Survival of Dominated Strategies under Evolutionary Dynamics

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a hypnodisk
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One role of evolutionary game theory is to provide foundations for traditional, rationality-based solution concepts.

example: rest points (of the dynamic) and Nash equilibria (of the underlying game)

Typically, \( NE \subseteq RP \).

For some general classes of dynamics, \( NE = RP \). (Sandholm (2005, 2006))

Is it possible to guarantee that all dynamically stable sets consist of Nash equilibria?

No. Hofbauer and Swinkels (1996) and Hart and Mas-Colell (2003) construct games in which most solutions of any reasonable evolutionary dynamic fail to converge to Nash equilibrium.
We therefore consider a more modest goal: elimination of strictly dominated strategies.

Dominance is the mildest requirement employed in standard game-theoretic analysis, so it is natural to expect evolutionary dynamics to accord with it.

And indeed:

1. Dominated strategies are eliminated by certain dynamics based on imitation, so long as play begins at an interior initial condition.
   - replicator dynamic (Akin (1980))
   - families of imitative dynamics (Samuelson and Zhang (1992), Hofbauer and Weibull (1996))

2. Clearly, dominated strategies are eliminated under the best response dynamic.
Still, there is no *a priori* reason to expect dominated strategies to be eliminated by evolutionary dynamics.

Evolutionary dynamics are based on the idea that agents switch to strategies whose *current* payoffs are *reasonably good*.

Even if a strategy is dominated, it can have reasonably good payoffs in many states.

Put differently: decision making in evolutionary models is “local”; domination is “global”.

We show that any evolutionary dynamic satisfying three natural conditions—*continuity*, *positive correlation*, and *innovation*—must sometimes fail to eliminate strictly dominated strategies.

Thus, under dynamics other than those noted above, we need not expect strictly dominated strategies to disappear.

Moral: Evolutionary dynamics provide surprisingly little support for a basic rationality postulate.
Population Games

We consider games played by a single unit-mass population of agents.

\[ S = \{1, \ldots, n\} \quad \text{strategies} \]

\[ X = \{x \in \mathbb{R}^n_+: \sum_i x_i = 1\} \quad \text{population states/mixed strategies} \]

\[ F_i: X \to \mathbb{R} \quad \text{payoffs to strategy } i \quad (\text{Lipschitz continuous}) \]

\[ F: X \to \mathbb{R}^n \quad \text{payoffs to all strategies = the payoff vector field} \]
example: random matching in symmetric two-player normal form games

\[ A \in \mathbb{R}^{n \times n} \quad \text{payoff matrix (} \Rightarrow \text{ bimatrix } (A, A') \text{)} \]

\[ A_{ij} = e_i \cdot Ae_j \quad \text{payoff for playing } i \in S \text{ against } j \in S \]

\[ F_i(x) = e_i \cdot Ax \quad \text{(expected) payoff for playing } i \text{ against } x \in X \]

\[ F(x) = Ax \quad \text{the payoff vector field} \]

note: \{Nash equilibria of } F \} = \{\text{symmetric Nash equilibria of } (A, A')\}
example: congestion games  
(Beckmann, McGuire, and Winsten (1956))

Home and Work are connected by 
paths \( i \in S \) consisting of links \( \phi \in \Phi \).

The payoff to choosing path \( i \) is 
\[-(\text{the delay on path } i) = -(\text{the sum of the delays on the links on path } i)\]

formally: \( F_i(x) = -\sum_{\phi \in \Phi_i} c_\phi(u_\phi(x)) \) \hspace{1cm} \text{payoff to path } i
\( x_i \) \hspace{1cm} \text{mass of players choosing path } i
\( u_\phi(x) = \sum_{i: \phi \in \Phi_i} x_i \) \hspace{1cm} \text{utilization of link } \phi
\( c_\phi(u_\phi) \) \hspace{1cm} \text{cost of delay on link } \phi

Examples: highway congestion \( \Rightarrow c_\phi \) increasing
positive externalities \( \Rightarrow c_\phi \) decreasing
Microfoundations for evolutionary dynamics

The choice procedure individual agents follow is called a revision protocol.

\[ \rho: \mathbb{R}^n \times X \to \mathbb{R}_{+}^{n \times n} \] 

\[ \rho_{ij}(F(x), x) \] 

revision protocol  
conditional switch rate from strategy \(i\) to strategy \(j\)

An all-purpose interpretation:

Each agent is equipped with a rate \(\lambda\) Poisson alarm clock \((\lambda = n \max_{ij,x} \rho_{ij}(F(x), x))\).

