dimensional (*global*) *stable manifold* M^s includes all solutions of (D) that converge to x^*, while the $n - k$ dimensional (*global*) *unstable manifold* M^u includes all solutions that converge to x^* as time runs backward. Among other implications of the existence of these manifolds, it follows that if x^* is hyperbolic and unstable, then the set M^s of states from which solutions converge to x^* is of measure zero, while the complement of this set is open, dense, and of full measure.

With 7.5: Local stability of ESS via linearization

We want to approximate the behavior of

(D) $\qquad \dot{x} = V(x)$

near x^* using the linear dynamic

(L) $\qquad \dot{y} = DV(x^*)y.$

Since $V\colon X \to TX$, $DV(x^*)$ maps TX into itself. Therefore, we can (and should) think of (L) as a dynamic on TX.

(Why must $DV(x^*)$ map TX into itself? Let $x \in X$ and $z \in TX(x)$, and write $V(x + \varepsilon z) = V(x) + \varepsilon DV(x)z + o(\varepsilon)$. Since $V(x)$ and $V(x + \varepsilon z)$ are both in TX, so is $DV(x)z$. Thus, in (L), $\dot{y}$ lies in TX whenever $y = DV(x^*)y$ lies in TX, implying that TX is invariant under (L).)

Consequently, rather than looking at all the eigenvalues of $DV(x^*)$, we should only consider those associated with the restricted linear map $DV(x^*)\colon TX \to TX$. One way to do this is to compute the eigenvalues of $DV(x^*)\Phi$, and to ignore the eigenvalue 0 corresponding to the eigenvector $\mathbf{1}$ (which is mapped to $\mathbf{0}$ by Φ).

The following result, called *Hines's lemma*, is often useful.

Lemma 0.37. *Suppose that $Q \in \mathbb{R}^{n\times n}$ is symmetric, satisfies $Q\mathbf{1} = \mathbf{0}$, and is positive definite with respect to TX, and that $A \in \mathbb{R}^{n\times n}$ is negative definite with respect to TX. Then each eigenvalue of the linear map $QA : TX \to TX$ has negative real part.*

The replicator dynamic

Theorem 0.38. *Let $x^* \in \operatorname{int}(X)$ be a regular ESS of F. Then x^* is linearly stable under the replicator dynamic.*

Proof. The single population replicator dynamic is given by

(R) $\qquad \dot{x}_i = V_i(x) = x_i\hat{F}_i(x).$

We saw in the proof of Theorem 0.42 that the derivative of $\hat{F}(x) = F(x) - \mathbf{1}\bar{F}(x)$ is

$$D\hat{F}(x) = DF(x) - \mathbf{1}(x'DF(x) + F(x)') = (I - \mathbf{1}x')DF(x) - \mathbf{1}F(x)'.$$

So

$$\begin{aligned} DV(x) &= D(\text{diag}(x)\hat{F}(x)) \\ &= \text{diag}(x)D\hat{F}(x) + \text{diag}(\hat{F}(x)) \\ &= \text{diag}(x)((I - \mathbf{1}x')DF(x) - \mathbf{1}F(x)') + \text{diag}(\hat{F}(x)) \\ &= Q(x)DF(x) - x\,F(x)' + \text{diag}(\hat{F}(x)), \end{aligned} \tag{19}$$

where we write $Q(x) = \text{diag}(x) - xx'$.

Since x^* is an interior Nash equilibrium, $F(x^*)$ is a constant vector, implying that $F(x^*)'\Phi = \mathbf{0}'$ and that $\hat{F}(x^*) = \mathbf{0}$. Thus, equation (19) becomes

$$DV(x^*)\Phi = Q(x^*)DF(x^*)\Phi. \tag{20}$$

Since the matrices $Q(x^*)$ and $DF(x^*)\Phi$ satisfy the conditions of Hines's Lemma, the eigenvalues of $DV(x^*)\Phi$ corresponding to directions in TX have negative real part. ■

One can also use linearization to establish stability of Nash equilibria that are not ESSs.

Example: Zeeman's game.

$$A = \begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}.$$

At the Nash equilibrium $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$,

$$DV(x^*)\Phi = Q(x^*)DF(x^*)\Phi = Q(x^*)A\Phi = \frac{1}{9}\begin{pmatrix} 4 & 9 & -13 \\ -5 & -9 & 14 \\ 1 & 0 & -1 \end{pmatrix}.$$

In addition to the irrelevant eigenvalue of 0 corresponding to eigenvector $\mathbf{1}$, this matrix has pair of complex eigenvalues, $-\frac{1}{3} \pm \text{i}\frac{\sqrt{2}}{3}$, corresponding to eigenvectors $(-2 \pm \text{i}(3\sqrt{2}), 1 \mp \text{i}(3\sqrt{2}), 1)'$ whose real and complex parts lie in TX. Thus x^* is linearly stable. (Show picture.)

Theorem 0.39. *Let x^* be a regular ESS of F. Then x^* is linearly stable under the replicator dynamic.*

Proof. Suppose without loss of generality that the support of x^* is $\{1, \ldots, n^*\}$, so that the number of unused strategies at x^* is $n^0 = n - n^*$.

One can show that $DV(x^*)$ must take the block diagonal form

$$DV(x^*) = \begin{pmatrix} Q^{++}D^{++} - (x^*)^+(\pi^+)' & Q^{++}D^{+0} - (x^*)^+(\pi^0)' \\ \mathbf{0} & \operatorname{diag}(\hat{F}(x^*)^0) \end{pmatrix}. \tag{21}$$

It is enough to show that if $v + \mathrm{i}w$ with $v, w \in TX$ is an eigenvector of $DV(x^*)$ with eigenvalue $a + \mathrm{i}b$, then $a < 0$.

