Large literature documenting increase in income and wealth inequality in the US. Started in the 1970s and continues today.

At same time, large increase in the returns to education.

Average wages of college graduates from increased by 60% for males and 90% for females from 1963 to 2002.

Average wages of high school graduates only increased by 20% for males and 50% for females over same period.

Main explanation: Skill-biased technical change. Skilled and unskilled labor are effectively different labor markets. Productivity changes have increased the relative demand for skilled labor.
Figure 4  Men's Earnings by Quantiles

Index of Real Wages of Full-Time Full-Year Men
Ages 22–65 by Specific Percentiles

Index, 1961=100
Figure 5  Women's Earnings by Quantiles

Index of Real Wages of Full-Time Full-Year Women Ages 22–65 by Specific Percentiles

Index, 1981=100
Figure 6 Men's Earnings by Education

Index of Mean of Real Wages of Full-Time Full-Year Men Ages 22–65 by Education Group

Index, 1963=100
Figure 7  Women's Earnings by Education

Index of Mean of Real Wages of Full-Time Full-Year Women
Ages 22–65 by Education Group

Index, 1963=100
Extend the previous to two sectors: skilled and unskilled.

Households: Assume both skilled and unskilled workers have same preferences given by:

\[ u(c, l) = c - \frac{(h - l)^2}{2} \]

Assume skilled workers own a share \( \beta \) of the capital stock, unskilled a share \( 1 - \beta \).

Wages \( w_s \) for skilled \( w_u \) for unskilled.
Skilled household problem (unskilled parallel):

$$\max_{N_s}\{ (N_s w_s + \beta rK) - (N_s)^2 / 2 \}$$

Optimality conditions:

$$\frac{u_l}{u_c} = N_s = w_s$$

So we get:

$$N_s = w_s$$
$$N_u = w_u$$
$$c_s = w_s^2 + \beta rK$$
$$c_u = w_u^2 + (1 - \beta) rK$$
Firms: Assume representative firm hires both skilled and unskilled labor. Each has different productivity \((z_s, z_u)\).

Firm substitutes between skilled and unskilled for total labor input.

\[
N = (z_s N_s^\rho + z_u N_u^\rho)^{1/\rho}
\]

where \(0 < \rho < 1\). Thus production is:

\[
Y = K^\alpha N^{1-\alpha}
= K^\alpha (z_s N_s^\rho + z_u N_u^\rho)^{\frac{1-\alpha}{\rho}}
\]
• Firms maximize profits:

\[ K^\alpha (z_s N_s^\rho + z_u N_u^\rho) \frac{1-\alpha}{\rho} - rK - w_s N_s - w_u N_u \]

• FOC’s – each type paid its marginal product:

\[
(1 - \alpha) K^\alpha (z_s N_s^\rho + z_u N_u^\rho) \frac{1-\alpha}{\rho} - 1 z_s N_s^{\rho-1} = w_s \\
(1 - \alpha) K^\alpha (z_s N_s^\rho + z_u N_u^\rho) \frac{1-\alpha}{\rho} - 1 z_u N_u^{\rho-1} = w_u 
\]

• Divide firm FOC’s:

\[
\frac{z_s N_s^{\rho-1}}{z_u N_u^{\rho-1}} = \frac{w_s}{w_u} 
\]
Characterize Equilibrium

- Equate to relative demands to labor supplies:

\[
\frac{zs N_s^{\rho-1}}{zu N_u^{\rho-1}} = \frac{N_s}{N_u}
\]

\[
\Rightarrow \frac{N_s}{N_u} = \frac{w_s}{w_u} = \left(\frac{zs}{zu}\right)^{1-\rho}
\]

- Skill-biased technical change: \(zs\) increases faster than \(zu\).
  Implies \(\frac{w_s}{w_u}\) increases \(\Rightarrow\) inequality.
  Also implies expansion in skill sector: \(\frac{N_s}{N_u}\) increases.
Figure 3.14  The effects of skill-biased technical change on wage inequality

(a) Skilled workers

1. Skill-biased technical change occurs

2. Skilled wages rise

2. Skilled employment rises

(b) Unskilled workers

1. Skill-biased technical change occurs

2. Unskilled wages fall

2. Unskilled employment falls
What Does This Analysis Miss?

- Captures broad aggregate facts. Misses on some dimensions.