When an \(i\) player’s clock rings, he receives a revision opportunity, during which

(i) he selects a strategy \(j \in S\) at random;

(ii) he switches to strategy \(j\) with probability proportional to \(\rho_{ij}(F(x), x)\).

Often, simpler interpretations work for specific revision protocols.
The mean dynamic

\[
\begin{align*}
\text{population game } F \\
\text{revision protocol } \rho \\
\text{population size } N
\end{align*}
\Rightarrow \text{Markov process } \{X_t^N\}
\]

Over finite time spans, the process \( \{X_t^N\} \) is well-approximated by solutions to an ODE. (Benaim and Weibull (2003), Sandholm (2003))

This ODE, the mean dynamic, is defined by the expected increments of \( \{X_t^N\} \).

(\text{MD}) \quad \dot{x}_i = V_i^F(x) = \sum_{j \in S} x_j \rho_{ij}(F(x), x) - x_i \sum_{j \in S} \rho_{ij}(F(x), x)

= \text{inflow into } i - \text{outflow from } i
Examples

**Revision protocol**

\[ \rho_{ij} = x_j [F_j - F_i]_+ \]

**Mean dynamic**

\[ \dot{x}_i = x_i (F_i(x) - \bar{F}(x)) \]

the replicator dynamic (Taylor and Jonker (1978))

\[ \rho_{ij} = B_j^F(x) \]

\[ \dot{x} \in B^F(x) - x \]

the best response dynamic (Gilboa and Matsui (1991))

\[ \rho_{ij} = [F_j - \bar{F}]_+ \]

\[ \dot{x}_i = [F_i(x) - \bar{F}(x)]_+ - x_i \sum_{j \in S} [F_j(x) - \bar{F}(x)]_+ \]

the Brown-von Neumann-Nash (BNN) dynamic (B-vN (1950))

\[ \rho_{ij} = [F_j - F_i]_+ \]

\[ \dot{x}_i = \sum_{j \in S} x_j [F_i(x) - F_j(x)]_+ - x_i \sum_{j \in S} [F_j(x) - F_i(x)]_+ \]

the pairwise difference (PD) dynamic (Smith (1984))
Survival of dominated strategies I: Families of simple revision protocols


We show that their construction also works for
- the pairwise difference dynamic
- families of dynamics that generalize the BNN and PD dynamics

Step 1: Cycling in bad RPS

Suppose agents are randomly matched to play bad Rock-Paper-Scissors.

\[
A = \begin{pmatrix}
0 & -2 & 1 \\
1 & 0 & -2 \\
-2 & 1 & 0
\end{pmatrix}
\]

Unless the initial state is the unique NE \( x^* = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \), play approaches a limit cycle. Proof: Use a Lyapunov function. (The function depends on the dynamic at issue).
the BNN dynamic in bad RPS

the PD dynamic in bad RPS

colors represent speeds (red = fastest, blue = slowest)
Step 2: Add a twin to bad RPS

Add a fourth strategy to bad RPS that duplicates Scissors.

\[
A = \begin{pmatrix}
0 & -2 & 1 & 1 \\
1 & 0 & -2 & -2 \\
-2 & 1 & 0 & 0 \\
-2 & 1 & 0 & 0
\end{pmatrix}
\]

Except from the line of NE \( x \in X: x = ((\frac{1}{3}, \frac{1}{3}, \frac{1}{6} - c, \frac{1}{6} + c)) \),
play converges to a cycle on the plane where \( x_3 = x_4 \).