If $v_j = w_j = 0$ whenever $j > n^*$, then the first n^* components of $v + \mathrm{i}w$ form an eigenvector of the upper left block of (21), and a variation on the proof of Theorem 0.38 shows that $a < 0$.

If not, then $v_j + \mathrm{i}w_j \neq 0$ for some $j > n^*$. In this case, since the lower right block of (21) is the diagonal matrix $\operatorname{diag}(\hat{F}(x^*)^0)$, the jth component of the eigenvector equation for $DV(x^*)$ is $\hat{F}_j(x^*)(v_j + \mathrm{i}w_j) = (a + \mathrm{i}b)(v_j + \mathrm{i}w_j)$. This implies that $a = \hat{F}_j(x^*)$ (and also that $b = w_j = 0$). But since x^* is a quasistrict equilibrium, $\hat{F}_j(x^*) < 0$, and so $a < 0$. ■

Other imitative dynamics

Theorem 0.40. *Assume that x^* is a hyperbolic rest point of both the replicator dynamic* (R) *and a given imitative dynamic* (1)*. Then x^* is linearly stable under* (R) *if and only if it is linearly stable under* (1)*. Thus, if x^* is a regular ESS that satisfies the hyperbolicity assumptions, it is linearly stable under* (1)*.*

Idea of proof. We focus on the case in which $x^* \in \operatorname{int}(X)$.

Observation 0.6 that any imitative dynamic (1) has monotone percentage growth rates:

$$\dot{x}_i = x_i G_i(x), \text{ where} \tag{22}$$

$$G_i(x) \geq G_j(x) \text{ if and only if } F_i(x) \geq F_j(x). \tag{23}$$

Property (23) imposes a remarkable amount of structure on the derivative matrix of the percentage growth rate function G at the equilibrium x^*:

Lemma 0.41. *Let x^* be an interior Nash equilibrium, and suppose that $\Phi DF(x^*)$ and $\Phi DG(x^*)$ define invertible maps from TX to itself. Then $\Phi DG(x^*)\Phi = c\,\Phi DF(x^*)\Phi$ for some $c > 0$.*

Since the linearizations of the imitative dynamic $W(x) = \operatorname{diag}(x)G(x)$ and the replicator dynamic $V(x) = \text{diag}(x)\hat{F}(x)$ are

$$DW(x^*)\Phi = Q(x^*)DG(x^*)\Phi = Q(x^*)\Phi DG(x^*)\Phi \text{ and}$$
$$DV(x^*)\Phi = Q(x^*)\Phi DF(x^*)\Phi,$$

the theorem is proved by applying Lemma 0.41. ■

The logit dynamic

It is a remarkable fact that the linearizations of the logit dynamic and the replicator dynamic at interior rest points differ only by a positive affine transformation: if

$$\dot{x}_i = V_i^{F,\eta}(x) = \frac{\exp(\eta^{-1}F_i(x))}{\sum_{j\in S}\exp(\eta^{-1}F_j(x))} - x_i,$$

has rest point $\tilde{x}^\eta$, then

$$DV^{F,\eta}(\tilde{x}^\eta) = \eta^{-1}Q(\tilde{x}^\eta)DF(\tilde{x}^\eta) - I.$$

See *PGED* Sec. 8.6.2 for details.

This fact is important for establishing striking links among the replicator dynamic, the logit dynamic, and the best response dynamic. See Gaunersdorfer and Hofbauer (1995), Hopkins (1999, 2002), and Hofbauer, Sorin, and Viossat (2009).

With 7.5: More on global convergence in contractive games

Integrable target dynamics

The BNN, best response, and logit dynamics can be expressed as target dynamics with target protocols that are reactive and only condition on the vector of excess payoffs: $\sigma_{ij}^F(x) = \tau_j(\hat{F}(x))$.

In general, such dynamics need not converge in contractive games.

Example: In good RPS, consider the protocol

$$\begin{pmatrix}\tau_R(\hat{\pi})\\ \tau_P(\hat{\pi})\\ \tau_S(\hat{\pi})\end{pmatrix} = \begin{pmatrix}[\hat{\pi}_R]_+g^\varepsilon(\hat{\pi}_S)\\ [\hat{\pi}_P]_+g^\varepsilon(\hat{\pi}_R)\\ [\hat{\pi}_S]_+g^\varepsilon(\hat{\pi}_P)\end{pmatrix}, \tag{24}$$

where for $\varepsilon > 0$, g^ε: $\mathbf{R} \to \mathbf{R}$ be a continuous decreasing function that equals 1 on $(-\infty, 0]$, equals ε^2 on $[\varepsilon, \infty)$, and is linear on $[0, \varepsilon]$. Under this protocol, the weight placed on a strategy is proportional to positive part of the strategy's excess payoff, as in the protocol for the BNN dynamic; however, this weight is only of order ε^2 if the strategy it beats in RPS has an excess payoff greater than ε. This protocol satisfies acuteness, and so the corresponding dynamic satisfies (PC) and (NS). But the dynamic cycles in Good RPS games when $\varepsilon > 0$ is small. (Show picture.)

But convergence can be ensured when τ is *integrable*: there is a C^1 *revision potential* $\gamma\colon \mathbb{R}^n \to \mathbb{R}$ such that

(25) $\quad \tau \equiv \nabla\gamma.$

This is weaker than *separability*:

(26) $\quad \tau_i(\hat{\pi})$ is independent of $\hat{\pi}_{-i}$.

