Endogenous Market Power*

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Abstract

In this paper we develop a framework to study markets with heterogeneous atomic traders. The competitive model is augmented as we provide traders with correct beliefs about their price impacts to define equilibrium with endogenously determined market power and show that such equilibrium exists in economies with smooth utility and cost functions and is generically determinate. Traders’ price impacts depend positively on the convexity of preferences or cost functions of the trading partners and are subject to mutual reinforcement. Compared to the competitive model, the volume of trade is reduced, and hence is Pareto inefficient. The price effects of non-competitive trading depend on the convexity of marginal utility or cost function.

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The problem of the exchange of goods among rational traders lies at the heart of economics. The study of this problem, originating with the work of Walras and refined by Fisher, Hicks, Samuelson, Arrow, and Debreu, provides a well-established methodology for analyzing market interactions in an exchange economy. The central concept of this approach is competitive equilibrium, wherein agents assume that individual trades cannot affect prices. Price-taking behavior is justified through the argument that the economy is so large that each individual trader is negligible and hence has no impact on price. Such an argument, however, cannot be applied to markets with atomic buyers and sellers who, have a non-negligible price impact. In this paper we modify the competitive framework by endowing all traders with correct beliefs about slopes of their market supplies/demands, called price impacts. This allows agents to respond optimally to market conditions in equilibrium. Since traders take the slopes of their market supplies/demands as given, we call them slope-takers.

The outcome of market interactions among slope-takers can be summarized as follows. Equilibrium price impacts increase with the convexity of preferences or cost functions of the trading

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partners and decrease with the number of traders. The volume of trade is smaller than the predicted by a competitive model. The price effect depends on the types of traders’ preferences. In a pure exchange economy the sign of the price bias, relative to the competitive model, depends on the convexity of marginal utility function. Finally, in large economies, endogenous price impacts are negligible, and slope-taking equilibrium converges to a competitive one as the number of traders increases to infinity.

The slope-taking, as discussed in this paper, naturally extends the competitive framework to markets with atomic traders, and it retains the following desirable properties of the Walrasian model. Slope-taking equilibrium exists and is generically locally unique under general conditions. Moreover, traders can derive information required to “play” a slope-taking equilibrium from the data available in anonymous markets (see [16]). The model is also tractable: In [5] we extend the theory of complete financial markets to economies with slope-takers.

The framework from this paper bridges the general equilibrium theory with the strategic literature based on Nash in Supply Functions games (see [6] and [11]) and also with conjectural variation literature (see [2] and [13]). We provide a detailed comparison of the slope-taking equilibrium with these two strands of literature in Section 4. The paper also contributes to the literature on market power in the general equilibrium framework that started in [12] and was followed by [1], [7], [8], [9], and [10]. These are highly complex models with equilibria that are not determinate.

1 Equilibrium with Slope-Takers

1.1 Motivating Example

We first explain the concept of a slope-taking equilibrium\(^1\) using a numerical example of a pure exchange economy, with \(I\) consumers characterized by identical quasilinear utility function, \(U_i(x_i, m_i) = x_i - 0.5v(x_i)^2 + m_i\), where \(v > 0\). The non-numéraire endowment of \(I/2\) consumers, called sellers, is given by \(e_i = 2\), while the endowment of the remaining \(I/2\) traders, referred to as buyers, is normalized to zero \(e_i = 0\).

The canonical model of market interactions, a competitive equilibrium, is based on the premise that the mechanism through which markets allocate resources is accurately captured by a Walrasian auction, in which traders choose demand functions, then a (nonstrategic) Walrasian auctioneer determines a market clearing price and trade occurs at this price.\(^2\) In economies with non-atomic traders, individual choices have no impact on the market clearing price \(\bar{p}\), and for each trader \(i\), it is individually optimal to use competitive demand function \(t_{Walras}^i(p) \equiv \max t_i U_i(e_i + t_i - pt_i)\). Demands \(t_{Walras}^i(\cdot) \equiv \{t_{Walras}^i(\cdot)\}_i\) when aggregated by the Walrasian auctioneer result in a com-

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1. The slope-taking equilibrium is defined using general equilibrium language. Rostek and Weretka [15] (Lemma 1) reformulate the notion of slope-taking equilibrium in strategic terms (which applies to the general setting from this paper) and allows a direct comparison between the slope-taking equilibrium and Nash in Supplies, and to show how the slope-taking approach refines the set of Nash equilibria in a Walrasian auction. In the current version of the paper we use some of these insights.

2. Alternatively, the competitive equilibrium is rationalized by a dynamic version of a Walrasian auction, called a tâtonnement process, see [16]. Our argument applies to a dynamic Walrasian auction as well.
petitive equilibrium, in which the market clearing price is given by \(\tilde{p}^{Walras} = 1 - v\), the non-numèraire consumption \(\tilde{x}_i^{Walras} \equiv e_i + t_i^{Walras} = 1\) is the same for all traders, and allocation is Pareto efficient.

Competitive theory is not suitable for application to economies with a finite number of traders. In a Walrasian auction, for each buyer \(i\), demands \(\{t_i^{Walras}(\cdot)\}_{i' \neq i}\) define imperfectly elastic market supply with slope \(M_i = v/(I - 1)\).\(^3\) On such a supply, \(t_i^{Walras}(\cdot)\) selects a price-trade point for which marginal utility is equal to price, a suboptimal outcome for a monopsony. By a symmetric argument, \(t_i^{Walras}(\cdot)\) is suboptimal for a seller, and \(t^{Walras}(\cdot)\) is not a Nash equilibrium. The framework from this paper corrects the price-taking “error” of atomic traders by endowing buyers and sellers with correct beliefs about their price impacts, thus allowing traders to use optimal demands in equilibrium. More specifically, to act optimally, each trader \(i\) estimates price impact \(\bar{M}_i\) from \(p_i(t_i) = \alpha + M_i t_i + \varepsilon\), and determines trading strategies accordingly. The collection of all optimal price-trade points for different values of \(\varepsilon\) defines a slope-taker demand \(\tilde{t}_i(p, \bar{M}_i) = (1 - ve_i - p)/(\bar{M}_i + v)\).\(^4\) Since \(\tilde{t}_i(\cdot, 0) = t_i^{Walras}(\cdot)\), a slope-taker demand naturally extends the notion of a competitive demand to environments in which traders face imperfectly elastic market supplies/demands. The positive price impact makes the slope-taker demand less responsive to changes in prices, and the demand function becomes steeper than the competitive demand. A slope-taking behavior is commonly observed in financial markets in which institutional investors use market impact models to determine trading strategies (see [15]).

Demand \(\tilde{t}_i(\cdot, \bar{M}_i)\) is optimal given \(\{t_{i'}(\cdot, M_{i'})\}_{i' \neq i}\); if price impact \(\bar{M}_i\) reflects the slope of the market supply/demand resulting from aggregation of other traders’ demand functions by a Walrasian auctioneer. In the symmetric equilibrium (\(M_i = M_{i'}\) for all \(i, i'\)), such slope is given by \((v + \bar{M}_i)/(I - 1)\). Consequently, \(\{\tilde{t}_i(\cdot, \bar{M}_i)\}_{i}\) are mutual best responses, if \(\bar{M}_i = v/(I - 2)\) for which slope-takers’ demands become

\[
\tilde{t}_i(p, \bar{M}_i) = \frac{I - 2}{(I - 1)v} (1 - ve_i - p).
\] (1)

We call the outcome of a Walrasian auction with demands (1), given by price \(\bar{p}\) trades \(\bar{t} = \{\tilde{t}_i\}_i\) and price impacts \(\bar{M} = \{\bar{M}_i\}_i\), a slope-taking equilibrium.

