Suggested Solutions for Problem Set #2

Problem 1

a. Uncertain. As mentioned in class, capital gains are very elastic in the short run, but in the long run they are fairly inelastic. So the tax will be inefficient in the short run but less inefficient in the long run.

b. False. Taxes on labor income decrease savings. Taxes on interest income have both income and substitution effects, and the net effect can go either way. So it’s not clear that labor income taxes would be better from the standpoint of encouraging savings.

Problem 2

a. If there is no capital gains taxation, it is as if the consumer had income of $100 in the first period. Using the short cut for demand given a Cobb-Douglas utility function, \( C = \frac{75m}{1} = 75 \), \( B = .25m(1 + r) = 27.5 \). So she sells $75 worth of the stock.

b. Now there is capital gains taxation, so we have to think more carefully about the budget constraint. Let \( N \) be the amount of stock she sells. We know two things must be true. \( C = N - .5(N - .6N) = .8N \) since for each dollar of stock she sells, the basis is sixty cents, so she has to pay fifty percent tax on capital gains of forty cents, for a total tax of twenty cents out of every dollar. Also, \( B = (100 - N)(1 + r) = 110 - 1.1N \) since any stock she doesn’t sell passes to her child, who won’t have to pay tax on it because of the step up of basis. If we solve one of these two equations for \( N \) and plug it into the other equation, we get the budget constraint:

\[
B + \frac{1.1}{.8}C = 110
\]

Solving this for \( B \) and substituting into the utility function gives

\[
U = .75 \ln C + .25 \ln 110 - \frac{11}{8} C
\]

\[
\frac{dU}{dC} = \frac{.75}{C} + \frac{-.25 \frac{11}{8}}{110 - \frac{11}{8} C} = 0
\]

\[
\frac{3}{C} = \frac{\frac{11}{110 - \frac{11}{8} C}}{\frac{11}{8} C}
\]

\[
3(110 - \frac{11}{8} C) = \frac{11}{8} C
\]

\[
330 = \frac{44}{8} C
\]
\[ C = 60 \]
\[ B = 27.5 \]

We know that \( C = 0.8N \) where \( N \) is the amount of stock sold, so \( N = \frac{60}{0.8} = 75 \). So she sells $75 of her stock holdings to consume today.

c. If she sells one dollar of her stock today, she gets to keep $0.80 after tax. So if she puts it in an investment that earns 20%, it will be worth $1.96 tomorrow. But if she doesn’t sell, the dollar of stock is worth $1.10 tomorrow. So she won’t sell it. This is an example of the lock-in effect discussed in class. There are more efficient uses for her money available, but she won’t choose them because of the capital gains tax structure.

**Problem 3**

The consumer faces the problem of maximizing \( U = \ln(C_1) + \ln(C_2) \), subject to two constraints. First, his income in the first period will either be spent on consumption or saved, so that \( C_1 + S = m \), where \( m \) is first period income.

Second, his consumption in the second period will be his savings and the interest earned on his savings: \( C_2 = S(1 + r) \). These two constraints can be combined into one by substituting for \( S \): \( C_2 = (m - C_1)(1 + r) \). This budget constraint can then be substituted into the utility function, and the utility maximizing level of \( C_1 \) will be the one where the derivative of this utility function is zero:

\[
U = \ln(C_1) + \ln((m - C_1)(1 + r))
\]

\[
\frac{dU}{dC_1} = \frac{1}{C_1} + \frac{-(1 + r)}{(m - C_1)(1 + r)} = 0
\]

\[
\frac{1}{C_1} = \frac{1}{m - C_1}
\]

\[
C_1 = m - C_1
\]

\[
C_1 = \frac{m}{2}
\]

This is the demand for \( C_1 \). Plugging this back into the budget constraint, the demand for \( C_2 \) is \( C_2 = \frac{m}{2}(1 + r) \). So with an income of $100 and an interest rate of 10%, \( C_1 = 50, C_2 = 50(1.1) = 55, S = 50 \).

b. If there is a tax on labor income, this just changes the consumer’s first period income to \( 100 - 100 \times 0.20 = 80 \). Using the demands from part (a), \( C_1 = 40, C_2 = 44, S = 40 \). There is only an income effect in this case.

