Lecture 8
Convergence, Optimal Growth

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Economics 702
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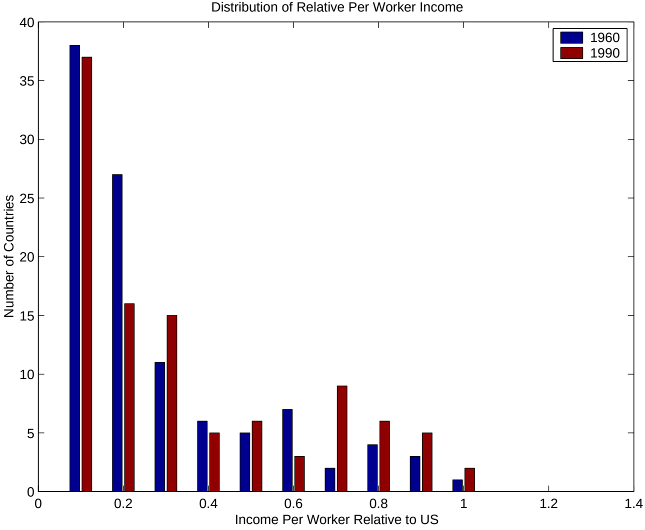
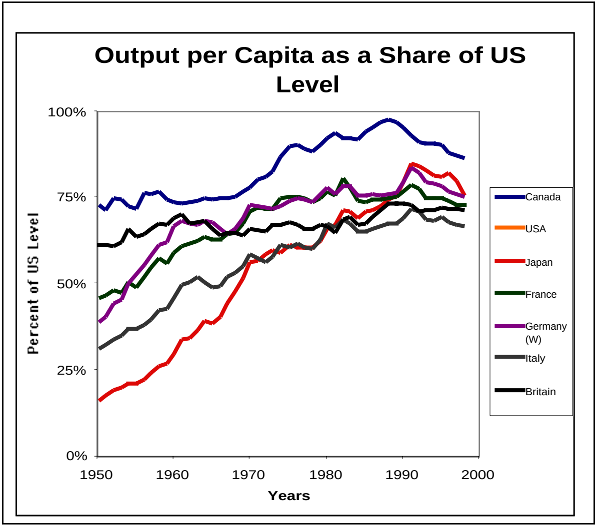
Some Development Facts

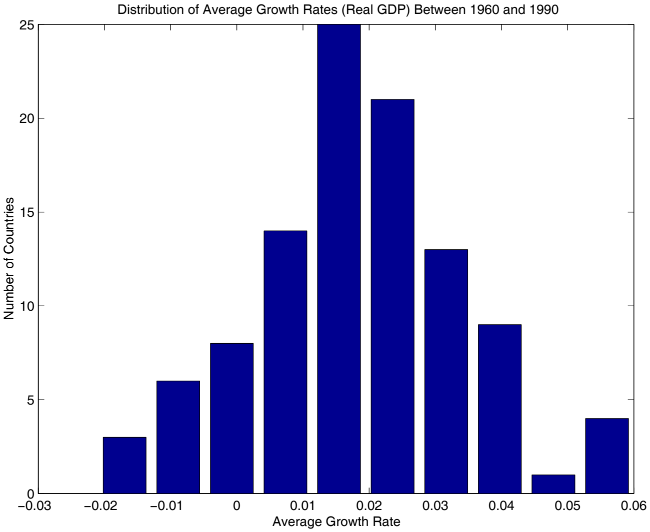
Stylized facts for cross-country comparisons.

1. Enormous variation of per worker income across countries.
2. Enormous variation in growth rates of per worker income across countries.
3. Growth rates are not constant over time for a given country.
4. Countries change their relative position in the international income distribution.


Distribution of Relative Per Worker Income
Number of Countries
Income Per Worker Relative to US
1960
1990
0
5
10
15
20
25
30
35
40
0
0.2
0.4
0.6
0.8
1
1.2
1.4



Output per Capita as a Share of US Level
100%
75%
50%
25%
0%
Percent of US Level
1950
1960
1970
1980
1990
2000
Years
Canada
USA
Japan
France
Germany (W)
Italy
Britain


Distribution of Average Growth Rates (Real GDP) Between 1960 and 1990
Number of Countries
0
5
10
15
20
25
−0.03
−0.02
−0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
Average Growth Rate

Evaluation of the Model: Development Facts

- Differences in income levels across countries explained in the model by differences in s, n and δ.
- Variation in growth rates: in model *permanent* differences can only be due to differences in rate of technological progress g.
  Temporary differences can be explained by transition dynamics.
- That growth rates are not constant over time for a given country can be explained by transition dynamics and/or shocks to n, s and δ.
- Changes in relative position: in the model countries whose s moves up, relative to other countries, move up in income distribution. Reverse with n.