(Why? (PD) \[
\dot{x}_i = \sum_{j \in S} x_j [F_i(x) - F_j(x)]_+ - x_i \sum_{j \in S} [F_j(x) - F_i(x)]_+ \\
\Rightarrow \dot{x}_i - \dot{x}_j = (x_j - x_i) \sum_{k \in S} [F_k(x) - F_i(x)]_+ .
\]
the PD dynamic in “bad RPS with a twin”
Step 3: A feeble twin

Uniformly reduce the payoff of the twin by $\varepsilon$:

$$A = \begin{pmatrix}
0 & -2 & 1 & 1 \\
1 & 0 & -2 & -2 \\
-2 & 1 & 0 & 0 \\
-2 - \varepsilon & 1 - \varepsilon & -\varepsilon & -\varepsilon
\end{pmatrix}$$

Since the dynamic depends continuously on the game's payoffs, the attracting interior limit cycle persists when $\varepsilon$ is slightly greater than zero.

$\therefore$ The feeble twin, a strictly dominated strategy, survives.
the PD dynamic in “bad RPS + twin”

the PD dynamic in “bad RPS + feeble twin”
Survival of dominated strategies II: The general result

Three conditions for evolutionary dynamics:

(C) **Continuity:** $\rho$ is Lipschitz continuous.

Small changes in aggregate behavior do not lead to large changes in agents’ responses.

Rules out protocols that are extremely sensitive to the exact values of $F(x)$ or $x$ (for example, the best response protocol).

(PC) **Positive correlation:** $\nu^F(x)'F(x) \geq 0$, with equality if and only if $x \in NE(F)$.

A weak payoff monotonicity condition for disequilibrium states (see Friedman (1991), Swinkels (1993), Sandholm (2001)).

Implies that $RP(\nu^F) = NE(F)$, ruling out purely imitative dynamics.

(IN) **Innovation:** If $x \notin NE(F)$, $x_i = 0$, and $i \in BR^F(x)$, then $\nu_i^F(x) > 0$.

If a disequilibrium state has an unused best response, some agents switch to it. Also rules out purely imitative dynamics.
**Theorem:** If the dynamic $V$ satisfies (C), (PC), and (IN), there are games $F$ in which a strictly dominated strategy survives from most initial conditions under $V^F$.

**Analysis**

The bad RPS construction only works under specific assumptions about the functional form of the revision protocol $\rho$.

Since we now impose little structure on $\rho$, we require a more elaborate construction.
Step 0: Potential games  (Monderer and Shapley (1996), Sandholm (2001))

\[ F \text{ is a potential game if there exists an } f : X \to \mathbb{R} \text{ s.t. } \nabla f(x) = F(x) \text{ for all } x \in X \]

\[ \iff \text{ externality symmetry: } DF(x) \text{ is symmetric for all } x \in X. \]

examples:  common interest games, congestion games, 
games generated by variable externality pricing schemes

In potential games, evolutionary dynamics are very well-behaved.

**Lemma:** If \( F \) is a potential game, and the dynamic \( V^F \) satisfies (PC), 
then nonstationary solutions of \( V^F \) ascend the potential function \( f \). 
(That is, \( f \) is a strict Lyapunov function for \( V^F \).)

\[ \text{Proof: } \frac{d}{dt} f(x_t) = \nabla f(x_t)' \dot{x}_t = F(x_t)' V^F(x_t) \geq 0. \]
example: coordination

\[ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ f(x) = \frac{1}{2} (x_1^2 + x_2^2 + x_3^2) \]

\[ F(x) = Cx \]

(projected onto \( TX \))
example: anti-coordination

\[-C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\]

\[f(x) = -\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\]

\[F(x) = -Cx\]
Step 1: Construction of the hypnodisk game

Begin with the coordination game \( F(x) = Cx \) above.