The slope-taking equilibrium differs qualitatively from its perfectly competitive counterpart (see Figure 1). Endogenously determined price impacts are non-negligible for the buyers and the sellers. Given the upward-sloping market supply (downward-sloping market demand), a set of all affordable combinations of \((t_i, m_i)\), a budget set of seller (buyer) \(i\), is strictly convex. At equilibrium, vectors normal to the boundary of budget sets are determined by marginal payments for non-numèraire and numèraire commodity \((\bar{p} + \bar{M}_i \tilde{t}_i, 1)\). Consequently, the slopes of budget sets and hence marginal rates of substitutions differ for two types of traders; equilibrium consumption \(\bar{x}_i = \frac{I - 2}{I - 1} + (1 - \frac{I - 2}{I - 1})e_i\)

\(^3\)Given demands \(\{t_{i'}(\cdot)\}_{i' \neq i}\), market supply of trader \(i\) is obtained by solving market clearing condition \(t_i + \sum_{i' \neq i} t_{i'}(p) = 0\) for price. Since competitive demand of each trader has slope \(1/v\), with \(I - 1\) trading partners, the slope of the market supply is \(v/(I - 1)\).

\(^4\)The slope-taker demand can be determined from \(1 - v (t_i + e_i) = p + M_i t_i\).
is smaller for the buyers than for the sellers, and allocation is Pareto inefficient. Finally, in the considered example, equilibrium price $\bar{p}$ coincides with the one from the competitive model.

With non-quadratic utilities $U_i(\cdot, \cdot)$, the key ingredient of price impact, the convexity of preferences $\bar{v}_i \equiv \partial^2 U_i(\bar{x}_i, \bar{m}_i)/\partial x_i$ depends on the equilibrium allocation. Allocation $\{\bar{x}_i\}_i$, convexity $\{\bar{v}_i\}_i$, and price impacts $\{\bar{M}_i\}_i$ are interdependent and are jointly determined as a nontrivial fixed point of the equilibrium conditions. In the remainder of the paper, we demonstrate that a slope-taking equilibrium is well defined in a general economy with asymmetric consumers and produces. We also characterize equilibrium properties.

### 1.2 Economy and Equilibrium

We focus on an economy $\mathcal{E}$, defined as follows. $\mathcal{E}$ is a one-period economy with a numéraire and non-numéraire, with two types of traders, $I$ consumers and $J$ producers. $\mathcal{I}$ is the set of all consumers, and $\mathcal{J}$ is a set of firms. Subscript $i \in \mathcal{I}$, for example, $x_i$, indicates that the variable refers to the $i^{th}$ consumer, and $j$ subscript denotes variables of $j^{th}$ firm. In particular, $x_i \in \mathbb{R}_+$ ($e_i \in \mathbb{R}_{++}$) is a consumption (endowment) by the $i^{th}$ consumer, and $x \in \mathbb{R}_+^I$ ($e \in \mathbb{R}_+^J$) is an allocation (initial allocation) of goods across consumers. Consumers’ endowments of numéraire are without loss of generality normalized to zero. For consumer $i$, a trade is defined as a net demand $t_i \equiv x_i - e_i$, and for firm $j$, a trade is a negative of a supply $t_j = -y_j$. The set of all traders in the economy is denoted by $\mathcal{N} \equiv \mathcal{I} \cup \mathcal{J}$, the number of traders is $N = I + J$, and the typical trader is indexed by $n = i, j$.

Consumer $i$ maximizes a quasi-linear utility function $U_i(x_i, m_i) = u_i(x_i) + m_i$, where the non-numéraire component $u_i : \mathbb{R}_+ \to \mathbb{R}$ is $C^2$, is strictly monotone, is strictly concave, and satisfies the Inada condition for all $i \in \mathcal{I}$.

Array $u = \{u_i\}_i$ specifies utility functions for all consumers. Firms produce the non-numéraire commodity, using numéraire as an input. Cost functions $c_j(\cdot)$ are assumed to be $C^2$, strictly monotone and strictly convex. We also assume zero inaction cost ($c_j(0) = 0$) and interiority $\lim_{y \to 0} c_j'(y) = 0$ and $0 < \lim_{y \to 0} c_j''(y) < \infty$. Array $c = \{c_j\}_j$ specifies cost functions for all $j \in \mathcal{J}$. By assumption, any economy $\mathcal{E} = (u, e, c)$ has two or more traders, $N \geq 2$, with at least one consumer, $I \geq 1$. An economy $\mathcal{E}$ that satisfies our assumptions is referred to as smooth.

We now define a slope-taking equilibrium $(\bar{p}, \bar{t}, \bar{M})$ in a smooth economy. For any $p, M_n > 0$, a slope-taker’s demand, $\bar{t}_n(p, M_n)$, assigns a trade $\bar{t}_n = \bar{t}_n(p, M_n)$ that solves $u_n'(e_n + \bar{t}_n) = p + M_n \bar{t}_n$ for a consumer and $c_n'(\bar{t}_n) = p + M_n \bar{t}_n$ for a producer. Demand $\bar{t}_n(\cdot, M_n)$ gives optimal trades of agent $n$ for different prices $p$ given price impact $M_n$. In a Walrasian auction, demand functions $\{\bar{t}_n(\cdot, M'_n)\}_{n' \neq n}$ define a market supply/demand for trader $n$. The slope of the supply gives the equilibrium price impact of trader $n$ and can be determined by the implicit function theorem applied to the market clearing condition $t_n + \sum_{n' \neq n} \bar{t}_n'(p, M'_n) = 0$.

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5 The results from this paper require weaker than Inada condition $\lim_{x \to 0} u_i'(x) > u_i'(e_i)$ for all $i$, which permits economies with quadratic utility functions as considered in the example from Section 1.1.
Definition 1. A vector \((\bar{p}, \bar{t}, \bar{M})\) is a slope-taking equilibrium if:
1. Markets clear, \(\sum_n \bar{t}_n = 0\);
2. For any \(n\), trade \(\bar{t}_n\) maximizes a utility or profit, given price and price impact \(\bar{t}_n = t_n(\bar{p}, \bar{M}_n)\);
3. For any \(n\), price impact \(\bar{M}_n\) satisfies
   \[
   \bar{M}_n = - \left( \sum_{n' \neq n} \left[ \frac{\partial t_{n'}(\bar{p}, \bar{M}_{n'})}{\partial p} \right] \right)^{-1}.
   \] (2)

The last condition assures that the equilibrium price impact estimates are correct, given the slope-taking demands of other traders. Slope-taking equilibrium maintains the key principles of the competitive theory, namely, optimization and market clearing. The only departure from the competitive equilibrium model is that, instead of \textit{a priori} defining traders’ price impacts to be equal to zero, \(\bar{M} \equiv 0\), the equilibrium price impacts reflect the true slopes of market supplies/demands.

One of the advantages of a slope-taking equilibrium framework over existing noncompetitive models such as Monopoly, Monopsony, Cournot, or Monopolistic Competition is that the model does not \textit{a priori} divide the traders into those who do and those who do not have market power, because “they are large or small.” Similar to prices and trades, for each trader \(n\), price impact \(\bar{M}_n\) is determined endogenously by market clearing and optimization by all agents. This is why we call this \textit{endogenous market power}.

2. Optimal Choice and Endogenous Market Power

2.1 Consumer Choice

Consider a linear market supply \(p_{\bar{p}, \bar{t}, \bar{M}_i}(t_i) = \bar{p} + \bar{M}_i (t_i - \bar{t}_i)\) of trader \(i\), where \((\bar{p}, \bar{t}_i, \bar{M}_i)\) are arbitrary predetermined parameters. A budget set \(B(\bar{p}, \bar{t}_i, \bar{M}_i)\) of consumer \(i\), a set of all affordable trades \((t_i, m_i)\), given market supply \(p_{\bar{p}, \bar{t}_i, \bar{M}_i}(\cdot)\), is determined by a budget constraint
   \[
   p_{\bar{p}, \bar{t}_i, \bar{M}_i}(t_i) t_i + m_i \leq 0.
   \] (3)

With \(\bar{M}_i > 0\), price depends linearly on trade, and therefore, the budget constraint is \textit{quadratic} in \(t_i\) and \textit{linear} in \(m_i\); thus Figure 1 displays this pattern in the shape of parabola.