If τ satisfies (26), then it satisfies (25) with

(27) $$\gamma(\hat{\pi}) = \sum_{i\in S} \int_0^{\hat{\pi}_i} \tau_i(s)\,\mathrm{d}s.$$

Roughly speaking, integrability requires that the weight placed on strategy j not provide systematic information about other strategies' excess payoffs.

Geometrically, integrability ensures that along a closed path around a GNSS, $V^F(x)$ tends to deviate from $F(x)$ in the direction of the GNSS.

Practically, the revision potential provides a building block for constructing Lyapunov functions.

We first consider integrable excess payoff dynamics. These include the BNN dynamic:

$$\tau_i(\hat{\pi}) = [\hat{\pi}_i]_+ \quad \Rightarrow \quad \gamma(\hat{\pi}) = \frac{1}{2}\sum_{i\in S}[\hat{\pi}_i]_+^2.$$

Theorem 0.42. *Let F be a C^1 contractive game, and let $\dot{x} = V^F(x)$ be the integrable excess payoff dynamic for F based on revision protocol τ with revision potential γ. Define the C^1 function $\Gamma : X \to \mathbb{R}$ by*

$$\Gamma(x) = \gamma(\hat{F}(x)).$$

Then Γ *is a strict Lyapunov function for* V_F*, and* $NE(F)$ *is globally attracting.*

In addition, if F *admits a unique Nash equilibrium, or if* τ *is separable, then* $NE(F)$ *is globally asymptotically stable.*

Proof that $NE(F)$ *is globally attracting*:

$$\begin{aligned}
D\bar{F}(x) &= D(x'F(x)) = x'DF(x) + F(x)'. \\
D\hat{F}(x) &= D(F(x) - \mathbf{1}\,\bar{F}(x)) \\
&= DF(x) - \mathbf{1}\,D\bar{F}(x) \\
&= DF(x) - \mathbf{1}(x'DF(x) + F(x)'). \\
\dot{\Gamma}(x) &= \nabla\Gamma(x)'\dot{x} \\
&= \nabla\gamma(\hat{F}(x))'D\hat{F}(x)\dot{x} \\
&= \tau(\hat{F}(x))'\left(DF(x) - \mathbf{1}\,(x'DF(x) + F(x)')\right)\dot{x} \\
&= \left(\tau(\hat{F}(x)) - \tau(\hat{F}(x))'\mathbf{1}x\right)' DF(x)\dot{x} - \tau(\hat{F}(x))'\mathbf{1}\,F(x)'\dot{x} \\
&= \dot{x}'DF(x)\dot{x} - (\tau(\hat{F}(x))'\mathbf{1})(F(x)'\dot{x}) \\
&\leq 0.
\end{aligned}$$

The inequality binds if and only if $x \in RP(V^F) = NE(F)$. That $NE(F)$ is globally attracting then follows from Theorem 0.26. ■

For the best response dynamic, the target protocol is the maximizer correspondence

$$M(\hat{\pi}) = \underset{y \in X}{\operatorname{argmax}}\, y'\hat{\pi}.$$

Is there a way to think of M as integrable?

Consider the maximum function

$$\mu(\hat{\pi}) = \max_{y \in X} y'\hat{\pi} = \max_{i \in S} \hat{\pi}_i.$$

Then when the unique optimal strategy under π is i, we have $\mu(\hat{\pi}) = \hat{\pi}_i$, so at such states we have $\nabla\mu(\hat{\pi}) = e_i = M(\hat{\pi})$.

Theorem 0.43. *Let* F *be a* C^1 *contractive game, and let* $\dot{x} \in V^F(x)$ *be the best response dynamic for* F*. Define the Lipschitz continuous function* $G\colon X \to \mathbb{R}_+$ *by*

$$G(x) = \max_{y \in X} (y - x)'F(x) = \max_{i \in S} \hat{F}_i(x).$$

Then $G^{-1}(0) = NE(F)$. Moreover, if $\{x_t\}_{t\geq 0}$ is a solution to V^F, then for almost all $t \geq 0$ we have that $\dot{G}(x_t) \leq -G(x_t)$, and so $NE(F)$ is globally asymptotically stable under V^F.

The target protocols for perturbed best response dynamics are perturbed maximizer function

$$\tilde{M}(\hat{\pi}) = \underset{y\in \mathrm{int}(X)}{\mathrm{argmax}}\; y'\hat{\pi} - v(y),$$

where v is an admissible deterministic perturbation. One can verify that the *perturbed maximum function*

$$\tilde{\mu}(\pi) = \max_{y\in \mathrm{int}(X)} y'\pi - v(y),$$

is a potential function for $\tilde{M}$.

Theorem 0.44. *Let F be a C^1 contractive game, and let $\dot{x} = V^{F,v}(x)$ be the perturbed best response dynamic for F generated by the admissible deterministic perturbation v. Define the function $\tilde{G}\colon \mathrm{int}(X) \to \mathbb{R}_+$ by*

$$\tilde{G}(x) = \tilde{\mu}(\hat{F}(x)) + v(x),$$

Then $G^{-1}(0) = PE(F, v)$, and this set is a singleton. Moreover, $\tilde{G}$ is a strict Lyapunov function for $V^{F,v}$, and so $PE(F, v)$ is globally asymptotically stable under $V^{F,v}$.