**Consumption Inequality.** Evidence that inequality in consumption was less than inequality in income (Kreuger & Perri, 2003). Here we have greater consumption inequality (since $\beta \approx 1$):

$$\frac{c_s}{c_u} = \frac{w_s^2 + \beta rK}{w_u^2 + (1 - \beta) rK} \approx \left( \frac{w_s}{w_u} \right)^2 + \frac{rK}{w_u^2}$$

- **Changes by Gender.** Most dramatic effects have been increase in female labor supply, especially in skilled labor. Hard to argue this was all from skill-biased technical change. Composition effects may be more important. (Eckstein & Nagypal, 2004)
Now start our analysis of dynamic general equilibrium models, which we will continue in the rest of the class.

Today start with optimal allocations, solving social planner problem. Later consider equilibrium analysis.

In going from static to dynamic model, the main difference is savings and investment.

Households no longer consume all income each period, save some for future consumption (or borrow against future income).

Firm no longer have fixed capital stock on hand each period, may choose to invest in order to build up future capital (or disinvest to allow future capital to fall).
Representative household lives infinite number of periods.

Utility function:

\[ V_0 = U(c_0) + \beta U(c_1) + \beta^2 U(c_2) + \beta^3 U(c_3) + \ldots = \sum_{t=0}^{\infty} \beta^t U(c_t) \]

\( c_t \) is consumption at date \( t \)

\( \beta \in (0, 1) \) **discount factor** measures household’s degree of impatience. Define \( \beta = \frac{1}{1+\theta} \), where \( \theta \) is **discount rate**

Preferences over \( \{c_0, c_1, \ldots\} \) satisfy the conditions discussed previously, i.e. monotonicity (\( U' > 0 \)) and convexity (\( U'' < 0 \)).
More on Preferences

\[ V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t) \]

- Abstract from labor/leisure tradeoff for now. Inelastic labor supply: work full time \( h \) hours, yields no utility.
- Consumption smoothing, partially offset by discounting.
- Assume all \( c_t \) are normal: more income \( \Rightarrow \) more consumption at each date \( t \).
- From vantage point of date 0, marginal utility of \( c_t \):

  \[ MU_{c_t} = \frac{\partial V_0}{\partial c_t} = \beta^t U'(c_t) \]

- Intertemporal marginal rate of substitution measures willingness to substitute consumption over time:

  \[ MRSC_{ct, ct+1} = \frac{MU_{ct}}{MU_{ct+1}} = \frac{U'(c_t)}{\beta U'(c_{t+1})} \]
Continue to abstract from labor for now. Assume $h = 1$ is supplied inelastically. Then production is:

$$y_t = F(k_t, 1) \equiv F(k_t)$$

where production function is same as before. Note since $N = 1$ fixed, diminishing marginal returns in $k$:

$$F'(k) > 0, \quad F''(k) < 0$$

For technical reasons, also assume Inada conditions:

$$\lim_{k \to 0} F'(k) = +\infty, \quad \lim_{k \to \infty} F'(k) = 0$$
Firms can now invest in order to expand future productivity.

Capital depreciates at rate $\delta$, and investment at $t$ increases $k_{t+1}$:

$$k_{t+1} = (1 - \delta)k_t + i_t$$

We abstract from government spending, so the feasibility or goods market clearing condition now includes investment and consumption:

$$y_t = c_t + i_t$$

Combining equations gives us the tradeoff between consumption and capital:

$$c_t = F(k_t) - k_{t+1} + (1 - \delta)k_t$$
In general \( \{k_t, c_t, y_t, i_t\} \) will vary over time. But let’s look for a steady state, where they are constant.

From the previous expression this implies:

\[
    c = F(k) - k + (1 - \delta)k = F(k) - \delta k
\]

In the steady state, consumption equals output minus replacement investment \( \delta k \).
Output, replacement investment, and consumption
The Golden Rule Allocation

- We now consider the social planner’s problem to determine the optimal allocation.
- We first focus on a simple objective, suppose that the planner wanted to maximize utility in the steady state. This is known as the “Golden Rule” allocation, as it treats consumption at all dates equally.

\[
\max_{c,k} U(c)
\]

subject to: \( c = F(k) - \delta k \)

- Since \( U(c) \) strictly increasing, this is equivalent to max \( c \) s.t. \( c = F(k) - \delta k \)
- First order condition determines golden rule capital \( k^\# \).

\[ F'(k^\#) = \delta \]
While the golden rule gives the maximal amount of steady state consumption, in general it is not optimal.