For any \(\bar{p}, \bar{M}_i > 0\), slope-taker’s demand \(\bar{t}_i(\bar{p}, \bar{M}_i)\) gives trade \(\bar{t}_i = \bar{t}_i(\bar{p}, \bar{M}_i)\), which maximizes utility function on a budget set \(B(\bar{p}, \bar{t}_i, \bar{M}_i)\). Since budget set itself depends on \(\bar{t}_i\), the trade is not simply an optimal trade given \(\bar{p}\) and \(\bar{M}_i\). Rather, it is an optimal response to the observed triple \((\bar{p}, \bar{t}_i, \bar{M}_i)\), as all three elements are needed to determine market supply \(p_{\bar{p}, \bar{t}_i, \bar{M}_i}(\cdot)\) and hence \(B(\bar{p}, \bar{t}_i, \bar{M}_i)\). Thus mathematically, trade \(\bar{t}_i\) and corresponding \(\bar{m}_i = -\bar{p}\bar{t}_i\) is a fixed point of a mapping given by
   \[
   \Gamma(\bar{t}_i, \bar{m}_i) \equiv \arg \max_{(t_i, m_i) \in B(\bar{p}, \bar{t}_i, \bar{M}_i)} U(t_i + e_i, m_i).
   \] (4)

and neither the existence nor uniqueness of \(\bar{t}_i\) is guaranteed by the standard arguments that use
the Weierstrass theorem and strict concavity of preferences. The next lemma establishes the two properties of trade \( \bar{t}_i \) in a smooth economy.

**Lemma 1.** For any \( \bar{p}, \bar{M}_i > 0 \), map \( \Gamma(\cdot, \cdot) \) has a unique fixed point.

The necessary condition for interior \( \bar{t}_i \) to be optimal on \( B(\bar{p}, \bar{t}_i, \bar{M}_i) \) is the tangency of a budget set with an upper contour set of the utility function at \( (\bar{t}_i, \bar{M}_i) \). Since the marginal utility of numéraire and its price are always equal to one, the tangency condition is equivalent to \( u'_i(\bar{t}_i + e_i) = \bar{p} + \bar{M}_i \bar{t}_i \). Lemma 1 implies that, for any \( p, M_i \), such an equation has a unique solution, and hence, there exists a function \( \bar{t}_i(\cdot, \cdot) \) mapping positive prices and price impacts into trades. Corollary 1 characterizes the properties of this function.

**Corollary 1.** Demand function \( \bar{t}_i(\cdot, \cdot) \) is continuous for all \( \bar{p}, \bar{M}_i > 0 \). This function is also differentiable for all \( \bar{p}, \bar{M}_i \geq 0 \).

For all interior trades, a derivative of a slope-taking demand function with respect to price is equal to \( \partial \bar{t}_i(p, M_i) / \partial p = - (M_i - u''_i)^{-1} \).

### 2.2 Firm Choice

In a competitive equilibrium, price-taking behavior assures that the marginal cost of production is equal to the marginal utilities of the consumers, and hence it does not matter in which of the two commodities the dividends are paid. When traders have market power, there is a wedge between the two values. If consumers are also shareholders, such a gap creates natural incentives to transfer non-numéraire goods from a firm directly to its owners and to bypass the markets. In addition, even if the dividends are paid solely in numéraire, there is no obvious objective function for the firm upon which all shareholders would agree. In particular, the level of production-maximizing profit does not coincide with the optimal production plan from the perspective of each individual shareholder, because, apart from receiving dividends, each consumer is tempted to use the firm as leverage to profitably affect market prices. The two considerations imply that, in a slope-taking setting, a complete characterization of a firm must explicitly specify which goods are transferred to owners as dividends and what the objective of a firm is. In this paper, we focus on incorporated firms, which by assumption maximize profits in terms of numéraire and pay dividends exclusively in this good (money).

Given \( (\bar{p}, \bar{t}_j, \bar{M}_j) \) and hence market demand curve \( p_{\bar{p}, \bar{t}_j, \bar{M}_j}(t_i) = \bar{p} + \bar{M}_j (t_j - \bar{t}_j) \), the profit of incorporated firm \( j \) is given by

\[
\pi_{\bar{p}, \bar{t}_j, \bar{M}_j}(t_j, m_j) = -p_{\bar{p}, \bar{t}_j, \bar{M}_j}(t_j)t_j - m_j. \tag{5}
\]

By convention, \( m_j \) is money spent on inputs, and trade \( t_j \) is a negative of the supply \( t_j \equiv -y_j \); hence, the expressions in the profit function are with a minus sign. Given \( \bar{p}, \bar{t}_j \) and \( \bar{M}_j \), equation (5) defines a map of iso-profit curves. Analogously to a consumer’s budget set, with non-negligible
price impact, any iso-profit curve has a parabolic shape, and the vector normal to it is equal to 
\(- (\bar{p} + \bar{M}_j \bar{t}_j, 1)\).

Similar to those of consumers, for any positive \(\bar{p}\) and \(\bar{M}_j\), \(\bar{t}_j(\bar{p}, \bar{M}_j)\) gives trade \(\bar{t}_j\), which, together with the required level of numéraire input to produce it, \(\bar{m}_j \equiv c_j(-\bar{t}_j)\), maximizes profit among all technologically feasible trades, given the inverse demand function \(p_h, M_j(\cdot)\). By identical arguments, as in the case of consumers, \(\bar{t}_j(\bar{p}, \bar{M}_j) \leq 0\) that assigns unique trade for any \(\bar{M}_j, \bar{p} > 0\) exists, and the function is differentiable for all prices and price impacts for which \(\bar{t}_j(\bar{M}_j, \bar{p}) < 0\). At \(\bar{t}_j = \bar{t}_j(\bar{M}_j, \bar{p})\), the production set must be tangent to the iso-profit curve, and hence the normal vectors are collinear: \(c_j'(-\bar{t}_j) = \bar{p} + \bar{M}_j \bar{t}_j\). The derivative of a slope-taking demand function with respect to price is given by \(\partial \bar{t}_j(p, M_j)/\partial p = -(M_j + c_j^n )^{-1}\).

### 2.3 Endogenous Market Power

We now characterize the endogenous market power of the traders. For any non-negative price impacts of other traders \(\bar{M}_{-n} = \{\bar{M}_{n'}\}_{n' \neq n}\) and slope-taking demands \(\{\bar{t}_{n'}(\cdot, \bar{M}_{n'})\}_{n' = n}\), the equilibrium price impact of trader \(n\) is given by (2). Characterization of slope-takers demands from previous sections then gives

\[
\bar{M}_n = \left(\sum_{n' \neq n} (\bar{M}_{n'} + v_{n'})^{-1}\right)^{-1} = \frac{1}{N-1} \mathcal{H}(\bar{M}_{n'} + v_{n'}| n' \neq n),
\]