Start twisting the portion of the vector field outside the inner circle in a clockwise direction. Exclude larger and larger inner circles as you twist, so that the outer circle is reached when the total twist is 180°.
The twist

the coordination game  \[\Rightarrow\]  the hypnodisk game
the hypnodisk game
Analytically: 

$$F(x) = \cos(\theta(x)) \begin{pmatrix} x_1 - \frac{1}{3} \\ x_2 - \frac{1}{3} \\ x_3 - \frac{1}{3} \end{pmatrix} + \frac{\sqrt{3}}{3} \sin(\theta(x)) \begin{pmatrix} x_2 - x_3 \\ x_3 - x_1 \\ x_1 - x_2 \end{pmatrix},$$

where $\theta(x) = \begin{cases} 
0 & \text{if } |x - x^*| \leq r; \\
1 & \text{if } |x - x^*| \geq R; \\
\text{smooth} & \text{in between.}
\end{cases}$

By construction, $F(x) = Cx$ inside the inner circle, and $F(x) = -Cx$ outside the outer circle.

Therefore, the annulus between the circles attracts almost all solutions to $V^F$. 
Step 2: Add a twin

Let \( \hat{F} \) be \( F \) augmented by a twin.

\[
\hat{F}(x) = \hat{C}x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} x
\]

In an inner cylinder \( I \), \( \hat{F}(x) = \hat{C}x \) is a potential game.

Outside an outer cylinder \( O \), \( \hat{F}(x) = -\hat{C}x \) is a potential game.

By (PC’), all solutions to \( V^{\hat{F}} \) that do not start on the line of NE \((\frac{1}{3}, \frac{1}{3}, \frac{1}{b} - c, \frac{1}{b} + c)\)

converge to \( D = O - I \), the difference of the two cylinders.

But unlike in the previous analysis, the plane where \( x_3 = x_4 \) typically is not attracting.
the cylinders $O$ and $I$, and their difference $D = O - I$
Step 3: The attractor $A$

We describe the possible limit behaviors of the dynamic using the notion of attractor-repeller pairs (Conley (1978)).

Define the flow from set $U \subset X$ under the dynamic $V$ by

$$\phi_t(U) = \{ \xi \in X : \text{there is a solution } \{x_s\} \text{ to } \dot{x} = V(x) \text{ with } x_0 \in U \text{ and } x_t = \xi \}.$$  

The set $NE$ is a repeller under $V$: there is a neighborhood $U$ of $NE$ such that

$$NE = \bigcap_{t < 0} \phi_t(\text{cl}(U)).$$

Let $A$ be the dual attractor of the repeller $NE$:

$$A = \bigcap_{t > 0} \phi_t(X - \text{cl}(U)).$$

Fact: $A$ is compact, and it is (forward & backward) invariant under $V$.

Also: By construction, $A \subset D$. 
Let \( Z = \{ x \in X : x_4 = 0 \} \) be the face of \( X \) on which the twin is unused.

**Lemma:** \( A \) and \( Z \) are disjoint.

**Proof:** Since strategies cannot become extinct in finite time, \( X - Z \) is forward invariant. Therefore, a backward solution trajectory from \( \xi \in Z \) cannot enter \( \text{int}(X) \).

Therefore, if \( \xi \in A \cap Z \), the entire backward orbit from \( \xi \) is in \( A \cap Z \), and so in \( D \cap Z \) (since \( A \subset D \)).

Since \( D \cap Z \) contains no rest points (by (PC)), the Poincaré-Bendixson Theorem implies that the backward orbit from \( \xi \) converges to a closed orbit in \( D \cap Z \) that circumnavigates \( I \cap Z \).

But this closed orbit must pass through a region in which strategies 3 and 4 are optimal, contradicting innovation (IN). (See the picture.)
The best response correspondence for the hypnodisk game.
Step 4: The feeble twin

The Lemma shows that in game \( \hat{F} \), the attractor \( A \) is a compact subset of \( \text{int}(X) \), and hence is bounded away from \( Z \).

Therefore, if we make the twin feeble, a continuity argument shows that it will survive.
Discussion

1. The hypnodisk game ensures the existence of a cycle under any evolutionary dynamic that satisfies (PC).

   As we have seen, survival results can be proved using simpler games if one fixes the evolutionary dynamic in advance.

2. Some purely imitative dynamics (e.g., the replicator dynamic) eliminate dominated strategies along interior solution trajectories.

   The results in this paper imply that if agents usually imitate but occasionally choose strategies directly, then dominated strategies can survive.
Inflow-outflow symmetry

What is special about imitative dynamics?