The Convergence Hypothesis

- Fact: Enormous variation in incomes per worker across countries
- Question: Do poor countries eventually catch up?
- Convergence hypothesis: They do, in the right sense.
- Main prediction of convergence hypothesis: Poor countries should grow faster (per capita) than rich countries.
- Why? Recall:

$$\frac{\dot{k}}{k} = sk^{\alpha-1} - (n+\delta)\,, \text{ and: } \frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k}$$

The Solow Model and Convergence

- Analyze countries with same s, n, δ, α, g
- Eventually same growth rate of output per worker and same level of output per worker (*absolute* convergence).
- Countries starting further below the balanced growth path (poorer countries) should grow faster than countries closer to balanced growth path.
- Seems to be the case for the sample of now industrialized countries.
- World capital markets should speed this process. Capital should flow from rich (high $K \Rightarrow$ low MPK) to poor countries (low K, high MPK).

Figure 1.a: Growth Rate Versus Initial Per Capita GDP

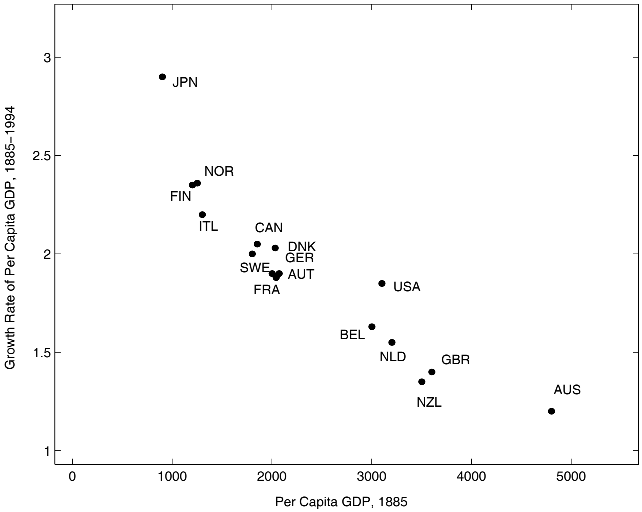


Figure 1.b: Growth Rate Versus Initial Per Capita GDP

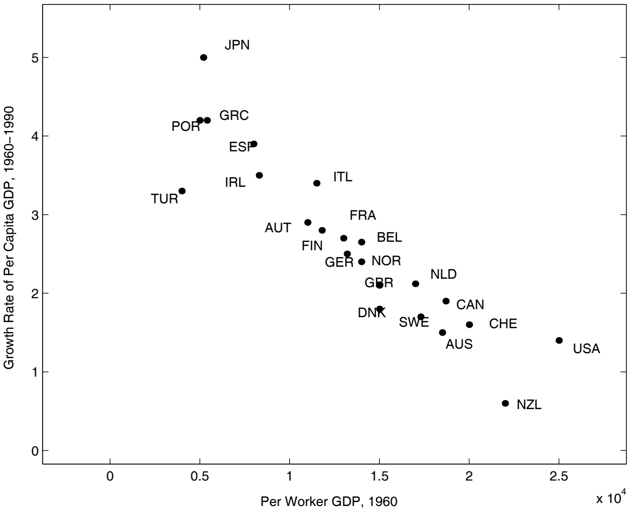