where \(v_{n'}\) is a positive scalar defined as \(v_i \equiv -u_i''(\bar{t}_i + e_i)\) for a consumer and \(v_j \equiv c_j''(-\bar{t}_j)\) for a producer, and \(\mathcal{H}(\cdot)\) is a harmonic average of \(N-1\) positive scalars. The solution to the system of consistency conditions for all traders defines equilibrium price impacts \(\bar{M}\). Inspection of the consistency condition (6) provides the following insights about equilibrium price impacts. The essential ingredient of a price impact for each trader is the convexity of the utility and cost functions of their trading partners. The steeper the marginal utility or cost of the other traders, the steeper the slope-takers’ demands, and hence the greater the price concession needed to make others willing to absorb the off-equilibrium deviation. It follows that the trader with the highest convexity has the smallest price impact. More generally, trader ranking with respect to the strength of price impact reverses the order of the convexities of their utility or cost functions. The dependence of the price impact on the convexity of preferences is standard and is shared by many noncompetitive models; for example, a model in which the demand in the industry results from aggregation of the individual demands of \(I\) competitive consumers who maximize quasilinear utility functions. The individual demand of a consumer coincides with her marginal (non-numéraire) utility, and hence, its slope is equal to the absolute value of \(u_i''\). The market demand is a sum of individual demands, and the slope of its inverse is given by \((1/I)\mathcal{H}(-u_i''| i \in I)\). Our model modifies this result in the following way: A market demand faced by trader \(n\) is a horizontal sum of individual (slope-taking) demands by all other traders in the economy, including producers, and therefore, we have \(N-1\) elements in a harmonic mean. Unlike in a standard model, here all traders choose demand functions
rather than quantities. In addition, the slopes of consumers’ individual demands are steeper than the competitive ones, \( \bar{M}_i - u_i'' \), as price impacts make consumer demands less elastic.

Price impacts are subject to a mutual reinforcement effect. Price impact \( M_n \) makes the slope-taking demand \( \bar{t}_n(\cdot, M_n) \) steeper, reinforcing price impacts and hence steepening demands of trading partners, which in turn increases \( M_n \). Such a positive mutual reinforcement effect amplifies the overall level of price impact in the economy.

It does not matter with regard to price impact whether the trader is a firm or a consumer. Moreover, consumers and firms with the same convexity have identical price impacts. As in the standard model of monopoly, price impact does not directly depend on the trader’s share in the total volume of trade. Note, however, that the same price impact affects traders’ decisions in a different way, depending on the individual volume of trade. In particular, when the traded quantity traded is large, the loss of profit is higher. Consequently, the same price impacts will affect large traders more strongly, and traders’ noncompetitive behavior will be more pronounced.

Finally, we would like to stress the role of the harmonic mean. The harmonic mean occurs naturally in the consistency condition, as it puts higher weight on traders with smaller convexities. Observe that such traders have the most elastic slope-takers demands and hence are key in price impact determination. The traders with approximately constant marginal utility or cost have almost perfectly elastic demands, making the price impacts of other traders negligible. This intuition is reflected in a basic property of the harmonic mean, that is, whenever one of its elements converges to zero, the value of the mean also becomes zero.

3 Properties of Equilibrium

This section, states five results that characterize a slope-taking equilibrium in a smooth economy. The first result demonstrates that the slope-taking behavior is rational in a Walrasian auction with atomic traders under general conditions. The two following results establish the existence and determinacy of an equilibrium. The fourth theorem is a version of the first welfare theorem, while the last finding shows the convergence of slope-taking equilibria to a unique competitive equilibrium in a large economy.

3.1 Rationality of Slope-taking Behavior

Consider a Walrasian auction, in which traders simultaneously choose demand functions, the Walrasian auctioneer finds a market clearing price at which trade among agents occurs. When \( N < \infty \), zero price impacts \( M = 0 \) violate the consistency condition, and the profile of competitive demands \( \{t_{n Walras}(\cdot)\}_n = \{\bar{t}_n(\cdot, 0)\}_n \) is not a Nash equilibrium in a smooth economy. Theorem 1 demonstrates that, if the traders’ optimization problems are convex, price impacts provide sufficient information for agents to respond optimally to market conditions.

For any equilibrium \( (\bar{p}, \bar{t}, \bar{M}) \), let \( \{\bar{t}_n(\cdot, M_n)\}_n \) be slope-takers’ demands. Suppose the market supply resulting from aggregation of \( \{\bar{t}_n(\cdot, M_n)\}_{n' \neq n} \) defines a globally convex optimization
problem for trader $n$; then:

**Theorem 1.** Profile of demands $\{\bar{t}_n(\cdot, \bar{M}_n)\}_n$ constitutes Nash equilibrium in a Walrasian auction, with the outcome given by $(\bar{p}, \bar{t}, \bar{M})$.

The problem of a buyer who faces an upward-sloping supply is convex unless the market supply is sufficiently concave, which in the presented model may occur (e.g., in a pure exchange economy with a small number of traders in which marginal utilities of the traders and hence their slope-takers' demands are very convex.)

By Theorem 1, consistent price impacts $\bar{M}$ summarize all the relevant information about the demands of the trading partners. In a complementary paper [16], we argue that traders can learn $\bar{M}$ from the data available in anonymous markets; in a quadratic example, we demonstrate that the process of learning, in which traders independently estimate and re-estimate their price impacts from past data on their own trades and prices, and act optimally given their estimates, the estimated price impacts eventually converge to $\bar{M}$. Thus, our model fits well with many nontransparent financial markets in which large traders use market impact models to determine their trading strategies.

### 3.2 Existence

A good economic model gives predictions for any economy, satisfying general assumptions. The next theorem establishes the existence of such equilibria in a smooth economy.

**Theorem 2.** In a smooth economy, an equilibrium exists if and only if $N > 2$.

The assumptions of Theorem 2 are relatively weak, as they only require that there be more than two agents trading in a market. For economies with two traders, consistency conditions become $\bar{M}_n = \bar{M}_n' + v_n'$. Geometrically, the two lines defined by the equations are parallel, and the system has no solution for any values of $v_1, v_2 > 0$. The mutual reinforcement of price impacts is without any discounting; price impacts explode to infinity, and equilibrium fails to exist.

Non-existence of equilibrium is contrary to a competitive model or traditional bargaining models. In Nash bargaining models, the outcome is Pareto efficient, while in this and virtually any other noncompetitive model based on uniform price (monopoly or Cournot), trade is associated with a dead-weight loss. In this context, the nonexistence of equilibrium can be interpreted as a manifestation of inefficiency in which the rigidity of the uniform price contract combined with reinforcement of price impacts prevents trade.

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6For the buyer $n$ who faces an upward-sloping supply $p_n(t_n)$, the optimization problem fails to be convex if $p_n''(t_n) < (u_n''(t_n + e_n) - 2p_n'(t_n))/t_n < 0$. Thus, the second-order condition holds for approximately linear supply, which in the model with slope-takers occurs when utility functions are close to quadratic. Moreover, since terms $p_n''(t_n), p_n'(t_n)$ converge to zero in the number of traders, the second-order condition is satisfied with sufficiently many trading partners, even if individual marginal utilities are convex.
3.3 Determinacy

The second desirable property of the model is that its predictions are sharp. Technically, the set of all equilibria should be small, possibly a singleton. It turns out that an equilibrium can be expressed as a solution to a system of nonlinear equations, where the number of equations is exactly equal to the number of endogenous unknowns. If no equation is (locally) redundant, the equilibria are locally unique. Unfortunately, local non-redundancy cannot be guaranteed for any arbitrary economy $E$. The next theorem formalizes the intuition that economies in which equilibria are not locally unique are not robust.

Fix economy $E$. For any $n$, let $\delta'_n, \delta''_n \in \mathbb{R}$ be two scalars, and finally let $\delta \equiv \{\delta'_n, \delta''_n\}_n$. By $E_\delta$, we denote an original economy with utilities perturbed by adding $\delta'_i x_i + \delta''_i (x_i)^2$ and cost functions by adding $\delta'_j y_j + \delta''_j (y_j)^2$. Let $\mathcal{P} \equiv B(0, r) \subset \mathbb{R}^{N \times 2}$ be an open ball, spanning a family of perturbed economies, with each member parameterized by $\delta \in \mathcal{P}$. We assume that the radius $r > 0$ is sufficiently small to preserve strict monotonicity and strict convexity of the preferences and cost functions on the relevant set of trades.\(^7\) Notice that $\mathcal{P}$ has a nonzero Lebesgue measure in $\mathbb{R}^{N \times 2}$ and contains the original economy (since $0 \in \mathcal{P}$).

