Lecture 11
Consumption under Uncertainty
Asset Pricing

Noah Williams

University of Wisconsin - Madison

Economics 702
Spring 2020
Now $x_{t+1}, r_{t+1}$ are random, unknown at $t$.

Agents form **expectations** of future income, maximize expected utility.

Can derive an Euler equation of the same form, but now must have expectations over $c_{t+1}$ and $r_{t+1}$:

$$u'(c_t) = \beta E_t [u'(c_{t+1})(1 + r_{t+1})]$$

Here $E_t(\cdot)$ represents the agent’s expectations, conditional on all information available at date $t$. 

Suppose again that $r_t = r$ and $\beta(1 + r) = 1$, so the Euler equation is:

$$u'(c_t) = E_t u'(c_{t+1})$$

Also suppose that agents have quadratic preferences, where $a > 0$ is a constant:

$$u(c_t) = c_t - \frac{a}{2} c_t^2,$$

So $u'(c_t) = 1 - ac_t$ and the Euler equation becomes:

$$c_t = E_t c_{t+1}$$

Also by the law of iterated expectations:

$$c_t = E_t c_{t+1} = E_t (E_{t+1} c_{t+2}) = E_t c_{t+2}$$
Implications

- With these preferences consumption is a random walk:

\[ c_{t+1} = c_t + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0 \]

- The best predictor of consumption one period ahead is current consumption. No other variables which are known at date \( t \) help predict consumption at \( t + 1 \).

- To express this another way, note that the present value budget constraint holds for any date \( t \):

\[
\sum_{s=0}^{\infty} \frac{E_t c_{t+s}}{(1 + r)^s} = \sum_{s=0}^{\infty} \frac{E_t x_{t+s}}{(1 + r)^s} + a_t(1 + r)
\]
Then note that $E_t c_{t+s} = c_t$ for all $s$. So then we have:

$$
ct \sum_{s=0}^{\infty} \frac{1}{(1 + r)^s} = \sum_{s=0}^{\infty} \frac{E_t x_{t+s}}{(1 + r)^s} + a_t(1 + r)
$$

$$
c_t = \frac{r}{1 + r} \sum_{s=0}^{\infty} \frac{E_t x_{t+s}}{(1 + r)^s} + ra_t
$$

Consumption depends on expectations of all future income.

Changes in consumption over time are driven by changes in expectations of future income. Information revealed about future income is the driver of consumption.

$$
c_{t-1} = E_{t-1} c_t = \frac{r}{1 + r} \sum_{s=0}^{\infty} \frac{E_{t-1} x_{t+s}}{(1 + r)^s} + ra_t
$$
A pure transitory income shock reveals at date $t$ that $x_t > E_{t-1}x_t$ is higher than anticipated, but $E_t x_{t+s}$ is unaffected for $s \geq 1$. Example: $x_t = x_{t-1} + v_t$, $x_{t+s} = x_{t-1}$.

$$c_t = c_{t-1} + \frac{r}{1 + r} v_t$$

A permanent income shock reveals at date $t$ that $x_t > E_{t-1}x_t$ is higher than anticipated, and $E_t x_{t+s}$ is also higher for $s \geq 1$. Example: $x_{t+s} = x_{t-1} + \Delta$

$$c_t = c_{t-1} + \Delta$$
With quadratic utility, uncertainty in income does not affect decisions:

$$c_t = \frac{r}{1 + r} \sum_{s=0}^{\infty} \frac{E_t x_{t+s}}{(1 + r)^s} + ra_t$$

This is a property known as certainty equivalence. Decisions are the same as if $x_t$ took on its expected value with certainty.

With more general preferences, variability of income would matter.

Suppose again that $r_t = r$ and $\beta(1 + r) = 1$, so the Euler equation is:

$$u'(c_t) = E_t u'(c_{t+1})$$

If $u'(c)$ is convex ($u'''(c) > 0$), then more uncertain income will lead to lower consumption today, more savings: precautionary savings.
Precautionary Savings

- Quadratic utility: $u'(c) = 1 - ac, \ u''(c) = -a < 0, \ u'''(c) = 0.$

  $$u'(c_t) = E_t u'(c_{t+1}) \Rightarrow c_t = E_t c_{t+1}$$

- Power utility: $u'(c) = c^{-\gamma}, \ u''(c) = -\gamma c^{-\gamma - 1} < 0, \ u'''(c) = -\gamma(-\gamma - 1)c^{-\gamma - 2} > 0.$

  $$u'(c_t) = E_t u'(c_{t+1}) > u'(E_t c_{t+1}) \Rightarrow c_t < E_t c_{t+1}$$
Implications for Consumption

- Uncertainty about future income will lead to more savings, to allow households to smooth potential consumption fluctuations.

- Periods of increased uncertainty will be characterized by reductions in household consumption.

