Lecture 11
Consumption under Uncertainty
Asset Pricing

Noah Williams

University of Wisconsin - Madison

Economics 702
Spring 2020

Consumption-Savings Under Uncertainty

- Now x_{t+1}, r_{t+1} are random, unknown at t.
- Agents form expectations of future income, maximize expected utility.
- Can derive an Euler equation of the same form, but now must have expectations over c_{t+1} and r_{t+1}:

$$u'(c_t) = \beta E_t \left[u'(c_{t+1})(1 + r_{t+1}) \right]$$

- Here $E_t(\cdot)$ represents the agent's expectations, conditional on all information available at date t.

Consumption-Savings Under Uncertainty: Hall (1978)

- Suppose again that $r_t = r$ and $\beta(1+r) = 1$, so the Euler equation is:

$$u'(c_t) = E_t u'(c_{t+1})$$

- Also suppose that agents have quadratic preferences, where $a > 0$ is a constant:

$$u(c_t) = c_t - \frac{a}{2} c_t^2,$$

- So $u'(c_t) = 1 - ac_t$ and the Euler equation becomes:

$$c_t = E_t c_{t+1}$$

- Also by the law of iterated expectations:

$$c_t = E_t c_{t+1} = E_t(E_{t+1} c_{t+2}) = E_t c_{t+2}$$

Implications

- With these preferences consumption is a random walk:

$$c_{t+1} = c_t + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0$$

- The best predictor of consumption one period ahead is current consumption. No other variables which are known at date t help predict consumption at $t+1$.
- To express this another way, note that the present value budget constraint holds for any date t:

$$\sum_{s=0}^{\infty} \frac{E_t c_{t+s}}{(1+r)^s} = \sum_{s=0}^{\infty} \frac{E_t x_{t+s}}{(1+r)^s} + a_t(1+r)$$

Permanent Income Theory Example

- Then note that $E_t c_{t+s} = c_t$ for all s. So then we have:

$$\begin{aligned} c_t \sum_{s=0}^{\infty} \frac{1}{(1+r)^s} &= \sum_{s=0}^{\infty} \frac{E_t x_{t+s}}{(1+r)^s} + a_t(1+r) \\ c_t &= \frac{r}{1+r} \sum_{s=0}^{\infty} \frac{E_t x_{t+s}}{(1+r)^s} + r a_t \end{aligned}$$

  Consumption depends on expectations of all future income.

- Changes in consumption over time are driven by changes in expectations of future income. Information revealed about future income is the driver of consumption.

$$c_{t-1} = E_{t-1} c_t = \frac{r}{1+r} \sum_{s=0}^{\infty} \frac{E_{t-1} x_{t+s}}{(1+r)^s} + r a_t$$

Permanent and Transitory Shocks

- A pure transitory income shock reveals at date t that $x_t > E_{t-1}x_t$ is higher than anticipated, but $E_t x_{t+s}$ is unaffected for $s \geq 1$. Example: $x_t = x_{t-1} + v_t$, $x_{t+s} = x_{t-1}$.

$$c_t = c_{t-1} + \frac{r}{1+r} v_t$$

- A permanent income shock reveals at date t that $x_t > E_{t-1}x_t$ is higher than anticipated, and $E_t x_{t+s}$ is also higher for $s \geq 1$. Example: $x_{t+s} = x_{t-1} + \Delta$

$$c_t = c_{t-1} + \Delta$$

Extensions of Permanent Income Theory

- With quadratic utility, uncertainty in income does not affect decisions:

$$c_t = \frac{r}{1+r} \sum_{s=0}^{\infty} \frac{E_t x_{t+s}}{(1+r)^s} + r a_t$$

- This is a property known as **certainty equivalence**. Decisions are the same as if x_t took on its expected value with certainty.
- With more general preferences, variability of income would matter.
- Suppose again that $r_t = r$ and $\beta(1+r) = 1$, so the Euler equation is:

$$u'(c_t) = E_t u'(c_{t+1})$$

- If $u'(c)$ is convex ($u'''(c) > 0$), then more uncertain income will lead to lower consumption today, more savings: **precautionary savings**

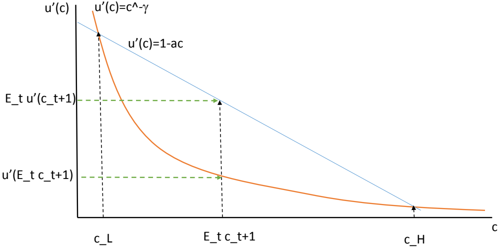
Precautionary Savings

- Quadratic utility: $u'(c) = 1 - ac$, $u''(c) = -a < 0$, $u'''(c) = 0$.

$$u'(c_t) = E_t u'(c_{t+1}) \Rightarrow c_t = E_t c_{t+1}$$

- Power utility: $u'(c) = c^{-\gamma}$, $u''(c) = -\gamma c^{-\gamma-1} < 0$, $u'''(c) = -\gamma(-\gamma - 1)c^{-\gamma-2} > 0$.

$$u'(c_t) = E_t u'(c_{t+1}) > u'(E_t c_{t+1}) \Rightarrow c_t < E_t c_{t+1}$$



Implications for Consumption

- Uncertainty about future income will lead to more savings, to allow households to smooth potential consumption fluctuations.
- Periods of increased uncertainty will be characterized by reductions in household consumption.
- Another complication we've abstracted from is borrowing constraints. These affect consumption in two ways:
  1. When household is constrained, consumption will closely follow income. Unable to smooth.
  2. Household will build up stock of assets to diminish the impact of the constraint.
- There is significant micro evidence for these effects on household consumption. Macro effects are less clear.