Impartial pairwise comparison dynamics

Pairwise comparison dynamics are defined using revision protocols $\rho_{ij}(\pi)$ that are sign preserving:

$$\mathrm{sgn}(\rho_{ij}(\pi)) = \mathrm{sgn}([\pi_j - \pi_i]_+) \quad \text{for all } i, j \in S.$$

To obtain a general convergence result for contractive games, we also require *impartiality*:

$$\rho_{ij}(\pi) = \phi_j(\pi_j - \pi_i) \quad \text{for some functions } \phi_j : \mathbb{R} \to \mathbb{R}_+. \tag{28}$$

In words: the function of the payoff difference $\pi_j - \pi_i$ that describes the conditional switch rate from i to j does not depend on an agent's current strategy i.

Theorem 0.45. *Let F be a C^1 contractive game, and let $\dot{x} = V_F(x)$ be an impartial pairwise comparison dynamic for F. Define the Lipschitz continuous function $\Psi : X \to \mathbb{R}_+$ by*

$$\Psi(x) = \sum_{i \in S} \sum_{j \in S} x_i \psi_j(F_j(x) - F_i(x)), \quad \textit{where} \quad \psi_k(d) = \int_0^d \phi_k(s)\,\mathrm{d}s.$$

Then $\Psi^{-1}(0) = NE(F)$*. Moreover,* $\dot{\Psi}(x) \leq 0$ *for all* $x \in X$*, with equality if and only if* $x \in NE(F)$*, and so* $NE(F)$ *is globally asymptotically stable.*

Both the form of the Lyapunov function and the proof that it is a Lyapunov function are more complicated than in the previous cases.

With 7.5: Local stability of ESS via Lyapunov functions

Recall that $x \in X$ a *regular ESS* if

(29) x is a *quasistrict equilibrium*: $F_i(x) = \bar{F}(x) > F_j(x)$ whenever $x_i > 0$ and $x_j = 0$.

(30) $z'DF(x)z < 0$ for all $z \in TX \setminus \{\mathbf{0}\}$ such that $z_i = 0$ whenever $x_i = 0$.

Theorem 0.46. *Let* x^* *be a regular ESS of* F*. Then* x^* *is asymptotically stable under the replicator dynamic for* F*.*

Proof. Use the Lyapunov function H_{x^*}. ■

Theorem 0.47. *Let* x^* *be a regular ESS of* F*. Then* x^* *is asymptotically stable under*

(*i*) *any separable excess payoff dynamic for* F*;*
(*ii*) *the best response dynamic for* F*;*
(*iii*) *any impartial pairwise comparison dynamic for* F*.*

Idea of proof. Augment the Lyapunov functions Γ, G, and Ψ from Theorems 0.42, 0.43, and 0.45 by adding

(31) $$\Upsilon_{x^*}(x) = C \sum_{j:\, x_j^* = 0} x_j,$$

which is a multiple of the total mass of agents using strategies unused in x^*. The resulting functions $\Gamma_{x^*} = \Gamma + \Upsilon_{x^*}$, $G_{x^*} = G + \Upsilon_{x^*}$, and $\Psi_{x^*} = \Psi + \Upsilon_{x^*}$ are strict local Lyapunov functions for x^*.

The reason for this, described for case (iii), is as follows: Condition (30) says that F is contractive on the face of X containing x^*, so Ψ decreases there. Condition (29) ensures that motion from interior states near x^* is toward this face, so Υ_{x^*} decreases there. Ψ need not decrease in $\mathrm{int}(X)$, but one can show that if C is large enough, the value of $C\Upsilon_{x^*}$ always falls fast enough to compensate for any growth in the value of Ψ, so that all told the value of Ψ_{x^*} falls. ■

With 8.1: More on strict dominance and imitative dynamics

Strategy i is *strictly dominated* by strategy j on $Y \subseteq X$ if there is a $j \in S$ with $F_j(x) > F_i(x)$ for all $x \in Y$.

Theorem 0.48. *Let $\{x_t\}$ be an interior solution trajectory of an imitative dynamic in game F. If strategy $i \in S$ is strictly dominated, then $\lim_{t\to\infty}(x_t)_i = 0$.*

Proof. By definition, an imitative dynamic can be written as

$$\dot{x}_i = x_i G_i(x), \tag{32}$$

where G is continuous and satisfies

$$G_k(x) \leq G_l(x) \text{ if and only if } F_k(x) \leq F_l(x) \quad \text{for all } x \in \mathrm{int}(X). \tag{33}$$

Now suppose $i \in S$ is strictly dominated by $j \in S$. Then equation (33) implies that $G_j(x) > G_i(x)$ for all $x \in X$. Thus since X is compact and G is continuous, we can find a $c > 0$ such that $G_j(x) - G_i(x) > c$ for all $x \in X$.

Now write $r = x_i/x_j$. Equation (32) and the quotient rule imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}r = \frac{\mathrm{d}}{\mathrm{d}t}\frac{x_i}{x_j} = \frac{\dot{x}_i x_j - \dot{x}_j x_i}{(x_j)^2} = \frac{x_i G_i(x) x_j - x_j G_j(x) x_i}{(x_j)^2} = r(G_i(x) - G_j(x)) \leq -cr. \tag{34}$$

To conclude, use (34) to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(r_t \mathrm{e}^{ct}) = \dot{r}_t \mathrm{e}^{ct} + c r_t \mathrm{e}^{ct} \leq 0,$$

which implies that $r_t \leq r_0 \mathrm{e}^{-ct}$. Since $(x_t)_j$ is bounded, $(x_t)_i$ must vanish. ■

Given a game F with strategy set S, let $D^1 \subset S$ be the set of strategies that are strictly dominated by another pure strategy, and let $S^1 = S \setminus D^1$ and $X^1 = \{x \in X \colon \sum_{i \in S^1} x_i = 1\}$.