Figure 1.c: Growth Rate Versus Initial Per Capita GDP

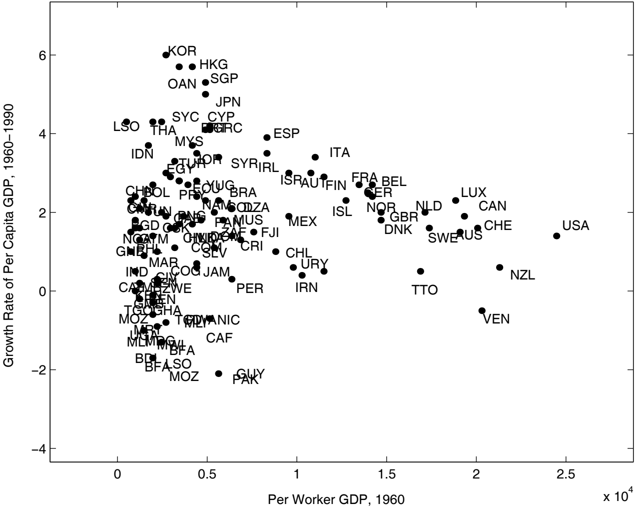
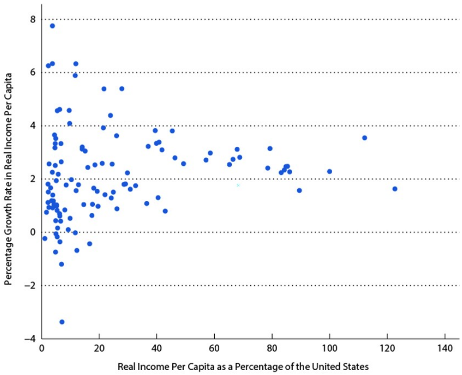
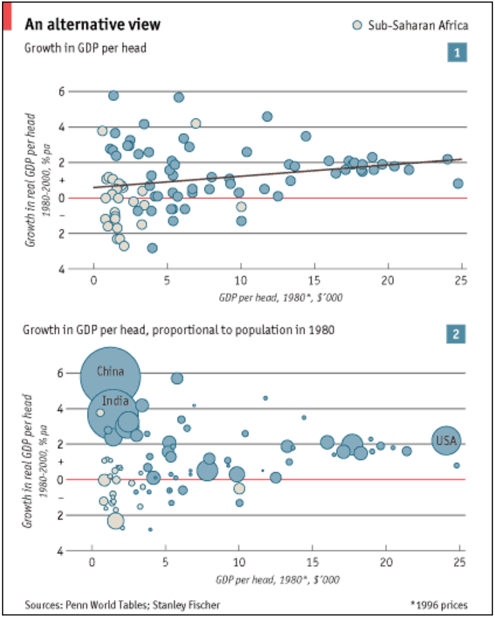


Figure 7.4 Growth Rate in Per Capita Income vs. Level of Per Capita Income



Pearson


An alternative view
Sub-Saharan Africa
Growth in GDP per head
1
6
4
2
+
0
-
2
4
Growth in real GDP per head
1980-2000, % pa
0
5
10
15
20
25
GDP per head, 1980*, $'000
Growth in GDP per head, proportional to population in 1980
2
China
India
USA
Sources: Penn World Tables; Stanley Fischer
*1996 prices

Conditional Convergence

- Countries with same g but potentially differing s, n, δ, α.
- Countries have different balanced growth path.
- Countries that start further below *their* balanced growth path (countries that are poor relative to their BGP) should grow faster than rich countries (relative to their BGP). This is called *conditional* convergence.
- Data for full sample lend support to conditional convergence.
- Industrialized countries as of 1885 or 1960: similar savings rates, population growth rates.
- US States: Strong evidence of convergence across states. Again similar technology, saving, population. Barro and Sala-i-Martin (1992)