A revision protocol $\rho$ generates a mean dynamic $V^F$ for any population game $F$.

\[
\dot{x}_i = V^F_i(x) = \sum_{j \in S} x_j \rho_{ji} - x_i \sum_{j \in S} \rho_{ij}
\]

$= \text{inflow into } i - \text{outflow from } i$

Notice that the outflow term is proportional to $x_i$, but that the inflow term is not.

Observation: If the functional form of $\rho_{ij}$ eliminates inflow-outflow asymmetry, the dynamic $V^F$ is one step closer to eliminating dominated strategies.
Inflow-outflow symmetry under imitative dynamics

ex. the replicator dynamic

\[ \rho_{ij} = x_j [F_j - F_i]_+ \]  
\[ (\rho_{ij} \text{ is proportional to } x_j) \]

\[ \Rightarrow \quad \dot{x}_i = \sum_{j \in S} x_j \rho_{ji} - x_i \sum_{j \in S} \rho_{ij} \]

\[ = \sum_{j \in S} x_j x_i [F_i - F_j]_+ - x_i \sum_{j \in S} x_j [F_j - F_i]_+ \]

\[ = x_i \sum_{j \in S} x_j (F_i - F_j) \]

\[ = x_i \hat{F}_i(x) \]  
\[ (\dot{x}_i \text{ is proportional to } x_i) \]

\[ \Rightarrow \quad \frac{d}{dt} \left( \frac{x_i}{x_j} \right) = \frac{x_i}{x_j} (F_i - F_j) \]

\[ \therefore \quad \text{If strategies } i \text{ and } j \text{ are twins, } \frac{x_i}{x_j} \text{ is constant along interior solution trajectories.} \]
the replicator dynamic in “RPS + twin”  
the replicator dynamic in “RPS + feeble twin”
Flows with a continuum of invariant planes are not robust.

Indeed, the main theorem tells us that if agents only imitate most of the time, dominated strategies can survive.

\[(.9 \times \text{replicator}) + (.1 \times \text{pairwise difference}) \]

in “slightly bad RPS”

\[(.9 \times \text{replicator}) + (.1 \times \text{pairwise difference}) \]

in “slightly bad RPS + feeble twin”
Inflow-outflow symmetry under an “abandonment dynamic”

ex. the projection dynamic (on \text{int}(X)) \quad \text{(Nagurney and Zhang (1997), Sandholm, Dokumaci, and Lahkar (2006)))}.

\[
\rho_{ij} = \frac{[F_j - F_i]}{nx_i} \quad \text{on \text{int}(X)} \quad (\rho_{ij} \text{ is inversely proportional to } x_i)
\]

\[
\Rightarrow \quad \dot{x}_i = \sum_{j \in S} x_j \rho_{ji} - x_i \sum_{j \in S} \rho_{ij}
\]

\[
= \sum_{j \in S} \frac{x_j [F_i - F_j]}{nx_j} - x_i \sum_{j \in S} \frac{[F_i - F_j]}{nx_i}
\]

\[
= \frac{1}{n} \sum_{j \in S} (F_i - F_j)
\]

\[
= F_i - \frac{1}{n} \sum_{j \in S} F_j \quad (\dot{x}_i \text{ does not depend explicitly on } x_i)
\]

\[
\Rightarrow \quad \frac{d}{dt}(x_i - x_j) = F_i - F_j
\]

\[\therefore\] If strategies \(i\) and \(j\) are twins, \(x_i - x_j\) is constant along interior solution trajectories.
the projection dynamic in “RPS + twin” the projection dynamic in “RPS + feeble twin”

(But: The projection dynamic has unusual boundary behavior: solution trajectories can enter and exit $\text{bd}(X)$.
Because of this, strictly dominated strategies can survive (Sandholm, Dokumaci, and Lahkar (2006)).)
In general, $\rho_{ij}$ is neither proportional to $x_j$ nor inversely proportional to $x_i$, so inflow-outflow symmetry does not hold.

Without this symmetry, dominance arguments have no bite: the “global” character of dominance is not captured by the “local” criteria used by agents in evolutionary models.