**Theorem 3.** There exists a subset $\hat{\mathcal{P}} \subset \mathcal{P}$ with full Lebesgue measure, such that for any $\delta \in \hat{\mathcal{P}}$, all equilibria of the economy $E_\delta$ are locally unique.

Theorem 3 can be interpreted in the following way: For any economy $E$, including the ones for which the equilibria are not determinate, for almost all economies with small perturbations of preferences and technology, all equilibria are locally unique. Consequently, the non-determinacy of slope-taking equilibria is not a robust phenomenon. One implication of this theorem is that, typically, the number of equilibria is, at most, finite. In the case of economies with quadratic utility and cost functions, the result can be strengthened as in such economies, slope-taking equilibria are *globally* unique.\(^8\)

It is well known that the games based on demand function have a continuum of Nash equilibrium outcomes (see [6], [11] and Section 4.1). As demonstrated in Theorem 3, slope-taking equilibria associated with Nash equilibrium $\{\bar{t}_n(\cdot, M_n)\}_n$ are generically locally unique, and hence the behavioral assumption of slope-taking refines the set of Nash equilibria (see Lemma 1 in Rostek and Weretka [15]).

3.4 Welfare

The first welfare theorem guarantees that, when markets are perfectly competitive, the equilibrium allocation is Pareto efficient. Theorem 4 provides an analogous though opposite result for interactions among slope-takers.

\(^7\)Such a sufficiently small radius exists by continuity of utility, cost functions, and the set of trades.

\(^8\)In [16] we demonstrate that, with quadratic utilities consistency condition is a contraction that maps (sufficiently large) compact set of price impacts into itself. Thus, there exists precisely one profile of price impacts $\bar{M}$ that satisfies consistency condition. The uniqueness of equilibrium then follows.
Theorem 4. In an equilibrium within a smooth economy, allocation is Pareto efficient if and only if \( J = 0 \) and the initial allocation is Pareto efficient.

It follows that, in economies with firms, equilibrium is always Pareto inefficient, and in a pure exchange economy, the allocation is inefficient as long as initially there are some gains to trade.

The negative message from Theorem 4 is partially counterbalanced by the next result, which asserts that, although equilibrium allocation is not Pareto efficient, in markets with a large number of traders, the inefficiency is negligible, as all equilibria converge to a singleton, the unique competitive equilibrium. To state the result formally, we take advantage of the construct of a \( k \)-replica economy.

Theorem 5. For any smooth economy, and any \( \varepsilon > 0 \), there exists \( k^\varepsilon \in \{2, 3, \ldots\} \) such that for any \( k \geq k^\varepsilon \), all equilibria in a \( k \)-replica economy satisfy for any \( n_a \)

1. \( \| \bar{M}^a(n_a(k^\varepsilon)) \| \leq \varepsilon; \)
2. \( \| \bar{t}^a(n_a(k^\varepsilon)) - t^\text{Walras} \| \leq \varepsilon; \)
3. \( \| \bar{p}(k^\varepsilon) - p^\text{Walras} \| \leq \varepsilon. \)

It is often claimed that the competitive equilibrium is a good predictor of market interactions if the following three preconditions are satisfied: 1) there are many traders, 2) these traders freely and optimally adjust their traded quantities to prices, 3) they trade in centralized anonymous markets. Slope-taking approach (Theorem 5) formalizes this claim using modern, game-theoretic language. Slope-taking equilibrium is rationalized by a Nash equilibrium in a game defined by a Walrasian auction. Moreover, in [16] we show that such equilibrium is a steady state of a learning process in which traders estimate price impacts from data available in anonymous markets. When the number of traders is large, the consistent price impacts become negligible, and the demands of the slope-taking traders, \( \bar{t}(\cdot, \bar{M}) \) converge (pointwise) to the Walrasian demands \( t^\text{Walras}(\cdot) \). It follows that, in large markets, slope-taking equilibrium becomes equivalent to a competitive auction (or tâtonnement process) proposed by Walras to rationalize a competitive equilibrium, and a slope-taking equilibrium \((\bar{p}, \bar{t}, \bar{M})\) coincides with the equilibrium with price takers \((p^\text{Walras}, t^\text{Walras}, 0)\). Thus, the competitive equilibrium approximates well the outcomes in markets with many slope-taking traders.

Can one empirically test for a slope-taking behavior, given information about trades, endowments, and prices from one trading period \((\bar{t}, e, \bar{p})\)? The triple \((\bar{p}, \bar{t}, \bar{M})\) is a slope-taking equilibrium in \( \mathcal{E} = (u, e, c) \) only if \((\bar{p}, \bar{t})\) is a competitive equilibrium in the economy with modified preferences and technology, \( \mathcal{E} = (\hat{u}, e, \hat{c}) \), where \( \hat{u}_i(x_i) = u_i(x_i) - \frac{1}{2} \bar{M}_i(x_i - e_i)^2 \), for any consumer \( i \in \mathcal{I} \), and \( \hat{c}_j(y_j) = c_j(y_j) + \frac{1}{2} \bar{M}_j(y_j)^2 \), for any firm \( j \in \mathcal{J} \). It follows that any allocation and price in a slope-taking equilibrium can be rationalized as a competitive equilibrium of some other economy. As a result, slope-taking equilibrium is empirically indistinguishable from competitive equilibrium, given that the data are from one period only.
3.5 Price Biases

In the quadratic and symmetric example in Section 1.1, we have shown that equilibrium price \( \bar{p} \) coincides with the competitive price. We now demonstrate that such an equality is nongeneric. Consider the economy from Section 1.1 in which buyers and sellers have quadratic utility functions in which convexity coefficients might be different for buyers \( v_i = v_b \) and sellers \( v_i = v_s \). The consistent price impacts can be found by solving (6) and slope-takers’ demands satisfy

\[
\tilde{t}_i(\bar{p}, \bar{M}_i) = \frac{1 - v_i e_i - \bar{p}}{v_i + M_i} = \frac{v_i}{v_i + M_i} \frac{1 - v_i e_i - \bar{p}}{v_i} = \frac{v_i}{v_i + M_i} \tilde{t}_i(\bar{p}, 0), \tag{7}
\]

where \( \tilde{t}_i(\bar{p}, 0) \) corresponds to a Walrasian demand.

When utilities are identical, the price impacts of both types of traders also coincide, \( \bar{M}_b = \bar{M}_s \), and therefore, the equilibrium market clearing condition implicitly defining the equilibrium price

\[
\sum_i \tilde{t}_i(\bar{p}, \bar{M}_i) = \frac{v_i}{v_i + M_i} \sum_i \tilde{t}_i(\bar{p}, 0) = 0. \tag{8}
\]

is satisfied if and only if \( \sum_i \tilde{t}_i(\bar{p}, 0) = 0 \). The unique price that solves the equality is \( \bar{p}^{Walras} \), and hence, \( \bar{p} = \bar{p}^{Walras} \). The intuition behind this result is as follows: The identical convexity for the buyer and seller leads to the same market power for both types of consumers. For any fixed price, the buyer with price impact reduces its demand by the same proportion \( v_i / (v_i + M_i) \) as the seller decreases its supply, the market clears, and the price remains unchanged.