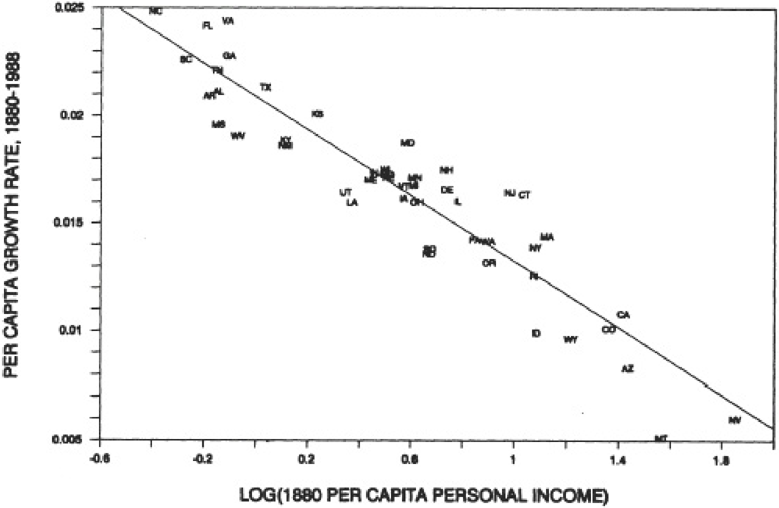


FIG. 1.—Growth rate from 1880 to 1988 vs. 1880 per capita income

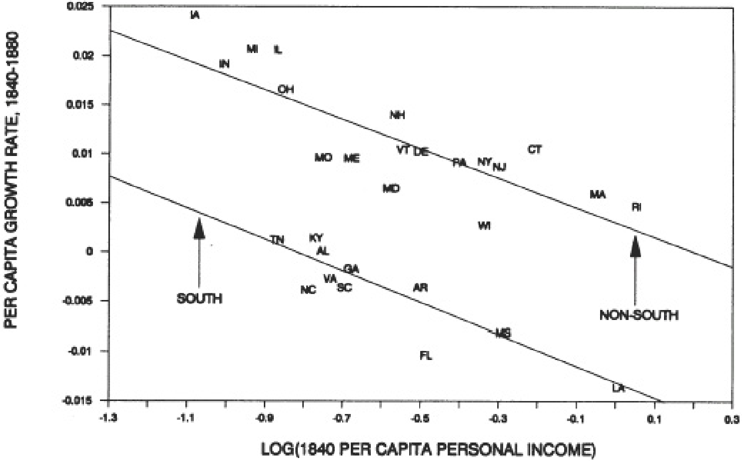


FIG. 2.—Growth rate from 1840 to 1880 vs. 1840 per capita income

Conclusion: The Solow Growth Model

- Offers a simple and elegant account of a number of growth facts.
- Leaves unexplained factors that make countries leave (or not attain) their BGP.
- Leaves unexplained why certain countries have higher s, n than others.
- Leaves unexplained technological progress, the source of growth.

Optimal Growth

- While the Solow model was useful for studying growth and convergence, it takes savings rates as constant and exogenous.
- Our previous analysis of optimal allocations showed how capital (and hence savings) are determined endogenously.
- We now add growth in technology and population to the model of optimal allocations to determine optimal growth.
- We again work in discrete time, so we let N_t and A_t and $A_t N_t$ evolve as:

$$\begin{aligned} N_t &= (1+n)N_{t-1}, \quad N_0 = 1 \\ A_t &= (1+g)A_{t-1}, \quad A_0 = 1 \\ A_t N_t &= (1+n)(1+g)A_{t-1}N_{t-1} \equiv (1+\eta)A_{t-1}N_{t-1} \end{aligned}$$

where $\eta = (1+n)(1+g) - 1 \approx n + g$

- As in our analysis of the Solow model, we use Cobb-Douglas production which implies:

$$\begin{aligned} Y &= K^{\alpha}(AN)^{1-\alpha} \\ \tilde{y} &= \frac{Y}{AN} = \tilde{k}^{\alpha} \end{aligned}$$

- Note that we can write consumption as:

$$C_t = \tilde{c}_t A_t N_t = \tilde{c}_t (1+\eta)^t$$

- We again use constant elasticity preferences (leaving off the -1 term), which implies

$$\begin{aligned} U(C_t) &= \frac{C_t^{1-\sigma}}{1-\sigma} = \frac{\left(\tilde{c}_t (1+\eta)^t\right)^{1-\sigma}}{1-\sigma} \\ &= (1+\eta)^{(1-\sigma)t} \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma} \end{aligned}$$

The Optimal Growth Path

- We also have the feasibility condition:

$$\begin{aligned} C_t &= F(K_t, N_t) - K_{t+1} + (1-\delta)K_t \\ \frac{C_t}{A_t N_t} &= \frac{K_t^\alpha (A_t N_t)^{1-\alpha}}{A_t N_t} - \frac{K_{t+1} A_{t+1} N_{t+1}}{A_{t+1} N_{t+1} A_t N_t} + (1-\delta)\frac{K_t}{A_t N_t} \\ \tilde{c}_t &= \tilde{k}_t^\alpha - (1+\eta)\tilde{k}_{t+1} + (1-\delta)\tilde{k}_t \end{aligned}$$

- So now let's consider the optimal allocation:

$$\max_{\{C_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(C_t)$$

subject to: $C_t = F(K_t, N_t) - K_{t+1} + (1-\delta)K_t, \quad \forall t, K_0$ given

- This can be re-written as:

$$\max_{\{\tilde{c}_t, \tilde{k}_{t+1}\}} \sum_{t=0}^{\infty} [\beta(1+\eta)^{1-\sigma}]^t \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma}$$

subject to: $\tilde{c}_t = \tilde{k}_t^\alpha - (1+\eta)\tilde{k}_{t+1} + (1-\delta)\tilde{k}_t$

Characterizing the Optimal Growth Path

- Form the Lagrangian with $\tilde{\beta} = \beta(1+\eta)^{1-\sigma}$:

$$\mathcal{L} = \max_{\{\tilde{c}_t, \tilde{k}_{t+1}\}} \sum_{t=0}^{\infty} \left(\tilde{\beta}^t \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma} + \lambda_t [\tilde{k}_t^{\alpha} - (1+\eta)\tilde{k}_{t+1} + (1-\delta)\tilde{k}_t - \tilde{c}_t] \right)$$

- First order conditions for any c_t, and for k_{t+1}, $t > 0$:

$$\begin{aligned} \tilde{\beta}^t \tilde{c}_t^{-\sigma} &= \lambda_t \\ -(1+\eta)\lambda_t + \lambda_{t+1}[\alpha \tilde{k}_{t+1}^{\alpha-1} + 1 - \delta] &= 0. \end{aligned}$$

