Lecture 7
The Solow Model and Convergence
Optimal Growth

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Re-introduce TFP. No limits to innovation.
Slightly simpler to introduce as labor-augmenting technical change. Aggregate production function:

\[ Y = K^\alpha (AN)^{1-\alpha} \]

Equivalent to \( zK^\alpha N^{1-\alpha} \) with \( z = A^{1-\alpha} \).

Key assumption: constant rate of technological progress:

\[ \frac{\dot{A}}{A} = g > 0 \]

TFP growth is exogenous here.
Balanced Growth Path

- Situation in which output per worker, capital per worker, and consumption per worker grow at constant rates.
- For Solow model, $y, k, c$ all grow same at constant rate $g$.
- Why? In balanced growth path $g_k$ constant, but:

$$\dot{k} = sy - (n + d)k \Rightarrow g_k = \frac{sy}{k} - (n + d)$$

So $y/k$ constant $\Rightarrow g_y = g_k$. But then:

$$y = \frac{Y}{N} = \frac{K^\alpha (AN)^{1-\alpha}}{N} = k^\alpha A^{1-\alpha}$$

- Take logs and differentiate:

$$g_y = \alpha g_k + (1 - \alpha)g_A$$

But we showed $g_k = g_y$ and assumed $g_A = g$, so

$$g_k = g_y = g_A = g.$$
In BGP all per-capita variables grow at rate $g$. Want to work with variables that are constant in long run. Define:

\[
\tilde{y} = \frac{y}{A} = \frac{Y}{AN}
\]

\[
\tilde{k} = \frac{k}{A} = \frac{K}{AN}
\]

Repeat the analysis with new variables:

\[
\tilde{y} = \tilde{k}^\alpha
\]

\[
\dot{\tilde{k}} = s\tilde{y} - (n + g + \delta)\tilde{k}
\]

\[
\ddot{\tilde{k}} = s\tilde{k}^\alpha - (n + g + \delta)\tilde{k}
\]
Balanced Growth Path Analysis I

- Solve for $\tilde{k}^*$ analytically

\[
0 = s\tilde{k}^{*\alpha} - (n + g + \delta)\tilde{k}^*
\]

\[
\tilde{k}^* = \left(\frac{s}{n + g + \delta}\right)^{\frac{1}{1-\alpha}}
\]

- Therefore

\[
\tilde{y}^* = \left(\frac{s}{n + g + \delta}\right)^{\frac{\alpha}{1-\alpha}}
\]
Balanced Growth Path Analysis II

\[
k(t) = A(t) \left( \frac{s}{n + g + \delta} \right)^{\frac{1}{1-\alpha}}
\]

\[
y(t) = A(t) \left( \frac{s}{n + g + \delta} \right)^{\frac{\alpha}{1-\alpha}}
\]

\[
K(t) = N(t) A(t) \left( \frac{s}{n + g + \delta} \right)^{\frac{1}{1-\alpha}}
\]

\[
Y(t) = N(t) A(t) \left( \frac{s}{n + g + \delta} \right)^{\frac{\alpha}{1-\alpha}}
\]
The fundamental equation of the Solow Model

$$\dot{k} = s\tilde{k}^\alpha - (n + g + \delta)\tilde{k}$$

Capital per effective labor in the balanced growth path:

$$\tilde{k}^* = \left(\frac{s}{n + g + \delta}\right)^{\frac{1}{1-\alpha}}$$

Along the balanced growth path, all per-capita variables grow at $g$, the rate of TFP growth.

In the BGP, all level variables grow at rate $n + g$. 
Stylized facts, originally due to Kaldor (1957). Empirical regularities of the growth process for the US and for most other industrialized countries

1. Output (real GDP) per worker $y = \frac{Y}{N}$ and capital per worker $k = \frac{K}{N}$ grow over time at relatively constant and positive rate.

2. They grow at similar rates, so that the ratio between capital and output, $\frac{K}{Y}$ is relatively constant over time.

3. The real return to capital $r$ (and the real interest rate $r - \delta$) is relatively constant over time.

4. The capital and labor shares are roughly constant over time.
US post-WWII data
The Solow model matches the stylized facts.

- Output and capital per worker grow at the same constant, positive rate in BGP of model. In long run model reaches BGP.
- Capital-output ratio $\frac{K}{Y}$ constant along BGP
- Interest rate constant in balanced growth path
- Capital share equals $\alpha$, labor share equals $1 - \alpha$ in the model (always, not only along BGP)
- Success of the model along these dimensions, but source of growth, technological progress, is left unexplained.
Some Development Facts

Stylized facts for cross-country comparisons.

1. Enormous variation of per worker income across countries.
2. Enormous variation in growth rates of per worker income across countries.
3. Growth rates are not constant over time for a given country.
4. Countries change their relative position in the international income distribution.
Distribution of Average Growth Rates (Real GDP) Between 1960 and 1990

Average Growth Rate

Number of Countries

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Differences in income levels across countries explained in the model by differences in $s$, $n$ and $\delta$.

Variation in growth rates: in model permanent differences can only be due to differences in rate of technological progress $g$. Temporary differences can be explained by transition dynamics.