- Another complication we’ve abstracted from is borrowing constraints. These affect consumption in two ways:
  1. When household is constrained, consumption will closely follow income. Unable to smooth.
  2. Household will build up stock of assets to diminish the impact of the constraint.

- There is significant micro evidence for these effects on household consumption. Macro effects are less clear.
We have thought about Euler equation as determining consumption given interest rates. But we can also use it to determine rates of return and so asset prices given consumption.

\[ u'(c_t) = \beta E_t [u'(c_{t+1})(1 + r_{t+1})] \]

Lucas (1978) looked at endowment economy model, so (aggregate) consumption was given exogenously, prices determined endogenously in equilibrium.

Generalization of Euler equation is the pricing relation for an asset with price \( p_t \) stochastic payoff \( x_{t+1} \) next period:

\[ p_t = E_t \left[ \frac{\beta u'(c_{t+1})}{u'(c_t)} x_{t+1} \right] = E_t (m_{t+1} x_{t+1}) \]

A return has price 1, payoff \( R_{t+1} = 1 + r_{t+1} \), i.e.

\[ R_{t+1} = \frac{p_{t+1} + d_{t+1}}{p_t} \]
Risk and Asset Prices

- **Risk Neutrality:**
  With linear utility $u'(c_t)$ constant, so risk free rate:
  \[ 1 = E_t(\beta R) \implies R = \frac{1}{\beta} \]

- So then for a stock which pays future dividends $\{d_{t+j}\}$:
  \[ p_t = E_t \left[ \sum_{j=1}^{\infty} \beta^j d_{t+j} \right] = E_t \left[ \sum_{j=1}^{\infty} \frac{d_{t+j}}{R^j} \right] \]

- **Risk Corrections:**
  Risk free rate:
  \[ 1 = E_t \left[ \frac{\beta u'(c_{t+1})}{u'(c_t)} R \right] \implies R = \frac{1}{E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right]} \]
  or $R = 1/E_t m_{t+1}$. 

Williams Economics 702
For general payoff $x_{t+1}$,

$$p_t = E_t(m_{t+1}x_{t+1}) = E_t m_{t+1} E_t x_{t+1} + \text{cov}_t (m_{t+1}, x_{t+1})$$

$$= \frac{E_t x_{t+1}}{R} + \text{cov}_t (m_{t+1}, x_{t+1})$$

$$= \frac{E_t x_{t+1}}{R} + \text{cov}_t \left( \frac{\beta u'(c_{t+1})}{u'(c_t)}, x_{t+1} \right)$$

$$= \frac{E_t x_{t+1}}{R} + \frac{\beta}{u'(c_t)} \text{cov}_t (u'(c_{t+1}), x_{t+1})$$

- The riskiness of a payoff only affects prices to the extent the risk is correlated with consumption.
- Assets that pay more when marginal utility is high (consumption is low) command higher prices.
Now assume $u(c) = c^{1-\gamma}/(1 - \gamma)$

Risk-free rate when $c_{t+1}$ known:

$$R = \frac{1}{E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]} = \frac{1}{\beta} \left( \frac{c_{t+1}}{c_t} \right)^\gamma$$

Define $r^f = R - 1$, $\beta = \frac{1}{1+\theta}$, then (net) stock return $r_{t+1}$ satisfies:

$$1 = E_t \left[ \frac{1}{1 + \theta} (1 + \Delta c_{t+1})^{-\gamma} (1 + r_{t+1}) \right]$$

Take 2nd order Taylor approximation of right side, unconditional expectations:

$$E(r) = \theta + \gamma E(\Delta c_t) + \gamma \text{cov}(r_t, \Delta c_t) - \frac{1}{2} \gamma (\gamma + 1) \sigma^2(\Delta c_t)$$
\[ E(r) = \theta + \gamma E(\Delta c_t) + \gamma \text{cov}(r_t, \Delta c_t) - \frac{1}{2} \gamma (\gamma + 1) \sigma^2(\Delta c_t) \]

For risk free rate \( \text{cov}(r_t, \Delta c_t) = 0 \) so:

\[ r^f = \theta + \gamma E(\Delta c_t) - \frac{1}{2} \gamma (\gamma + 1) \sigma^2(\Delta c_t) \]

So excess return on risk assets can be written:

\[ \frac{E(r_t) - r^f}{\sigma(r)} = \gamma \sigma(\Delta c_t) \text{corr}(\Delta c_t, r_t) \]

Left side known as Sharpe ratio
At tempted Resolutions of Equity Premium

- Consumption based model fails empirically in explaining premium on stocks vs. bonds.
- Change preferences: recursive preferences, ambiguity/robustness, habit persistence
- Change constraints: Limited participation, transaction costs, incomplete markets
- Change shocks: disaster models, long-run risk, learning