c. If there is a tax on interest income instead, this changes the after-tax interest rate the consumer faces, so that now the consumer gets \( (1 + r(1 - \tau)) = 1.08 \). Again using the demands from part (a), \( C_1 = 50, C_2 = 54, S = 50 \). In this case, there are both income and substitution effects on savings. The substitution effect decreases savings, and the income effect increases savings. Because of the shape of the utility function in this case, they are the same size and thus cancel each other out, leaving savings unchanged.
Problem 4

a. First, write the budget constraint. Since the effective wage is $3(6 - $3 childcare costs), his consumption is $C = 3(160 - L)$ since $(160 - L)$ is the number of hours he works. Plugging this budget constraint into the utility function gives

\[ U = \frac{2}{3} \ln 3(160 - L) + \frac{1}{3} \ln L \]

\[ \frac{dU}{dL} = \frac{2}{3} \frac{-3}{3(160 - L)} + \frac{1}{3} \frac{1}{L} = 0 \]

\[ \frac{2}{3} \frac{1}{160 - L} = \frac{1}{3} \frac{1}{L} \]

\[ \frac{2}{3} L = \frac{1}{3} (160 - L) \]

\[ 3L = 160 \]

\[ L = 53.33 \]

\[ C = 320 \]

b. Because the government reduces labor income $1 for every dollar of labor income, there’s no incentive for the individual to work if he is getting money from the government. If he works in the labor market, he will have 53.33 hours of leisure and consume $320. Otherwise, he will work 0 hours and get $210 from the government. He will work if

\[ \frac{2}{3} \ln 320 + \frac{1}{3} \ln 53.33 \geq \frac{2}{3} \ln 210 + \frac{1}{3} \ln 160 \]

\[ 5.17 \geq 5.26 \]

So he won’t work and will have 160 hours of leisure and $210 of consumption.

c. In this case the individual is deciding between 160 hours of leisure and $100 or 53.33 hours of leisure and $320. He will work if

\[ \frac{2}{3} \ln 320 + \frac{1}{3} \ln 53.33 \geq \frac{2}{3} \ln 100 + \frac{1}{3} \ln 160 \]

\[ 5.17 \geq 4.76 \]

So he will work.

d. Now the effective wage is $6 instead of $3. In part (a) we saw that the individual, if working, chooses 53.33 hours of leisure, regardless of the effective wage. So now his choices are to work and have 53.33 hours of leisure and $640 of consumption, or to have 160 hours of leisure and $210. He will work if

\[ \frac{2}{3} \ln 320 + \frac{1}{3} \ln 53.33 \geq \frac{2}{3} \ln 210 + \frac{1}{3} \ln 160 \]

\[ 5.63 \geq 5.26 \]

So he will work. The total government cost, since he is working, is $3 per hour he works, or $320. In part b, the total cost is $210. So in part (b) the ratio of the individual’s income to the government’s cost is 1. In this part, the ratio is $640/320 = \frac{1}{2}$. 
Problem 5

a. Karl is more impatient, because an extra dollar of consumption in the second period increases his utility less.

b. The budget constraint is given by $C_1 + \frac{C_2}{(1+r)} = m$. Megan maximizes

$$U = \sqrt{C_1} + \sqrt{(m-C_1)((1+r)}}$$

$$\frac{dU}{dC_1} = \frac{1}{2\sqrt{C_1}} + \frac{1}{2\sqrt{(m-C_1)((1+r)}} = 0$$

$$\frac{1}{2\sqrt{C_1}} = \frac{1}{2\sqrt{(m-C_1)}}$$

$$2\sqrt{(m-C_1)} = 2\sqrt{C_1}\sqrt{(1+r)}$$

$$m - C_1 = C_1(1 + r)$$

$$C_1^{Megan} = \frac{m}{2+r}$$

This is Megan’s demand for $C_1$. Plugging this into the budget constraint gives

$$C_2^{Megan} = \frac{(1 + r)^2m}{2 + r}$$

Karl maximizes

$$U = \sqrt{C_1} + \frac{1}{2}\sqrt{(m-C_1)((1+r)}}$$

$$\frac{dU}{dC_1} = \frac{1}{2\sqrt{C_1}} + \frac{1}{4\sqrt{(m-C_1)((1+r)}} = 0$$

