Financial Markets

Price Processes:

$$S_t = \left[S_t^0, ..., S_t^N \right]$$

Example: (Black-Scholes, $S^0 := B,\ S^1 := S$)

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

Portfolio:

$$h_t = \left[h_t^0, ..., h_t^N \right]$$

$h_t^i =$ number of units of asset i at time t.

Value Process:

$$V_t^h = \sum_{i=0}^{N} h_t^i S_t^i = h_t S_t$$

Self Financing Portfolios

Definition: (intuitive)
A portfolio is **self-financing** if there is no exogenous infusion or withdrawal of money. "The purchase of a new asset must be financed by the sale of an old one."

Definition: (mathematical)
A portfolio is **self-financing** if the value process satisfies

$$dV_t = \sum_{i=0}^{N} h_t^i dS_t^i$$

Major insight:
If the price process S is a **martingale**, and if h is **self-financing**, then V is a **martingale**.

NB! This simple observation is in fact the basis of the following theory.

Arbitrage

Recall that h is an arbitrage if

- h is self financing
- $V_0 = 0$.
- $V_T \geq 0, \; P - a.s.$
- $P\left(V_T > 0\right) > 0$

Major insight

This concept is invariant under an **equivalent change of measure!**

Martingale Measures

Definition: A probability measure Q is called an **equivalent martingale measure** (EMM) if and only if it has the following properties.

- Q and P are equivalent, i.e.

$$Q \sim P$$

- The normalized price processes

$$Z_t^i = \frac{S_t^i}{S_t^0}, \quad i = 0, \ldots, N$$

  are **Q-martingales**.

Wan now state the main result of arbitrage theory.

First Fundamental Theorem

Theorem: The market is arbitrage free

iff

there exists an equivalent martingale measure.

Proof that EMM implies no arbitrage

Assume that there exists an EMM denoted by Q. Assume that $P(V_T \geq 0) = 1$ and $P(V_T > 0) > 0$. Then, since $P \sim Q$ we also have $Q(V_T \geq 0) = 1$ and $Q(V_T > 0) > 0$.

Recall:

$$dV_t^Z = \sum_1^N h_t^i dZ_t^i$$

Q is a martingale measure

$$\Downarrow$$

V^Z is a Q-martingale

$$\Downarrow$$

$$V_0 = V_0^Z = E^Q\left[V_T^Z\right] > 0$$

$$\Downarrow$$

No arbitrage

Choice of Numeraire

The **numeraire** price S_t^0 can be chosen arbitrarily. The most common choice is however that we choose S^0 as the **bank account**, i.e.

$$S_t^0 = B_t$$

where

$$dB_t = r_t B_t dt$$

Here r is the (possibly stochastic) short rate and we have

$$B_t = e^{\int_0^t r_s ds}$$

Example: The Black-Scholes Model

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

Look for martingale measure. We set $Z = S/B$.

$$dZ_t = Z_t(\alpha - r)dt + Z_t \sigma dW_t,$$

Girsanov transformation on $[0, T]$:

$$
\begin{cases}
dL_t = L_t \varphi_t dW_t, \\
L_0 = 1.
\end{cases}
$$

$$dQ = L_T dP, \quad \text{on } \mathcal{F}_T$$

Girsanov:

$$dW_t = \varphi_t dt + dW_t^Q,$$

where W^Q is a Q-Wiener process.

The Q-dynamics for Z are given by

$$dZ_t = Z_t \left[\alpha - r + \sigma\varphi_t\right] dt + Z_t \sigma dW_t^Q.$$

Unique martingale measure Q, with Girsanov kernel given by

$$\varphi_t = \frac{r - \alpha}{\sigma}.$$

Q-dynamics of S:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q.$$

Conclusion: The Black-Scholes model is free of arbitrage.

Pricing

We consider a market $B_t, S_t^1, \ldots, S_t^N$.

Definition:
A **contingent claim** with **delivery time** T, is a random variable

$$X \in \mathcal{F}_T.$$

"At $t = T$ the amount X is paid to the holder of the claim".

Example: (European Call Option)

$$X = \max\left[S_T - K, 0\right]$$

Let X be a contingent T-claim.

Problem: How do we find an arbitrage free price process $\Pi_t\left[X\right]$ for X?

Solution

The extended market

$$B_t, S_t^1, \ldots, S_t^N, \Pi_t[X]$$

must be arbitrage free, so there must exist a martingale measure Q for $(S_t, \Pi_t[X])$. In particular

$$\frac{\Pi_t[X]}{B_t}$$

must be a Q-martingale, i.e.

$$\frac{\Pi_t[X]}{B_t} = E^Q\left[\left.\frac{\Pi_T[X]}{B_T}\right| \mathcal{F}_t\right]$$

Since we obviously (why?) have

$$\Pi_T[X] = X$$

we have proved the main pricing formula.

Risk Neutral Valuation

Theorem: For a T-claim X, the arbitrage free price is given by the formula

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right]$$

Hedging

Def: A portfolio is a **hedge** against X ("replicates X") if

- h is self financing
- $V_T = X, \quad P - a.s.$

Def: The market is **complete** if every X can be hedged.

Pricing Formula:
If h replicates X, then a natural way of pricing X is

$$\Pi_t[X] = V_t^h$$

When can we hedge?

Existence of hedge

$$\Updownarrow$$

Existence of stochastic integral representation

Black-Scholes Model

Q-dynamics

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^Q, \\ dZ_t &= Z_t \sigma dW_t^Q \end{aligned}$$

$$M_t = E^Q\left[e^{-rT} X \middle| \mathcal{F}_t \right],$$

Representation theorem for Wiener processes

$\Downarrow$

there exists g such that

$$M_t = M(0) + \int_0^t g_s dW_s^Q.$$

Thus

$$M_t = M_0 + \int_0^t h_s^1 dZ_s,$$

with $h_t^1 = \frac{g_t}{\sigma Z_t}$.

Result:
X can be replicated using the portfolio defined by

$$\begin{aligned} h_t^1 &= g_t/\sigma Z_t, \\ h_t^B &= M_t - h_t^1 Z_t. \end{aligned}$$

Moral: The Black Scholes model is complete.

Main Results

- The market is arbitrage free $\Leftrightarrow$ There exists a martingale measure Q
- The market is complete $\Leftrightarrow$ Q is unique.
- Every X must be priced by the formula

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t \right]$$

  for some choice of Q.

- In a non-complete market, different choices of Q will produce different prices for X.
- For a hedgeable claim X, all choices of Q will produce the same price for X:

$$\Pi_t[X] = V_t = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t \right]$$