Iterating, let $D^k \subset S^{k-1}$ be the set of strategies that strictly dominated on X^{k-1}, and let $S^k = S^{k-1} \setminus D^k$ and $X^k = \{x \in X\colon \sum_{i \in S^k} x_i = 1\}$.

Theorem 0.49. *Let $\{x_t\}$ be an interior solution trajectory of an imitative dynamic in game F. If $i \in D^k$, then $\lim_{t\to\infty}(x_t)_i = 0$.*

Proof. Since strategies in D^2 are strictly dominated in X^1, they are also strictly dominated in a neighborhood O of X^1 in X. Therefore, since each interior solution converges to X^1, it is in O from some finite time T onward, so the previous proof can be used to show that the mass on strategies in D^2 vanishes. Repeat as necessary. ∎

Example: The sets S^k need not be locally stable. Consider the game

$$F(x) = Ax \;\text{ with }\; A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 4 & 0 \end{pmatrix},$$

Evidently $S^1 = \{2, 3\}$. But X^1 is not even Lyapunov stable: when no one plays strategy 2, strategy 1 is always superior to strategy 3, so solutions in the interior of the face connecting e_3 and e_1 move from the former to the latter.

However, if a game is dominance solvable, meaning that $S^k = \{i\}$, then state e_i is a strict equilibrium, and so is easily shown to be asymptotically stable.

For more on asymptotic stability of faces under imitative dynamics, see Ritzberger and Weibull (1995).

For work on domination by mixed strategies and imitative dynamics, see Hofbauer and Weibull (1996) and Viossat (2012).

With 8.3: More on iterated p-dominance and sampling best response dynamics

Consider the following revision protocol: Agents receive revision opportunities at rate 1. When an agent receives a revision opportunity, he takes a sample of size k from the population and plays a best response to the distribution of strategies in his sample.

Let $\mathbb{Z}^{n,k}_{+} = \{z \in \mathbb{Z}^n_{+}\colon \sum_{i\in S} z_i = k\}$ be the set of possible outcomes of samples of size k.

For convenience, we focus on games F in which each such outcome generates a unique best response: that is, games for which the pure best response correspondence $b^F\colon X \Rightarrow S$ is single valued on $\frac{1}{k}\mathbb{Z}^{n,k}_{+}$.

For such games, we define the *k-sampling best response function* $B^{F,k}\colon X \to X$ by

$$B_i^{F,k}(x) = \sum_{z \in \mathbb{Z}_+^{n,k}\colon b^F(\frac{1}{k}z) = \{i\}} \binom{k}{z_1 \; \ldots \; z_n} x_1^{z_1} \cdots x_n^{z_n}.$$

The *k-sampling best response dynamic* is

$$(\mathrm{S}^k) \qquad \dot{x} = B^{F,k}(x) - x.$$

For $p \in [0,1]$, we call strategy $i \in S$ is a *p-dominant equilibrium* of F if it is the unique optimal strategy whenever $x_i \geq p$. Note that 1-dominant equilibria are strict equilibria, while 0-dominant equilibria correspond to strictly dominant strategies.

Theorem 0.50. *Suppose that strategy i is $\frac{1}{k}$-dominant in game F. Then state e_i is asymptotically stable, and attracts solutions from all initial conditions with $x_i > 0$.*

Proof. It is sufficient to consider a coordination game with $S = \{0,1\}$ and in which strategy 1 is $\frac{1}{k}$-dominant but not $\frac{1}{k+1}$-dominant; for instance, $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & k-1+\varepsilon \end{smallmatrix}\right)$.

Since the sample is of size k, an agent will only choose strategy 0 if all of the agents in his sample choose strategy 0. Thus if $x_1 \in (0,1)$,

$$\begin{aligned} \dot{x}_1 &= B_1^{F,k}(x) - x_1 \\ &= \left(1 - (1 - x_1)^k\right) - x_1 \\ &= (1 - x_1) - (1 - x_1)^k \\ &> 0. \quad \blacksquare \end{aligned}$$

To obtain more general results, we consider an iterative solution concept based on p-best responses. $S^* \subseteq S$ is a *p-best response set* of F if all best responses to x are in S^* whenever $\sum_{i \in S^*} x_i \geq p$. $S^* \subseteq S$ is an *iterated p-best response set* of F if there exists a sequence $S^0, \ldots, S^m$ with $S = S^0 \supseteq \cdots \supseteq S^m = S^*$ such that S^ℓ is a p-best response set in the restricted game $F|_{S^{\ell-1}}$ for each $\ell = 1, \ldots, m$. Strategy $i \in S$ is an *iterated p-dominant equilibrium* if $\{i\}$ is an iterated p-best response set.

Theorem 0.51. *Let S^* be an iterated $\frac{1}{k}$-best response set. Then $X_{S^*} = \{x \in X\colon \operatorname{support}(x) \subseteq S^*\}$ is almost globally asymptotically stable under* (S^k). *In particular, if i is an iterated $\frac{1}{k}$-dominant equilibrium, then e_i is almost globally asymptotically stable.*

To prove that S^* is asymptotically stable, one first shows that at states where only actions in $S^{\ell-1}$ and some actions in S^ℓ receive mass, the mass on actions in S^ℓ must increase over time. This implies that $\mathcal{X}_{S^\ell} = \{x \in X\colon \text{support}(x) \subseteq S^\ell\}$ is asymptotically stable relative to $\mathcal{X}_{S^{\ell-1}}$. Then asymptotic stability of S^* can be established by repeated application of the *transitivity theorem* (Conley (1978))—see below.

The proof that S^* is almost globally attracting requires a detailed analysis that we omit.