$$4\sqrt{(m-C_1)} = 2\sqrt{C_1}\sqrt{(1+r)}$$

$$4m - 4C_1 = C_1(1 + r)$$

$$C_1^{Karl} = \frac{4m}{5+r}$$

This is Karl’s demand for $C_1$. Plugging this into the budget constraint gives

$$C_2^{Karl} = \frac{(1 + r)^2m}{5 + r}$$

So with the income and interest rate given in the problem, $C_1^{Megan} = 47.6$, $C_2^{Megan} = 57.6$, $C_1^{Karl} = 78.4$, $C_2^{Karl} = 23.7$.

c. $(1 + r) \times (1 + s) = 1 + s + r + rs$

So for Karl’s consumption in the two periods to be equal, $C_1^{Karl} = C_2^{Karl}$

$$\frac{4m}{5 + s + r + rs} = \frac{((1 + r)(1 + s))^2m}{5 + s + r + rs}$$

$$4m = ((1 + r)(1 + s))^2m$$
\[ 2 = (1 + r)(1 + s) \]
\[ 1 + s = 1.818 \]
\[ s = 0.818 \]
\[ (1 + r)(1 + s) = 2 \]

With the subsidy, \( C_1^{Karl} = \frac{4(100)}{5} = 33.33 \).

d. Because of the potentially confusing wording of this part, any serious attempt to answer it received full credit. Karl’s relative consumption is

\[ \frac{C_1}{C_2} = \frac{4m}{(1+r)^2m} = \frac{4}{(1+r)^2} \]

We can see from this that if there are only income effects, relative consumption will remain constant. So there need to be both income and substitution effects for Karl, but only income effects for Megan.

One level of subsidized saving that works is 33.33, the level that Karl chooses to save under the subsidy. If the government subsidizes this level of saving, \( C_1^{Karl} = 33.33 \) and \( C_2^{Karl} = 33.33 \), so \( \frac{C_1}{C_2} = 1 \). Without the subsidy, Karl’s relative consumption was \( \frac{C_1}{C_2} = \frac{78.4}{237} = 3.3 \). Megan is already saving more than this, so she has only income effects. The budget constraint she faces is

\[
\begin{align*}
C_2 &= 140 - 1.1C_1 \text{ if } C_1 \leq 66.6 \\
C_2 &= 200 - 2C_1 \text{ if } C_1 > 66.6
\end{align*}
\]

Megan ends up on the upper segment of the budget constraint. Rewriting this as \( C_1 + \frac{C_2}{2} = 127.3 \) shows that having the IRA-style savings plan is for Megan like a $27.30 increase in her income. So \( C_1^{Megan} = \frac{127.3}{2} = 63.6 \) and \( C_2^{Megan} = \frac{(1.1)^2 127.3}{21} = 73.33 \) and \( \frac{C_1}{C_2} = \frac{60.6}{73.33} = 0.826 = \frac{47.6}{57.6} \).

e. Karl has the utility function \( U_{Karl} = 25C_1 + \ln(C_2) \). He faces a budget constraint of the form \( C_1 + \frac{C_2}{1+r} = m \). Solving for \( C_2 \) and plugging into the utility function yields

\[ U_{Karl} = 25C_1 + \ln((m - C_1)(1 + r)) \]

\[ \frac{dU_{Karl}}{dC_1} = 25 - \frac{(1 + r)}{(m - C_1)(1 + r)} = 0 \]

\[ 25 = \frac{1}{(m - C_1)} \]

\[ C_1 = m - \frac{1}{25} \]

\[ C_2 = \frac{1}{25}(1 + r) \]

Since \( C_1 \) doesn’t depend on \( r \), the subsidy will have no effect on Karl’s savings. There’s no subsidy you can implement to make him save more.
f. For this utility function, income and substitution effects cancel each other out, leaving savings the same regardless of the interest rate.

g. This would only be the case when period 2 consumption is a Giffen good. For a Giffen good, the amount consumed falls when the price rises. A subsidy to saving effectively decreases the price of second period consumption. In practice, it is inconceivable that aggregate consumption would be a Giffen good.