Now suppose that traders’ convexities differ; for example \( v_b > v_s \). Although in an asymmetric case, equilibrium does not have a closed form solution, it can be shown that \( v_b > v_s \) implies \( M_s > M_b \) where \( \bar{M}_s \) and \( \bar{M}_b \) are price impacts of sellers and buyers, respectively. Intuitively, when the buyers’ marginal utility is steeper than the sellers’, the sellers’ deviation is associated with a higher price concession than that of the buyers. Consequently, inequality \( v_b / (v_b + M_b) > v_s / (v_s + M_s) \) holds. This fact and the market clearing condition

\[
0 = \sum_i \tilde{t}_i(\bar{p}, \bar{M}_i) = \sum_i \frac{v_i}{v_i + M_i} \tilde{t}_i(\bar{p}, 0), \tag{9}
\]

imply that, at the slope-taking equilibrium price, \( \bar{p} \), the Walrasian excess demand is strictly negative: \( \sum_i \tilde{t}_i(\bar{p}, 0) < 0 \). This is because, in the last expression in (9), the constants for buyers are above the ones for sellers. Since the Walrasian aggregate demand function strictly decreases in price, in order to establish the market clearing in a competitive framework, the Walrasian price must go down. One can conclude that \( \bar{p} > \bar{p}^{Walras} \). By an analogous argument, the price bias is negative when the seller’s convexity is greater, \( v_b < v_s \), that is, \( \bar{p} < \bar{p}^{Walras} \). This example illustrates that, in a slope-taking equilibrium, price can be greater, the same, or below the competitive one, depending on the values of the parameters of the considered economy.

In general, when consumers have identical, quasi-linear, and concave, but not necessarily quadratic utility functions, their Walrasian consumption is the same, \( \bar{x}_i = \bar{x}^{Walras} \). In a slope-
taking equilibrium, the sellers are given incentives to reduce their supply, while the buyers cut down on their demands. This leads to a disparity between the consumption of the buyers and the sellers, $\bar{x}_b < \bar{x}^{Walras} < \bar{x}_s$. If the second derivative of the utility function is increasing, $(u'' > 0)$, then the second derivative of the utility function of the buyer at the equilibrium consumption, $v_b = |u''(\bar{x}_b)|$, exceeds the one for the seller $v_s = |u''(\bar{x}_s)|$. The quadratic example suggests that such endogenous asymmetry in convexity may lead to a positive price bias over the price predicted by a competitive model, $\bar{p} > \bar{p}^{Walras}$. To make this observation formal, for any slope-taking equilibrium $(\bar{p}, \bar{t}, \bar{M})$, we partition the set of all traders into two groups: buyers, $I_b = \{i \in I | t_i \geq 0\}$, and sellers, $I_s = \{i \in I | t_i < 0\}$.

**Proposition 1.** Consider a smooth pure exchange economy with Pareto-inefficient initial allocation, in which utility functions are the same for all consumers. Then, in any slope-taking equilibrium $(\bar{p}, \bar{t}, \bar{M})$, for any $b \in I_b$ and $s \in I_s$,

1. $u''' > 0$ implies upward price bias $\bar{p} > \bar{p}^{Walras}$ and $\bar{M}_s > \bar{M}_b$;
2. $u''' = 0$ implies no price bias $\bar{p} = \bar{p}^{Walras}$ and $\bar{M}_s = \bar{M}_b$;
3. $u''' < 0$ implies downward price bias $\bar{p} < \bar{p}^{Walras}$ and $\bar{M}_s < \bar{M}_b$.

There are two effects generating the price bias. First, with the positive third derivative, $u''' > 0$, the sellers tend to reduce their supply more than the buyers, given any level of price impact $M_i$, and hence the price of the good must go up to clear the market. This is a direct effect. Second, the endogenous asymmetry in convexity implies that the sellers’ price impacts exceed those of the buyers, further increasing the price. For example, with CARA or CRRA utility functions, the third derivative is positive; therefore, in a pure exchange economy with such utility functions, one should expect upward price biases and a high relative market power for the sellers.

4 Slope-taking Equilibrium and Other Solution Concepts

We now discuss how the framework with slope-takers complements the literature based on Nash in Supplies (Klemperer and Meyer [11] model) and the Bresnahan [2] consistent conjectural variation model.

4.1 Nash in Supplies

In their classic paper [11], Klemperer and Meyer (KM) study a model of oligopolistic industry with $I$ identical producers who face market demand and whose strategies are supply functions. To “discipline” submitted supply functions, KM introduce uncertainty to demand and show in the model with identical producers with quadratic cost functions and exogenous linear demand with additive unbounded shock that the equilibrium is unique ($I^{KM}(\cdot)$ equilibrium). The slope-taking approach complements this result in the following ways: In the quadratic example with identical producers, the unique equilibrium coincides with the slope-taking equilibrium from this paper.
Thus, the learning model from [16] based on re-estimation of the price impacts that provides a behavioral foundation is applicable to equilibrium $\bar{t}^{KM}(\cdot)$.

The main advantage of the slope-taking approach is tractability. In settings with identical traders, $\bar{t}^{KM}(\cdot)$ is found by solving a nonlinear differential equation with an endogenously determined boundary condition, which makes the characterization of an equilibrium very challenging. For example, even in the simple model with identical producers with non-quadratic cost functions, the uniqueness (or even determinacy), let alone characterization of an equilibrium, has not been established. The model becomes even more complicated with heterogenous traders, as an equilibrium is given by the solution to $I$ nonlinear differential equations, and there are no results that characterize equilibrium even with $I \geq 3$ heterogenous traders. In contrast, the slope-taking equilibrium is well defined, and its properties are characterized in quite general settings.

Finally, in non-quadratic settings, $\bar{t}^{KM}(\cdot)$ equilibrium is demanding in terms of the level of rationality it requires from the traders. To play $\bar{t}^{KM}(\cdot)$, the least the traders have to know is their true price impacts for any possible realizations of the market supply, including the extreme ones. Since extreme realizations are infrequent, the price impacts for such realizations are not easily estimable from the data available in anonymous markets. Given the limited information, traders typically extrapolate their price impact from the typical realizations to the extreme ones, which this paper assumes. Such an approximation is almost without loss of utility for the traders as long as the trades are concentrated around typical values.

### 4.2 Conduct Parameter and Conjectural Variation

In the empirical industrial organization literature, the competitiveness of an oligopolistic industry is often measured by the conduct parameter $\theta_j$ estimated using the first-order condition

$$\bar{p} + \theta_j q_j \frac{\partial P}{\partial Q} = MC(q_j).$$

Parameter $\theta_j$ gives the markup of producer $j$ over the marginal cost, and its estimated value typically varies (strictly) between zero and one. The conduct parameter is sometimes interpreted as a conjecture of firm $j$ about the reaction of the industry to increased production of $j$; i.e., $\theta_j$ tells how much total production in the industry goes up should trader $j$ increase sales by one (See [4]). Such an interpretation is problematic from the theoretical point of view, as no satisfactory game-theoretic model exists that explains how such different reactions could arise in equilibrium. The standard static games rationalize only two extreme values $\theta_j = 0$ (full reaction, Bertrand competition) and $\theta_j = 1$ (no reaction, Cournot). In the early '80s, conjectural variation literature emerged with an objective to rationalize the intermediate values of the conduct parameters ([2], [13]). This literature did not fully specify a strategic environment for traders, and hence the conjectural variation approach could be only considered as a reduced form of some underlying game-theoretic model, which has not been provided.

The slope-taking equilibrium has a game theoretic foundation and, when specialized to the
oligopolistic industry, it results in a conduct parameter that ranges strictly between zero and one. The value of the conduct parameter decreases in the convexity of the cost functions and increases in the number of competitors. In the slope-taking model, a positive “reaction” of the competing firms occurs because traders have downward-sloping demands. By increasing quantity, trader \( j \) depresses the market clearing price, which crowds out the sales of the other producers. The behavioral interpretation of the conduct parameter as a conjectured reaction of the competing firms does not apply to the model with slope-taking firms. Here, agents trade in anonymous markets and have no information about other traders’ quantities or identities.