The results above can be extended to cases where sample sizes are random (and often large), and to cases in which agents weigh their samples against a prior distribution—see Oyama, Sandholm, and Tercieux (2012) for details.

With 8.3: A transitivity theorem for asymptotic stability

For $Z \subset Y \subset X$, we say that Z is *asymptotically stable in* Y if Y is forward invariant and Z is asymptotically stable with respect to the dynamical system restricted to Y.

Theorem 0.52 (Transitivity theorem)**.** *Suppose $\dot{x} = V(x)$ is Lipschitz continuous and forward invariant on X. Let $C \subset B \subset A$ be compact sets. If B is backward invariant and asymptotically stable in A, and if C is asymptotically stable in B, then C is asymptotically stable in A.*

With 9.1: More on games with nonconvergent dynamics

The hypercycle system

Consider the game

$$F(x) = Ax = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} x.$$

The replicator dynamic for this game is known as the *hypercycle system*, was introduced by Eigen and Schuster (1979) to model of cyclical catalysis in a collection of polynucleotides during prebiotic evolution.

The unique Nash equilibrium of F is the barycenter $x^* = \frac{1}{n}\mathbf{1}$. Let $\dot{x} = R(x)$ denote the replicator dynamic for F. To evaluate the stability of this equilibrium, we compute the eigenvalue/eigenvector pairs of $DR(x^*)$:

$$(\lambda_k, v_k) = \left(\frac{1}{n}\iota_n^{(n-1)k} - \frac{2}{n^2}\sum_{j=0}^{n-1} \iota_n^{jk},\ (1, \iota_n^k, \ldots, \iota_n^{(n-1)k})' \right), \quad k = 0, \ldots, n-1.$$

where $\iota_n = \exp(\frac{2\pi \mathrm{i}}{n}) = \cos(\frac{2\pi}{n}) + \mathrm{i}\sin(\frac{2\pi}{n})$ is the nth root of unity. Eigenvalue $\lambda_0 = \frac{1}{n} - \frac{2}{n} = -\frac{1}{n}$ corresponds to eigenvector $v_0 = \mathbf{1}$ and so has no bearing on the stability analysis. For $k \geq 1$, the sum in the formula for λ_k vanishes, leaving us with $\lambda_k = \frac{1}{n}\iota_n^{(n-1)k} = \frac{1}{n}\iota_n^{-k}$. The stability of x^* therefore depends on whether any λ_k with $k > 0$ has positive real part. This largest real part is negative when $n \leq 3$, zero when $n = 4$, and positive when $n \geq 5$. It follows that x^* is asymptotically stable when $n \leq 3$, but unstable when $n \geq 5$.

The local stability results can be extended to global stability results using the Lyapunov function $\mathcal{H}(x) = -\sum_{i\in S} \log x_i$, and that global stability can also be proved when $n = 4$. When $n \geq 5$, it is possible to show that the boundary of X is repelling, as it is in the lower dimensional cases, and that the dynamic admits a stable periodic orbit.

The Hofbauer-Swinkels game

Consider the game

$$F^\varepsilon(x) = A^\varepsilon x = \begin{pmatrix} 0 & 0 & -1 & \varepsilon \\ \varepsilon & 0 & 0 & -1 \\ -1 & \varepsilon & 0 & 0 \\ 0 & -1 & \varepsilon & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

When $\varepsilon = 0$, the payoff matrix $A^\varepsilon = A^0$ is symmetric, so F^0 is a potential game with potential function $f(x) = \frac{1}{2}x'A^0x = -x_1x_3 - x_2x_4$. The function f attains its minimum of $-\frac{1}{4}$ at states $v = (\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $w = (0, \frac{1}{2}, 0, \frac{1}{2})$, has a saddle point with value $-\frac{1}{8}$ at the Nash equilibrium $x^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, and attains its maximum of 0 along the closed path of Nash equilibria γ consisting of edges $\overline{e_1e_2}$, $\overline{e_2e_3}$, $\overline{e_3e_4}$, and $\overline{e_4e_1}$. Thus if $\dot{x} = V^{F^0}(x)$ satisfies (NS) and (PC), then all solutions whose initial conditions ξ satisfy $f(\xi) > -\frac{1}{8}$ converge to γ. (In fact, if x^* is a hyperbolic rest point of V^{F^ε}, then the stable manifold theorem (see the discussion after Theorem 0.35) tells us that the set of initial conditions from which solutions converge to x^* is a manifold of dimension no greater than 2, and hence has measure zero.)

Now suppose that $\varepsilon > 0$. If our revision protocol satisfies continuity (C), then by the discussion at the beginning of this section, the attractor γ of V^{F^0} continues to an attractor γ^ε of V^{F^ε}; γ^ε is contained in a neighborhood of γ, and its basin approximates that of γ. At the same time, the unique Nash equilibrium of F^ε is the central state x^*. Thus if we fix $\delta > 0$, then if $\varepsilon > 0$ is sufficiently small, solutions to $\dot{x} = V^{F^\varepsilon}(x)$ from all initial conditions

x with $f(x) > -\frac{1}{8} + \delta$ converge to an attractor γ^ε on which f exceeds $-\delta$; in particular, γ^ε contains neither Nash equilibria nor rest points.

(Show pictures.)

Mismatching Pennies

Mismatching Pennies is a three-player normal form game in which each player has two strategies, Heads and Tails. Player p receives a payoff of 1 for choosing a different strategy than player $p+1$ and a payoff of 0 otherwise, where players are indexed modulo 3. If we let F be the population game generated by matching in Mismatching Pennies, then for each population $p \in \mathcal{P} = \{1, 2, 3\}$ we have that

$$F^p(x) = \begin{pmatrix} F^p_H(x) \\ F^p_T(x) \end{pmatrix} = \begin{pmatrix} x^{p+1}_T \\ x^{p+1}_H \end{pmatrix}.$$

The unique Nash equilibrium of F is the central state $x^* = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$.