- These imply the Euler equation:

$$(1+\eta)\tilde{c}_t^{-\sigma} = \tilde{\beta}\tilde{c}_{t+1}^{-\sigma}[\alpha \tilde{k}_{t+1}^{\alpha-1} + 1 - \delta]$$

Optimal Balanced Growth

- Look for a steady state of the transformed optimal allocation.

$$\begin{aligned}(1+\eta)(\tilde{c}^*)^{-\sigma} &= \tilde{\beta}(\tilde{c}^*)^{-\sigma}[\alpha(\tilde{k}^*)^{\alpha-1}+1-\delta] \\ (1+\eta) &= \tilde{\beta}[\alpha(\tilde{k}^*)^{\alpha-1}+1-\delta]\end{aligned}$$

- Or, recalling that $\beta = 1/(1+\theta)$:

$$\begin{aligned}f'(\tilde{k}^*) &= \frac{1+\eta}{\beta(1+\eta)^{1-\sigma}}+\delta-1 \\ &= \frac{(1+\theta)}{(1+\eta)^{-\sigma}}+\delta-1 \\ &\approx \delta+\theta+\sigma\eta \\ &\approx \delta+\theta+\sigma(n+g)\end{aligned}$$

Optimal Balanced Growth

- Therefore we have capital per unit of effective labor in the balanced growth path:

$$\begin{aligned}\tilde{k}^* &= \left(\frac{\alpha}{\delta - 1 + (1+\theta)(1+\eta)^{\sigma}}\right)^{\frac{1}{1-\alpha}} \\ &\approx \left(\frac{\alpha}{\delta + \theta + \sigma(n+g)}\right)^{\frac{1}{1-\alpha}}\end{aligned}$$

- This generalizes the solution we had for the optimal allocation without growth.
- As in the Solow model, along a balanced growth path all level variables are growing at rate $\eta \approx n + g$.
- Unlike the Solow model, the steady state depends on the household preferences, as the savings rates are determined optimally.

Qualitative Dynamics

- We can analyze the qualitative dynamics just as we did without productivity growth.
- The key equations of the model are now:

$$\begin{aligned} U'(\tilde{c}_t) &= \beta(1+\eta)^{1-\sigma}U'(\tilde{c}_{t+1})[f'(\tilde{k}_{t+1})+1-\delta] \\ (1+\eta)\tilde{k}_{t+1} &= (1-\delta)\tilde{k}_t + f(\tilde{k}_t) - \tilde{c}_t \end{aligned}$$

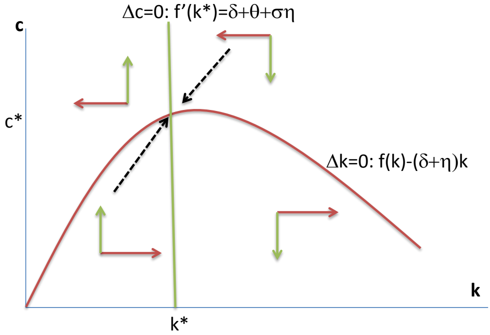
- The dynamics work in much the same way, only now they depend on η. So we can analyze the effects of a change in n or g which lead to a change in η.
- In steady state, $\Delta\tilde{c}_{t+1} = 0$, and

$$f'(\tilde{k}^*) \approx \delta + \theta + \sigma\eta$$

- Also in steady state $\Delta\tilde{k}_{t+1} = 0$, so:

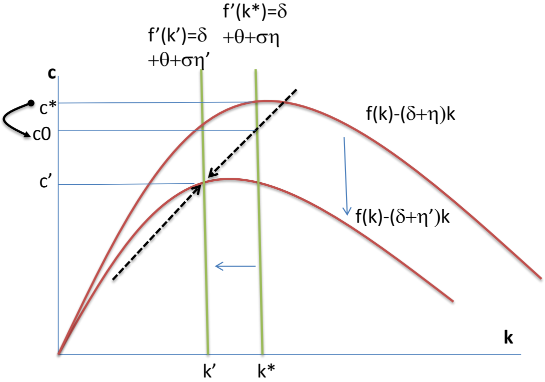
$$\tilde{c} = f(\tilde{k}) - (\delta+\eta)\tilde{k}$$

Phase Diagram of Optimal Growth Model



Phase diagram of the optimal growth model

Effect of an Increase in n or g



Phase diagram: An increase in the growth rate η to η’. As before, initial effect depends on the slope of the saddle path.