It is also instructive to contrast the consistency condition in the conjectural variation model of Bresnahan, [2] with the consistency of price impacts of slope-taking equilibrium. In doing so, we restrict attention to a model in which conjectural equilibrium is well defined (exists and is unique); i.e., the duopoly with producers having identical quadratic cost functions who face a linear demand\(^9\) (see the debate between Bresnahan [2], [3], and Robson [14]). In both approaches, trader \( j \) has a conjecture about the reaction of other traders to a unilateral deviation and acts optimally given such belief, both in and out of equilibrium. In the model of duopoly, conjectures are constant in both models, and the two approaches are outcome-equivalent.

The key modeling difference that makes the slope-taking approach more general is as follows: In the conjectural variation model, trader \( j \) has a conjecture about the best response of his opponent. It is not clear how such a formulation in terms of best responses can be generalized to allow for more than two heterogenous producers.\(^{10}\) With many traders, after a unilateral deviation of \( j \), it is reasonable to assume that agent \( i \) should not only best respond to the deviation of \( j \) but should also take into account the adjustments of the other traders (\( \neq i,j \)) and \textit{vice versa}. In the slope-taking equilibrium, (by assumption) a unilateral deviation of \( j \) triggers an outcome in which all other traders simultaneously behave optimally given what the other traders are doing, and such an adjustment of the whole system defines the reaction of the market. Thus, after unilateral deviation, each trader \( i \neq j \) responds optimally not only to the deviation of trader \( i \), but also to the optimal adjustments of the remaining traders (\( \neq i,j \)) given the price impacts. Consistent price impact is not a simple sum of the best responses of other traders \( i \neq j \) to a unilateral deviation of \( j \). Our approach allows one to study markets with an arbitrary number of heterogenous traders, consumers, and producers.

\(^{9}\)Another feature that distinguishes the slope-taking equilibrium from Bresnahan’s conjectural variation is that it defines equilibrium in models with non-quadratic utility functions.

\(^{10}\)Perry [13] extends the conjectural variation approach to a model of oligopolistic industry with \( I > 2 \) producers with identical quadratic costs by assuming symmetry of the responses, invoking the symmetry of the model. The general consistency condition for the oligopolistic industry with many heterogenous traders has not been formulated.
5 Extensions

5.1 Many Commodities

The slope-taking framework straightforwardly extends to economies with many \((L > 1)\) commodities. In the definition of equilibrium (Definition 1), prices and individual trades are vectors \(\bar{p}, \bar{t}_n \in \mathbb{R}^L\) and price impacts \(M_n\) are \(L \times L\), positive definite matrices. Budget constraints and profit functions are quadratic forms, budget sets and iso-profit curves are elliptic paraboloids, and vectors orthogonal to these sets are given by the vectors of marginal payments for all commodities. With additively separable utility, markets for different commodities are independent, and \(L\)-good economy is equivalent to \(L\) economies, each with one good. Thus all the results from Section 3 automatically hold in e.g., two-period Arrow-Debreu financial economies with Neumann-Morgenstern preferences. With non-separable utilities, there are cross-market price impacts, and the results from this paper are not directly applicable to the multi-good setting.\(^{11}\)

5.2 Free Entry

In the long-run equilibrium proposed by Marshall, free-entry condition and price taking imply that the price is equal to a minimal average cost, and firms produce at the minimal efficient scale. The number of active firms is determined by the ratio of the aggregate demand at the equilibrium price divided by the minimal efficient scale. The Marshallian model suffers from the following drawbacks: The equilibrium is well defined only for the technology characterized by \(U\)-shaped average cost, and hence, the model does not give predictions for industries with, for example, increasing returns to scale. In addition, the model assumes price-taking behavior even when the predicted number of firms is small. In this section, we modify the competitive free-entry model by introducing price impacts.

We consider a pure exchange economy \((u_i, e_i)_i\) and an infinite pool of entrepreneurs who have access to the technology they may potentially use to establish a firm. The technology is given by cost function \(c\). We model entry as a two-stage game. In the first stage, entrepreneurs simultaneously choose whether to pay fixed cost \(F\) to establish a firm, and in the second stage, a Walrasian auction takes place. We determine market structures of the industry by looking at a Subgame Perfect Nash equilibrium in which in the second stage, “slope-taking” Nash equilibrium is played.

**Constant Marginal Cost:** An active or inactive monopoly is the only market structure compatible with cost function \(c(y_j) = \bar{c} \times y_j\), in which marginal cost \(\bar{c}\) is constant and entry cost \(F > 0\). With two or more operating firms, constant marginal cost makes positive price impacts

\(^{11}\text{With non-separable utilities, equilibrium price impacts inherit two properties of the Hessians of the utility function: positive definiteness and symmetry. \cite{15} characterizes a unique equilibrium in a dynamic economy with \(L\) commodities (assets) and non-separable quadratic utilities, which using their Lemma 1 can be mapped here. More generally, under mild conditions of strict monotonicity and concavity (convexity) of preferences (of cost functions) the results from Theorems 3, 4, and 5 can be extended to all non-separable economies with many commodities. The existence of slope-taking equilibrium requires additional assumptions that guarantee interiority of the solution and can be established using the same arguments as proof of Theorem 2.}\)
inconsistent.

**Proposition 2.** Consider a cost function that satisfies $c''(\cdot) = 0$. The consistent price impacts satisfy $\bar{M}_j > 0$ if and only if there is one firm $J = 1$ in the industry.

Since strictly positive price impacts $\bar{M}_j > 0$ are necessary for strictly positive profit in the second stage, entering the market is not a best response if one or more of the other producers choose entry. Consequently, a monopoly is the richest market structure compatible with such a cost function. New firms are not willing to challenge the incumbent, as their entry in slope-taking equilibrium triggers perfectly competitive interactions, resulting in strictly negative profits for all firms.

**Increasing Marginal Cost:** Consider $c(y_j)$, which is strictly convex and for which marginal cost satisfies $c'(0) = 0$. With $F > 0$, average cost is either monotonically decreasing or $U$-shaped and has a well-defined, minimal efficient scale. In the equilibrium with entry, such a technology induces a monopolistic or oligopolistic market structure in which all firms have positive price impacts and typically positive profits.

Industries with $U$-shaped average costs have already been analyzed in a Marshallian framework. For example, it is known that, with quadratic variable cost function $c(y_j) = vy_j^2/2$, in the competitive equilibrium with entry, each firm produces at the minimal efficient scale, $\sqrt{2F/v}$, and price is equal to a minimal average cost, $\sqrt{2Fv}$. The number of consumers per each firm $I/J$ can be found as a ratio of the individual production and the aggregate demand in the economy. A competitive equilibrium with entry is not consistent unless the economy is large. As with the predicted small number of active firms, the assumption of price taking is not justified. Now, using the model of entry, we calculate equilibrium with entry in an economy with slope-takers for consumers’ utility functions $U(x_i, m_i) = 15x_i^2 - 5x_i^2 + m_i$, zero endowments, and quadratic cost function with $F = 10$ and $v = 10$. The results are given in Table 1.

<table>
<thead>
<tr>
<th>$J$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>10</th>
<th>100</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>18</td>
<td>34</td>
<td>50</td>
<td>66</td>
<td>165</td>
<td>1649</td>
</tr>
<tr>
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<td>18</td>
<td>17</td>
<td>24.5</td>
<td>16.66</td>
<td>22.0</td>
<td>16.5</td>
</tr>
<tr>
<td>$\bar{y}$</td>
<td>1.34</td>
<td>1.38</td>
<td>1.41</td>
<td>1.38</td>
<td>1.41</td>
<td>1.41</td>
</tr>
<tr>
<td>$\bar{\pi}$</td>
<td>0.070</td>
<td>0.026</td>
<td>0.380</td>
<td>0.008</td>
<td>0.290</td>
<td>0.002</td>
</tr>
</tbody>
</table>

Notes: The first row, $J$, reports the number of firms in the industry for which the simulation was conducted. $I$ represents the number of consumers compatible with $J$ firms; $I/J$ is the number of consumers per firm. $\bar{y}$ gives the range of production levels; $\bar{p}$ denotes a price, and $\bar{\pi}$ is profit.

