VI. Applications

1. Merton portfolio selection in finite horizon

An agent invests at any time $t$ a proportion $\alpha_t$ of his wealth $X$ in a stock of price $S$ and $1 - \alpha_t$ in a bond of price $S^0$ with interest rate $r$. The investor faces the portfolio constraint that at any time $t$, $\alpha_t$ is valued in $A$ closed convex subset of $\mathbb{R}$.

Assuming a Black-Scholes model for $S$ (with constant rate of return $\mu$ and volatility $\sigma > 0$), the dynamics of the controlled wealth process is:

$$dX_t = X_t \alpha_t dS_t + \frac{X_t (1 - \alpha_t)}{S_t} dS_t^0$$

$$= X_t (r + \alpha_t (\mu - r)) dt + X_t \alpha_t \sigma dW_t.$$

The preferences of the agent is described by a utility function $U$: increasing and concave function. The performance of a portfolio strategy is measured by the expected utility from terminal wealth $\rightarrow$ Utility maximization problem at a finite horizon $T$:

$$v(t, x) = \sup_{\alpha \in A} \mathbb{E}[U(X_t^{t,x})], \quad (t, x) \in [0, T] \times (0, \infty).$$

$\rightarrow$ Standard stochastic control problem
HJB equation for Merton’s problem

\[ v_t + r x v_x + \sup_{a \in A} \left[ \left( a(\mu - r)xv_x + \frac{1}{2} x^2 a^2 \sigma^2 v_{xx} \right) \right] = 0, \quad (t, x) \in [0, T) \times (0, \infty) \]

\[ v(T, x) = U(x), \quad x > 0. \]

- The case of CRRA utility functions:

\[ U(x) = \frac{x^p}{p}, \quad p < 1, \quad p \neq 0 \]

→ Relative Risk Aversion: \(-xU''(x)/U'(x) = 1 - p.

- We look for a candidate solution to HJB in the form

\[ w(t, x) = \varphi(t) U(x). \]

Plugging into HJB, we see that \( \varphi \) should satisfy the ODE:

\[ \varphi'(t) + \rho \varphi(t) = 0, \quad \varphi(T) = 1, \]

where

\[ \rho = rp + p \sup_{a \in A} \left[ a(\mu - r) - \frac{1}{2} a^2 (1 - p) \sigma^2 \right], \]

→

\[ \varphi(t) = e^{\rho(T-t)}. \]
The value function is equal to
\[ v(t, x) = e^{\rho(T-t)U(x)}, \]
and the optimal control is constant (in proportion of wealth invested)
\[ \hat{a} = \arg\max_{a \in A} [a(\mu - r) - \frac{1}{2}a^2(1 - p)\sigma^2]. \]

When \( A = \mathbb{R} \) (no portfolio constraint), the values of \( \rho \) and \( \hat{a} \) are explicitly given by
\[ \rho = \frac{(\mu - r)^2}{2\sigma^2} \frac{p}{1 - p} + rp. \]
and
\[ \hat{a} = \frac{\mu - r}{\sigma^2(1 - p)}, \]
2. Merton portfolio/consumption choice on infinite horizon

In addition to the investment \( \alpha \) in the stock, the agent can also consume from his wealth:

\[ (c_t)_{t \geq 0} \text{ consumption per unit of wealth} \]

- The wealth process, controlled by \((\alpha, c)\) is governed by:

\[
dX_t = X_t (r + \alpha_t (\mu - r) - c_t) \, dt + X_t \alpha_t \sigma dW_t.
\]

- The preferences of the agent is described by a utility \( U \) from consumption, and the goal is to maximize over portfolio/consumption the expected utility from intertemporal consumption up to a random time horizon:

\[
v(x) = \sup_{(\alpha, c)} E \left[ \int_0^\tau e^{-\beta t} U(c_t X^x_t) \, dt \right], \quad x > 0.
\]

We assume that \( \tau \) is independent of \( \mathcal{F}_\infty \) (market information), and follows \( \mathcal{E}(\lambda) \).

- Infinite horizon stochastic control problem:

\[
v(x) = \sup_{(\alpha, c)} E \left[ \int_0^\infty e^{-(\beta + \lambda) t} U(c_t X^x_t) \, dt \right], \quad x > 0.
\]
The HJB equation is given by:

\[
(\beta + \lambda)v - rxv' - \sup_{a \in A} \left[ a(\mu - r)v' + \frac{1}{2}a^2x^2\sigma^2v'' \right] - \sup_{c \geq 0} \left[ U(cx) - cxv' \right] = 0, \quad x > 0.
\]

- Explicit solution for CRRA utility function: \( U(x) = \frac{x^p}{p} \).

Under the condition that \( \beta + \lambda > \rho \), we have

\[
v(x) = K U(x), \quad \text{with} \quad K = \left( \frac{1 - p}{\beta + \lambda - \rho} \right)^{1-p}.
\]

The optimal portfolio/consumption strategies are:

\[
\hat{a} = \arg \max_{a \in A} \left[ a(\mu - r) - \frac{1}{2}a^2(1 - p)\sigma^2 \right],
\]

\[
\hat{c} = \frac{1}{x}(v'(x))^{\frac{1}{p-1}} = K^{\frac{1}{p-1}}.
\]