VI. Applications

1. Merton portfolio selection in finite horizon

An agent invests at any time t a proportion α_t of his wealth X in a stock of price S and $1-\alpha_t$ in a bond of price S^0 with interest rate r. The investor faces the portfolio constraint that at any time t, α_t is valued in A closed convex subset of $\mathbb{R}$.

▶ Assuming a Black-Scholes model for S (with constant rate of return μ and volatility $\sigma > 0$), the dynamics of the controlled wealth process is:

$$\begin{aligned} dX_t &= \frac{X_t\alpha_t}{S_t}dS_t + \frac{X_t(1-\alpha_t)}{S_t^0}dS_t^0 \\ &= X_t\left(r + \alpha_t(\mu - r)\right)dt + X_t\alpha_t\sigma dW_t. \end{aligned}$$

• The preferences of the agent is described by a utility function U: increasing and concave function. The performance of a portfolio strategy is measured by the expected utility from terminal wealth → Utility maximization problem at a finite horizon T:

$$v(t,x) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}[U(X_T^{t,x})], \quad (t,x) \in [0,T]\times(0,\infty).$$

→ Standard stochastic control problem

HJB equation for Merton's problem

$$v_t + rxv_x + \sup_{a \in A} \Big[a(\mu - r)xv_x + \frac{1}{2}x^2a^2\sigma^2 v_{xx} \Big] = 0, \ (t,x) \in [0,T) \times (0,\infty)$$

$$v(T,x) = U(x), \quad x > 0.$$

• *The case of CRRA utility functions*:

$$U(x) = \frac{x^p}{p}, \quad p < 1, \ p \neq 0$$

→ Relative Risk Aversion: $-xU''(x)/U'(x) = 1 - p$.

▶ We look for a candidate solution to HJB in the form

$$w(t,x) = \varphi(t)U(x).$$

Plugging into HJB, we see that φ should satisfy the ODE:

$$\varphi'(t) + \rho\varphi(t) = 0, \qquad \phi(T) = 1,$$

where

$$\rho = rp + p \sup_{a \in A} \big[a(\mu - r) - \frac{1}{2}a^2(1-p)\sigma^2 \big],$$

→

$$\varphi(t) = e^{\rho(T-t)}.$$

▶ The value function is equal to

$$v(t,x) \;=\; e^{\rho(T-t)}U(x),$$

and the optimal control is constant (in proportion of wealth invested)

$$\hat{a} \;=\; \arg\max_{a\in A}\big[a(\mu-r)-\frac{1}{2}a^2(1-p)\sigma^2\big].$$

When $A=\mathbb{R}$ (no portfolio constraint), the values of ρ and $\hat{a}$ are explicitly given by

$$\rho \;=\; \frac{(\mu-r)^2}{2\sigma^2}\frac{p}{1-p}+rp.$$

and

$$\hat{a} \;=\; \frac{\mu-r}{\sigma^2(1-p)},$$

2. Merton portfolio/consumption choice on infinite horizon

In addition to the investment α in the stock, the agent can also consume from his wealth:

$\rightarrow (c_t)_{t\geq 0}$ consumption per unit of wealth

▶ The wealth process, controlled by (α, c) is governed by:

$$dX_t = X_t \left(r + \alpha_t(\mu - r) - c_t\right) dt + X_t \alpha_t \sigma dW_t.$$

• The preferences of the agent is described by a utility U from consumption, and the goal is to maximize over portfolio/consumption the expected utility from intertemporal consumption up to a random time horizon:

$$v(x) = \sup_{(\alpha,c)} \mathbb{E}\left[\int_0^{\tau} e^{-\beta t} U(c_t X_t^x) dt\right], \quad x > 0.$$

We assume that τ is independent of $\mathcal{F}_\infty$ (market information), and follows $\mathcal{E}(\lambda)$.

▶ Infinite horizon stochastic control problem:

$$v(x) = \sup_{(\alpha,c)} \mathbb{E}\left[\int_0^{\infty} e^{-(\beta+\lambda) t} U(c_t X_t^x) dt\right], \quad x > 0.$$

HJB equation

$$(\beta+\lambda)v - rxv' - \sup_{a\in A}[a(\mu-r)v' + \frac{1}{2}a^2x^2\sigma^2v''] - \sup_{c\geq 0}[U(cx) - cxv'] \; = \; 0, \quad x > 0.$$

• Explicit solution for CRRA utility function: $U(x) = x^p/p$.

Under the condition that $\beta+\lambda > \rho$, we have

$$v(x) \; = \; K\,U(x), \quad \text{with } K \; = \; \left(\frac{1-p}{\beta+\lambda-\rho}\right)^{1-p}.$$

The optimal portfolio/consumption strategies are:

$$\hat{a} \; = \; \arg\max_{a\in A}[a(\mu-r) - \frac{1}{2}a^2(1-p)\sigma^2]$$
$$\hat{c} \; = \; \frac{1}{x}(v'(x))^{\frac{1}{p-1}} \; = \; K^{\frac{1}{p-1}}.$$