If households are impatient ($\beta < 1$) then they value current consumption more than future consumption. So the timing of consumption matters.

So now let’s consider the optimal allocation:

$$\max_{\{c_t, k_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to: $c_t = F(k_t) - k_{t+1} + (1 - \delta)k_t$, $\forall t, k_0$ given
Characterizing the Optimal Allocation

- Form the Lagrangian with multipliers \( \{\lambda_t\} \) on the constraints:

\[
\mathcal{L} = \max_{\{c_t, k_t\}} \sum_{t=0}^{\infty} \left( \beta^t U(c_t) + \lambda_t [F(k_t) - k_{t+1} + (1 - \delta)k_t - c_t] \right)
\]

- First order conditions for any \( c_t \), and for \( k_{t+1}, t > 0 \):

\[
\beta^t U'(c_t) = \lambda_t
\]

\[
- \lambda_t + \lambda_{t+1} [F'(k_{t+1}) + 1 - \delta] = 0.
\]

- Note that if there were a finite terminal date \( T \), we would have \( k_{T+1} = 0 \). Consume everything in last date.

- For infinite horizon problem need a similar condition known as **transversality** condition:

\[
\lim_{T \to \infty} \beta^T U'(c_T)k_{T+1} = 0
\]

Value in utility terms of capital goes to zero at infinity.
The Euler Equation

- Combine the two optimality conditions to get:

  \[ U'(c_t) = \beta U'(c_{t+1})[F'(k_{t+1}) + 1 - \delta] \]

- This is known as an **Euler equation** and is a key condition for optimality in dynamic models.

- Can also be written:

  \[ MRS_{c_t, c_{t+1}} = \frac{U'(c_t)}{\beta U'(c_{t+1})} = F'(k_{t+1}) + 1 - \delta \]

- Here \( F'(k_{t+1}) + 1 - \delta \) is the slope of the production possibility frontier for \( c_t, c_{t+1} \). To see this note:

  \[
  c_{t+1} = F(k_{t+1}) - k_{t+2} + (1 - \delta)k_{t+1} \\
  = F(F(k_t) - c_t + (1 - \delta)k_t) - k_{t+2} + (1 - \delta)[F(k_t) - c_t + (1 - \delta)k_t]
  \]

- Take derivative with respect to \( c_t \): \( F'(k_{t+1}) + 1 - \delta \)
Optimal choice

$c_{t+1}$

$c^*_{t+1}$

$c^*_t$

$1 + r_{t+1}$

$U(c_t) + b U(c_t=1)$

Optimal choice
More on The Euler Equation

- Can also interpret \( F'(k_{t+1}) - \delta \) as holding period return \( r_{t+1} \) on capital.

\[
U'(c_t) = \beta U'(c_{t+1})[1 + r_{t+1}]
\]

- Recall that \( U \) is concave, so \( U'' < 0 \) or in other words \( U'(c) \) is decreasing.

- So if:

\[
\begin{align*}
\beta(1 + r_{t+1}) &> 1, \quad U'(c_t) > U'(c_{t+1}), \Rightarrow c_t < c_{t+1} \\
\beta(1 + r_{t+1}) &< 1, \quad U'(c_t) < U'(c_{t+1}), \Rightarrow c_t > c_{t+1} \\
\beta(1 + r_{t+1}) &= 1, \quad U'(c_t) = U'(c_{t+1}), \Rightarrow c_t = c_{t+1}
\end{align*}
\]

- Behavior of consumption over time depends on rate of time preference relative to interest rate.

- If equal, perfect consumption smoothing.
Look for a steady state of the optimal allocation.

\[ U'(c^*) = \beta U'(c^*)[F''(k^*) + 1 - \delta] \]

Or, recalling that \( \beta = 1/(1 + \theta) \):

\[ F'(k^*) = \frac{1}{\beta} + \delta - 1 = \delta + \theta \]

From the previous expression we also have:

\[ c^* = F(k^*) - \delta k^* \]

The optimal steady state is only equal to the golden rule if \( \theta = 0 \). And since \( F''(k) < 0 \) we have:

\[ F'(k^\#) = \delta < \delta + \theta = F'(k^*), \Rightarrow k^\# > k^* \]
Optimal steady state consumption and capital

$F(k) \quad dK$

$y$ $k$

$c^* + dk^*$ $c^* + dk^*$

$k^*$ $k^*$

Williams Economics 312/702
F(k)-dk

Optimal steady state and golden rule

$\text{Optimal steady state and golden rule}$