Asset Pricing

- We have thought about Euler equation as determining consumption given interest rates. But we can also use it to determine rates of return and so **asset prices** given consumption.

$$u'(c_t) = \beta E_t \left[u'(c_{t+1})(1 + r_{t+1}) \right]$$

- Lucas (1978) looked at endowment economy model, so (aggregate) consumption was given exogenously, prices determined endogenously in equilibrium.
- Generalization of Euler equation is the pricing relation for an asset with price p_t stochastic payoff x_{t+1} next period:

$$\begin{aligned} p_t &= E_t \left[\frac{\beta u'(c_{t+1})}{u'(c_t)} x_{t+1} \right] \\ &= E_t(m_{t+1} x_{t+1}) \end{aligned}$$

- A return has price 1, payoff $R_{t+1} = 1 + r_{t+1}$, i.e. $R_{t+1} = \frac{p_{t+1} + d_{t+1}}{p_t}$.

Risk and Asset Prices

- **Risk Neutrality**:
  With linear utility $u'(c_t)$ constant, so risk free rate:

$$1 = E_t(\beta R) \quad \Rightarrow R = \frac{1}{\beta}$$

- So then for a stock which pays future dividends $\{d_{t+j}\}$:

$$p_t = E_t\left[\sum_{j=1}^{\infty} \beta^j d_{t+j}\right] = E_t\left[\sum_{j=1}^{\infty} \frac{d_{t+j}}{R^j}\right]$$

- **Risk Corrections**:
  Risk free rate:

$$1 = E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)} R\right] \quad \Rightarrow R = \frac{1}{E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)}\right]}$$

or $R = 1/E_t m_{t+1}$.

Risk and Asset Prices

- For general payoff x_{t+1},

$$
\begin{aligned}
p_t &= E_t(m_{t+1}x_{t+1}) \\
&= E_t m_{t+1} E_t x_{t+1} + cov_t(m_{t+1}, x_{t+1}) \\
&= \frac{E_t x_{t+1}}{R} + cov_t(m_{t+1}, x_{t+1}) \\
&= \frac{E_t x_{t+1}}{R} + cov_t\left(\frac{\beta u'(c_{t+1})}{u'(c_t)}, x_{t+1}\right) \\
&= \frac{E_t x_{t+1}}{R} + \frac{\beta}{u'(c_t)} cov_t(u'(c_{t+1}), x_{t+1})
\end{aligned}
$$

- The riskiness of a payoff only affects prices to the extent the risk is correlated with consumption.
- Assets that pay more when marginal utility is high (consumption is low) command higher prices.

Power Utility and Risk-Free Rates

Now assume $u(c) = c^{1-\gamma}/(1-\gamma)$
Risk-free rate when c_{t+1} known:

$$R = \frac{1}{E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]} = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma}$$

Define $r^f = R - 1$, $\beta = \frac{1}{1+\theta}$, then (net) stock return r_{t+1} satisfies:

$$1 = E_t\left[\frac{1}{1+\theta}(1+\Delta c_{t+1})^{-\gamma}(1+r_{t+1})\right]$$

Take 2nd order Taylor approximation of right side, unconditional expectations:

$$E(r) = \theta + \gamma E(\Delta c_t) + \gamma cov(r_t, \Delta c_t) - \frac{1}{2}\gamma(\gamma+1)\sigma^2(\Delta c_t)$$

Power Utility and the Equity Premium

$$E(r) = \theta + \gamma E(\Delta c_t) + \gamma cov(r_t, \Delta c_t) - \frac{1}{2}\gamma(\gamma+1)\sigma^2(\Delta c_t)$$

For risk free rate $cov(r_t, \Delta c_t) = 0$ so:

$$r^f = \theta + \gamma E(\Delta c_t) - \frac{1}{2}\gamma(\gamma+1)\sigma^2(\Delta c_t)$$

So excess return on risk assets can be written:

$$\frac{E(r_t) - r^f}{\sigma(r)} = \gamma\sigma(\Delta c_t) corr(\Delta c_t, r_t)$$

Left side known as Sharpe ratio

Attempted Resolutions of Equity Premium

- Consumption based model fails empirically in explaining premium on stocks vs. bonds.
- Change preferences: recursive preferences, ambiguity/robustness, habit persistence
- Change constraints: Limited participation, transaction costs, incomplete markets
- Change shocks: disaster models, long-run risk, learning