In this game, almost all solutions of the replicator dynamic converge to a six-segment heteroclinic cycle that agrees with the best response cycle in the underlying normal form game. Similarly, almost all solutions of the best response dynamic converge to a six-sided closed orbit. (Show pictures.)

The next proposition shows that these examples are not exceptional.

Proposition 0.53. *Let $V_{(\cdot)}$ be an evolutionary dynamic that is generated by a C^1 revision protocol ρ and that satisfies Nash stationarity* (NS). *Let F be Mismatching Pennies, and suppose that the unique Nash equilibrium x^* of F is a hyperbolic rest point of $\dot{x} = V_F(x)$. Then x^* is unstable under V_F, and there is an open, dense, full measure set of initial conditions from which solutions to V_F do not converge.*

Proposition 0.53 is remarkable in that it does not require the dynamic to satisfy a payoff monotonicity condition. Instead, it takes advantage of the fact that by definition, the revision protocol for population p does not condition on the payoffs of other populations. In fact, the specific payoffs of Mismatching Pennies are not important to obtain the instability result; any three-player game whose unique Nash equilibrium is interior works equally well. The proof of the theorem makes these points clear.

Proof. For ε close to 0, let F^ε be generated by a perturbed version of Mismatching Pennies in which player 3's payoff for playing H when player 1 plays T is not 1, but $\frac{1+2\varepsilon}{1-2\varepsilon}$. Then like Mismatching Pennies itself, F^ε has a unique Nash equilibrium, here given by $((\frac{1}{2} + \varepsilon, \frac{1}{2} -$

$\varepsilon), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$.

Since there are two strategies per player, it will simplify our analysis to let $y^p = x^p_H$ be the proportion of population p players choosing Heads, and to focus on the new state variable $y = (y^1, y^2, y^3) \in Y = [0, 1]^3$. If $\dot{y} = \hat{V}_{F^\varepsilon}(y)$ is the dynamic $\dot{x} = V_{F^\varepsilon}(x)$ expressed in terms of y, then Nash stationarity (NS) tells us that

$$\hat{V}_{F^\varepsilon}(\tfrac{1}{2} + \varepsilon, \tfrac{1}{2}, \tfrac{1}{2}) = \mathbf{0} \tag{35}$$

whenever $|\varepsilon|$ is small. By definition, the law of motion for population 1 does not depend directly on payoffs in the other populations, regardless of the game at hand. Therefore, since changing the game from F^ε to F^0 does not alter population 1's payoff function, equation (35) implies that

$$\hat{V}^1_{F^0}(\tfrac{1}{2} + \varepsilon, \tfrac{1}{2}, \tfrac{1}{2}) = 0$$

whenever $|\varepsilon|$ is small. This observation and the fact that the dynamic is differentiable at $y^* = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ imply that

$$\frac{\partial \hat{V}^1_{F^0}}{\partial y^1}(y^*) = 0.$$

Repeating this argument for the other populations shows that the trace of $D\hat{V}_{F^0}(y^*)$, and hence the sum of the eigenvalues of $D\hat{V}_{F^0}(y^*)$, is 0. Since y^* is a hyperbolic rest point of $\hat{V}_{F^0}$, it follows that some eigenvalue of $D\hat{V}_{F^0}(y^*)$ has positive real part, and thus that y^* is unstable under $\hat{V}_{F^0}$. Thus, the stable manifold theorem (see the discussion after Theorem 0.35) tells us that the set of initial conditions from which solutions converge to y^* is of dimension at most 2, and that its complement is open, dense, and of full measure in Y. ■

Hypnodisk

Hypnodisk is a three-strategy population game with nonlinear payoffs. Near the barycenter $x^* = \frac{1}{3}\mathbf{1}$, the payoffs are those of the coordination game

$$F^C(x) = Cx = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Near the boundary of X, the payoffs are those of the anticoordination game $F^{-C}(x) = -Cx$. In between, the payoffs are chosen so that F is continuous, and so that the unique Nash equilibrium of the resulting game is x^*. This is accomplished using a simple geometric construction. (Show pictures.)

With the construction complete, the following result is easy to prove.

Proposition 0.54. *Let $V^{(\cdot)}$ be a Lipschitz evolutionary dynamic that satisfies* (NS) *and* (PC)*, and let H be the hypnodisk game. Then every solution to $\dot{x} = V^H(x)$ other than the stationary solution at x^* approaches a limit cycle.*

With 9.1: Attractors and continuation

A set $\mathcal{A} \subseteq X$ is an *attractor* of the flow ϕ if it is nonempty, compact, and invariant under ϕ, and if there is a neighborhood U of $\mathcal{A}$ such that

$$\lim_{t\to\infty} \sup_{x\in U} \text{dist}(\phi_t(x), \mathcal{A}) = 0.$$

Put differently, attractors are asymptotically stable sets that are also invariant under the flow. The set $B(\mathcal{A}) = \{x \in X : \omega(x) \subseteq \mathcal{A}\}$ is called the *basin* of $\mathcal{A}$.