That growth rates are not constant over time for a given country can be explained by transition dynamics and/or shocks to $n$, $s$ and $\delta$.

Changes in relative position: in the model countries whose $s$ moves up, relative to other countries, move up in income distribution. Reverse with $n$. 
The Convergence Hypothesis

- Fact: Enormous variation in incomes per worker across countries
- Question: Do poor countries eventually catch up?
- Convergence hypothesis: They do, in the right sense.
- Main prediction of convergence hypothesis: Poor countries should grow faster (per capita) than rich countries.
- Why? Recall:

\[
\frac{\dot{k}}{k} = sk^{\alpha-1} - (n + \delta), \quad \text{and:} \quad \frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}
\]
The Solow Model and Convergence

- Analyze countries with same $s$, $n$, $δ$, $α$, $g$
- Eventually same growth rate of output per worker and same level of output per worker (absolute convergence).
- Countries starting further below the balanced growth path (poorer countries) should grow faster than countries closer to balanced growth path.
- Seems to be the case for the sample of now industrialized countries.
- World capital markets should speed this process. Capital should flow from rich (high $K \Rightarrow$ low $MPK$) to poor countries (low $K$, high $MPK$).
Figure 1.a: Growth Rate Versus Initial Per Capita GDP

Per Capita GDP, 1885

Growth Rate of Per Capita GDP, 1885−1994

0 1000 2000 3000 4000 5000

1

1.5

2

2.5

3

JPN

FIN

NOR

ITL

SWE

CAN

FRA

DNK

AUT

GER

BEL

USA

NLD

NZL

GBR

AUS
Figure 1.c: Growth Rate Versus Initial Per Capita GDP
Per Worker GDP, 1960
Growth Rate of Per Capita GDP, 1960−1990
0 ... RWA
GNB COM
CAF
MWI
TCD
UGAMLI
BDI BFA
LSO
MLI
BFAMOZ
CAF
An alternative view

Growth in GDP per head

Growth in real GDP per head 1980-2000, % pa

Sub-Saharan Africa

Sources: Penn World Tables; Stanley Fischer

*1996 prices
Countries with same $g$ but potentially differing $s, n, \delta, \alpha$.

Countries have different balanced growth path.

Countries that start further below their balanced growth path (countries that are poor relative to their BGP) should grow faster than rich countries (relative to their BGP). This is called conditional convergence.

Data for full sample lend support to conditional convergence.

Industrialized countries as of 1885: similar savings rates, population growth rates.

Fig. 1. — Growth rate from 1880 to 1988 vs. 1880 per capita income.
Fig. 2.—Growth rate from 1840 to 1880 vs. 1840 per capita income
Conclusion: The Solow Growth Model

- Offers a simple and elegant account of a number of growth facts.
- Leaves unexplained factors that make countries leave (or not attain) their BGP.
- Leaves unexplained why certain countries have higher $s, n$ than others.
- Leaves unexplained technological progress, the source of growth.
While the Solow model was useful for studying growth and convergence, it takes savings rates as constant and exogenous.

Our previous analysis of optimal allocations showed how capital (and hence savings) are determined endogenously.

We now add growth in technology and population to the model of optimal allocations to determine optimal growth.

We again work in discrete time, so we let $N_t$ and $A_t$ and $A_t N_t$ evolve as:

\[
N_t = (1 + n) N_{t-1}, \quad N_0 = 1
\]
\[
A_t = (1 + g) A_{t-1}, \quad A_0 = 1
\]
\[
A_t N_t = (1 + n)(1 + g) A_{t-1} N_{t-1} \equiv (1 + \eta) N_{t-1}
\]

where $\eta = (1 + n)(1 + g) - 1 \approx n + g$
As in our analysis of the Solow model, we use Cobb-Douglas production which implies:

\[ Y = K^\alpha (AN)^{1-\alpha} \]

\[ \tilde{y} = \frac{Y}{AN} = \tilde{k}^\alpha \]

Note that we can write consumption as:

\[ C_t = \tilde{c}_t A_t N_t = \tilde{c}_t (1 + \eta)^t \]

We again use constant elasticity preferences (leaving off the -1 term), which implies

\[ U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma} = \frac{(\tilde{c}_t (1 + \eta)^t)^{1-\sigma}}{1-\sigma} \]

\[ = (1 + \eta)^{(1-\sigma)t} \tilde{c}_t^{1-\sigma} \frac{1}{1-\sigma} \]
The Optimal Growth Path

- We also feasibility condition:

\[
C_t = F(K_t, N_t) - K_{t+1} + (1 - \delta)K_t
\]

\[
\frac{C_t}{A_t N_t} = \frac{K_t^{\alpha}(A_t N_t)^{1-\alpha}}{A_t N_t} - \frac{K_{t+1} A_{t+1} N_{t+1}}{A_{t+1} N_{t+1} A_t N_t} + (1 - \delta) \frac{K_t}{A_t N_t}
\]

\[
\tilde{c}_t = \tilde{k}_t^\alpha - (1 + \eta)\tilde{k}_{t+1} + (1 - \delta)\tilde{k}_t
\]