A key property of attractors for the current context is known as *continuation*. Let $\dot{x} = V^\varepsilon(x)$ be a one-parameter family of differential equations on $\mathbb{R}^n$ with unique solutions $x_t = \phi_t^\varepsilon(x_0)$ such that $(\varepsilon, x) \mapsto V^\varepsilon(x)$ is continuous. Then $(\varepsilon, t, x) \mapsto \phi_t^\varepsilon(x)$ is continuous as well. Suppose that $X \subset \mathbb{R}^n$ is compact and forward invariant under the semi-flows ϕ^ε. For $\varepsilon = 0$ we omit the superscript in ϕ.

Fix an attractor $\mathcal{A} = \mathcal{A}^0$ of the flow ϕ^0. Then as ε varies continuously from 0, there exist attractors $\mathcal{A}^\varepsilon$ of the flows ϕ^ε that vary upper hemicontinuously from $\mathcal{A}$; their basins $B(\mathcal{A}^\varepsilon)$ vary lower hemicontinuously from $B(\mathcal{A})$. Thus, if we slightly change the parameter ε, the attractors that exist under ϕ^0 continue to exist, and they do not explode. (See *PGED*, Sec. 9.B.)

With 9.1: The Poincaré-Bendixson theorem

The celebrated *Poincaré-Bendixson Theorem* characterizes the possible long run behaviors of such dynamics, and provides a simple way of establishing the existence of periodic orbits. Recall that a *periodic* (or *closed*) *orbit* of a differential equation is a nonconstant solution $\{x_t\}_{t\geq 0}$ such that $x_T = x_0$ for some $T > 0$.

Theorem 0.55 (The Poincaré-Bendixson Theorem). *Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be Lipschitz continuous, and consider the differential equation $\dot{x} = V(x)$.*

(*i*) *Let $x \in \mathbb{R}^2$. If $\omega(x)$ is compact, nonempty, and contains no rest points, then it is a periodic orbit.*

(*ii*) *Let $Y \subset \mathbb{R}^2$. If Y is nonempty, compact, forward invariant, and contains no rest points, then it contains a periodic orbit.*

Theorem 0.55 tells us that in planar systems, the only possible ω-limit sets are rest points, sequences of trajectories leading from one rest point to another (called *heteroclinic cycles* where there are multiple rest points in the sequence and *homoclinic orbits* when there is just one), and *periodic orbits*. In part (i) of the theorem, the requirement that $\omega(x)$ be compact and nonempty are automatically satisfied when the dynamic is forward invariant on a compact set—see Proposition 0.24.

With 9.2: More on survival of dominated strategies

We have seen that that the best response dynamic and all imitative dynamics eliminate dominated strategies, at least along solutions starting from most initial conditions. Should we expect elimination to occur more generally?

Not necessarily. Evolutionary dynamics describe the aggregate behavior of agents who employ simple revision protocols, switching to strategies whose current payoffs are good, though not necessarily optimal. In some cases—in particular, when solutions converge—these simple rules are enough to ensure individually optimal long run behavior. When a solution trajectory of an evolutionary dynamic converges, the payoffs to each strategy converge as well; because payoffs become fixed, even simple rules are enough to ensure that only optimal strategies are chosen. But when solutions do not converge, payoffs remain in flux. In this situation, it is not obvious whether choice rules favoring strategies whose current payoffs are relatively high will necessarily eliminate strategies that perform well at many states, but that are never optimal.

To work toward a formal result, we introduce a new condition on evolutionary dynamics.

(IN) *Innovation* If $x \notin NE(F)$, $x_i = 0$, and $i \in \operatorname*{argmax}_{j \in S} F_j(x)$, then $V_i^F(x) > 0$.

Innovation (IN) requires that when a non-Nash population state includes an unused optimal strategy, this strategy's growth rate must be positive. In other words, if an unplayed strategy is sufficiently rewarding, some members of the population will discover it and select it. This condition excludes dynamics based purely on imitation; however, it includes hybrid dynamics that combine imitation with a small amount of direct selection of candidate strategies.

Theorem 0.56. *Suppose the evolutionary dynamic $V^{(\cdot)}$ is based on a Lipschitz continuous protocol and satisfies* (NS), (PC), *and* (IN). *Then there is a game F such that under $\dot{x} = V^F(x)$, along solutions from most initial conditions, there is a strictly dominated strategy played by a fraction of the population bounded away from* 0.

Idea of proof. Start with the hypnodisk game. Then add a twin strategy, 4, that duplicates strategy 3. The set of Nash equilibria of game is the line segment $NE = \{x^* \in X : x_1^* = x_2^* = x_3^* + x_4^* = \frac{1}{3}\}$. Solutions to dynamics satisfying (PC) and (NS) with initial conditions off of this line segment must approach an attractor $\mathcal{A}$ that is bounded away from NE and from the face of X on which strategy 4 is unused.

Now make the twin feeble by uniformly reducing its payoff by d. Doing so changes the dynamic continuously, and so moves the attractor $\mathcal{A}$ upper-hemicontinuously to $\mathcal{A}^d$; likewise, its basin moves lower-hemicontinuously (see the beginning of this section and *PGED* Sec. 9.B). Thus $\mathcal{A}^d$ is still in the interior of X, implying that the strictly dominated twin strategy survives from solutions starting at most initial conditions. ■

(Show pictures of the proof.)

(Show pictures of the argument for bad RPS/Smith.)

Although the theorem only says that the proportion of agents playing the dominated strategy is bounded away from zero, numerical analysis reveals that this proportion need not be small. For bad RPS with a Twin under the Smith dynamic, Twin is recurrently played by at least 10% of the population when $d \leq .31$, by at least 5% of the population when $d \leq .47$, and by at least 1% of the population when $d \leq .66$. These values of d are surprisingly large relative to the base payoff values of 0, −2, and 1.