- So now let’s consider the optimal allocation:

\[
\max \{ \tilde{c}_t, K_{t+1} \} \sum_{t=0}^{\infty} \beta^t U(C_t)
\]

subject to: \( C_t = F(K_t, N_t) - K_{t+1} + (1 - \delta)K_t \), \( \forall t, K_0 \) given

- This can be re-written as:

\[
\max \{ \tilde{c}_t, \tilde{k}_{t+1} \} \sum_{t=0}^{\infty} [\beta(1 + \eta)^{1-\sigma}]^t \frac{\tilde{c}_t^{1-\sigma}}{1 - \sigma}
\]

subject to: \( \tilde{c}_t = \tilde{k}_t^\alpha - (1 + \eta)\tilde{k}_{t+1} + (1 - \delta)\tilde{k}_t \)
Characterizing the Optimal Growth Path

- Form the Lagrangian with $\tilde{\beta} = \beta (1 + \eta)^{1-\sigma}$:

$$
\mathcal{L} = \max_{\{\tilde{c}_t, \tilde{k}_{t+1}\}} \sum_{t=0}^{\infty} \left( \tilde{\beta}^t \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma} + \lambda_t [\tilde{k}_t^\alpha - (1 + \eta)\tilde{k}_{t+1} + (1 - \delta)\tilde{k}_t - \tilde{c}_t] \right)
$$

- First order conditions for any $c_t$, and for $k_{t+1}$, $t > 0$:

$$
\tilde{\beta}^t \tilde{c}_t^{-\sigma} = \lambda_t \\
-(1 + \eta)\lambda_t + \lambda_{t+1} [\alpha \tilde{k}_{t+1}^{\alpha-1} + 1 - \delta] = 0.
$$

- These imply the Euler equation:

$$
(1 + \eta)\tilde{c}_t^{-\sigma} = \tilde{\beta} \tilde{c}_{t+1}^{-\sigma} [\alpha \tilde{k}_{t+1}^{\alpha-1} + 1 - \delta]
$$
Optimal Balanced Growth

- Look for a steady state of the transformed optimal allocation.

\[(1 + \eta)(\tilde{c}^*)^{-\sigma} = \tilde{\beta}(\tilde{c}^*)^{-\sigma}[\alpha(\tilde{k}^*)^{\alpha-1} + 1 - \delta]\]

\[(1 + \eta) = \tilde{\beta}[\alpha(\tilde{k}^*)^{\alpha-1} + 1 - \delta]\]

- Or, recalling that \(\beta = 1/(1 + \theta)\):

\[f'(\tilde{k}^*) = \frac{1 + \eta}{\beta(1 + \eta)^{1-\sigma}} + \delta - 1\]

\[= \frac{(1 + \theta)}{(1 + \eta)^{-\sigma}} + \delta - 1\]

\[\approx \delta + \theta + \sigma \eta\]

\[\approx \delta + \theta + \sigma (n + g)\]
Therefore we have capital per unit of effective labor in the balanced growth path:

\[
\tilde{k}^* = \left( \frac{\alpha}{\delta - 1 + (1 + \theta)(1 + \eta)^\sigma} \right)^{\frac{1}{1-\alpha}}
\]

\[
\approx \left( \frac{\alpha}{\delta + \theta + \sigma(n + g)} \right)^{\frac{1}{1-\alpha}}
\]

This generalizes the solution we had for the optimal allocation without growth.

As in the Solow model, along a balanced growth path all level variables are growing at rate \( \eta \approx n + g \).

Unlike the Solow model, the steady state depends on the household preferences, as the savings rates are determined optimally.
We can analyze the qualitative dynamics just as we did without productivity growth.

The key equations of the model are now:

\[ U'(\tilde{c}_t) = \beta (1 + \eta)^{1-\sigma} U'(\tilde{c}_{t+1})[f'(\tilde{k}_{t+1}) + 1 - \delta] \]

\[ (1 + \eta)\tilde{k}_{t+1} = (1 - \delta)\tilde{k}_t + f(\tilde{k}) - \tilde{c}_t \]

The dynamics work in much the same way, only now they depend on \( \eta \). So we can analyze the effects of a change in \( n \) or \( g \) which lead to a change in \( \eta \).

In steady state, \( \Delta \tilde{c}_{t+1} = 0 \), and

\[ f'(\tilde{k}^*) \approx \delta + \theta + \sigma \eta \]

Also in steady state \( \Delta \tilde{k}_{t+1} = 0 \), so:

\[ \tilde{c} = f(\tilde{k}) - (\delta + \eta)\tilde{k} \]
Phase Diagram of Optimal Growth Model

Phase diagram of the optimal growth model

\[ \Delta c = 0: f'(k^*) = \delta + \theta + \sigma \eta \]

\[ \Delta k = 0: f(k) - (\delta + \eta)k \]

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Effect of an Increase in $n$ or $g$

Phase diagram: An increase in the growth rate $\eta$ to $\eta'$. As before, initial effect depends on the slope of the saddle path.