The first row defines the market structure in question. Column $J = 1$ represents a monopoly, $J = 2$ is a duopoly, and $J > 2$ is an oligopoly with $J$ firms. In column $\infty$ we report the predictions of the competitive equilibrium with entry. Row $I$ gives the numbers of consumers supporting the considered market structure, and row $I/J$ is the number of consumers per each active firm. The last three rows show the equilibrium level of individual production, price, and profit, respectively.
The profits of oligopolistic firms are positive, and the price is above the minimal average cost. In a small economy, the number of consumers per active firm is significantly higher than the value predicted by the Marshallian equilibrium. For example, a monopoly serves from 18 to 33 consumers and not predicted 16.48. One of the reasons for this is that indivisibility of a fixed cost prevents the other firms from entering the industry, even if the incumbent has a relatively high profit. In addition, in a small economy, consumers have some market power, and hence, their aggregate demand is below the competitive demand. On the other hand, the simulation shows that the Marshallian equilibrium is a good approximation whenever the number of operating firms is large. In an economy with more than 100 active firms, the gap in the predictions becomes negligible.

Finally, observe that, when entry cost becomes negligible \((F \rightarrow 0)\), the number of firms in equilibrium converges to infinity \((J \rightarrow \infty)\), price impacts become negligible, and the economy becomes perfectly competitive.

References


Appendix

Proof: Lemma 1. A trade that maximizes preferences $U_i(\cdot, \cdot)$ on a budget set $B(\bar{p}, \bar{t}_i, \bar{M}_i)$, $(t_i, m_i)$ is a solution to program $\max_{t_i, m_i} U_i(t_i + e_i, m_i)$, subject to $(t_i, m_i) \in B(\bar{p}, \bar{t}_i, \bar{M}_i)$ and $t_i \geq -e_i$. $U_i(\cdot, \cdot)$ is strictly concave in $t_i$, and constraints are convex, given that $\bar{M}_i$ is positive. In addition, by strict monotonicity of preferences in money, in an optimum budget constraint must be satisfied with equality. The necessary (and sufficient) Khun-Tucker optimality condition is given by

$$u_i' - \bar{p} - \bar{M}_i(2t_i - \bar{t}_i) \leq 0,$$

$$p_{\bar{p}, \bar{t}_i, \bar{M}_i}(t_i) t_i + m_i = 0,$$

and $p_{\bar{p}, \bar{t}_i, \bar{M}_i}(t_i) = \bar{p} + \bar{M}_i(t_i - \bar{t}_i)$, where the first inequality holds with equality for all commodities such that $t_i > -e_i$. By the definition of a trade, $\bar{t}_i$ is optimal on the budget set spanned by itself, and therefore the additional condition is $t_i = \bar{t}_i$. Thus, (11) reduces to two conditions

$$u_i' - \bar{M}_i \bar{t}_i \leq \bar{p},$$

$$\bar{m}_i + \bar{p} \bar{t}_i = 0.$$

For any positive $\bar{M}_i$, equation (12) has a unique solution $\bar{t}_i$. $\bar{M}_i > 0$, therefore the left hand side is decreasing in $\bar{t}_i$. Given Inada condition, this function attains infinity at $\bar{t}_i \to -e_i$ and is equal to zero for some finite value of $\bar{t}_i$. Therefore, for any $p$ from interval $[0, \infty)$, there exists a unique positive value of $\bar{t}_i$ that solves inequality (12) with equality. Given $\bar{t}_i$ that solves (12), a unique $\bar{m}_i$ is determined by the second equality (12). This shows that, for any $\bar{p} \geq 0$ and positive $\bar{M}_i$, there exists a unique fixed point $(\bar{t}_i, \bar{m}_i)$.

Proof: Corollary 1. Lemma 1 defines demand function $\bar{t}_i(\bar{p}, \bar{M}_i)$. In addition, since $\bar{t}_i > -e_i$, the non-numéraire demand is implicitly defined by $u_i'(\bar{t}_i + e_i) = \bar{p} + \bar{M}_i \bar{t}_i$. The derivative of the implicit function with respect to $\bar{t}_i$ is given by $\bar{M}_i - u''_i$, which is positive. From the implicit function
theorem it follows that there exist neighborhoods of \( \tilde{t}_i \) and \( \bar{p} \) and a differentiable bijection, \( \tilde{t}_i^* (\bar{p}, \bar{M}_i) \), solving (12). \( \tilde{t}_i^* (\cdot, \cdot) \) coincides with a slope-taking demand function. Since the argument holds for any interior trade, \( \tilde{t}_i (\bar{p}, \bar{M}_i) \) is smooth. The implicit function theorem also implies that the derivative of the slope-taking demand function with respect to price is \( \partial^2 \tilde{t}_i (p, M_i) / \partial p^2 = -(M_i - u_i'')^{-1} \).

**Proof:** Theorem 1. \( \tilde{t}_n (\cdot, \bar{M}_n) \) is the best response to \( \{ \tilde{t}_n' (\cdot, \bar{M}_n') \}_{n' \neq n} \) if it results in an outcome of a Walrasian auction \( \bar{p}, \bar{t}_n \) that maximizes the utility of trader \( n \). Since \( \{ \tilde{t}_n' (\cdot, \bar{M}_n') \}_{n' \neq n} \) gives rise to trade for which \( \bar{p}, \bar{t}_n \) equalizes the marginal utility of \( n \) with the marginal revenue, given market supply defined by \( \{ \tilde{t}_n' (\cdot, \bar{M}_n') \}_{n' \neq n} \) and hence the first-order optimality condition is satisfied at \( \bar{t}_n \). Moreover, since the optimization problem is globally convex, \( \tilde{t}_n \) is optimal and \( \tilde{t}_n (\cdot, \bar{M}_n) \) is a best response to \( \{ \tilde{t}_n' (\cdot, \bar{M}_n') \}_{n' \neq n} \).

**Proof:** Theorem 2. We establish the existence of equilibrium for economies with more than two traders in the following steps. First we construct a truncated set of trades that is non-empty, convex, and compact. Then we define three continuous functions and two non-empty, convex, and compact sets of the second derivatives, and the price impacts respectively. We show that a continuous function that consists of the three components defined in the previous steps has a fixed point. Next, we show that truncation of the trade set is without loss of generality, which finally allows us to prove that the fixed point implies a slope-taking equilibrium. In the last step, we argue that a slope-taking equilibrium does not exist when there are only two traders.

**Step 1. Truncated space of trades, \( \mathcal{T} \).**

(Argument for Step 1): Given the upper bound on a price as \( \bar{u} \equiv \max_i u_i' (e_i) \), for any consumer \( i \), a lower bound of the consumption in equilibrium \( x_i^{\min} \) is determined by \( u_i' (x_i^{\min}) = \bar{u} \). By Inada, \( x_i^{\min} \) is well defined and \( 0 < x_i^{\min} \leq e_i \). Similarly, for firm \( j \) we define an upper bound on the production \( y_j^{\max} \), as \( c_j' (y_j^{\max}) = \bar{u} \). By monotonicity and convexity, production \( y_j^{\max} > 0 \) exists and is unique. A truncated space of trades, denoted by \( \mathcal{T} \), is a set of all trades such that markets clear and individual trades are within the following bounds

\[
\mathcal{T} \equiv \{ t \in \mathbb{R}^N | \sum_n t_n = 0, \text{ and } t_i \geq -e_i + \frac{1}{2} x_i^{\min}, \text{ and } 0 \geq t_j \geq -y_j^{\max} \}. \quad (13)
\]

Set \( \mathcal{T} \) is non-empty (for example, \( t = 0 \) is a member of \( \mathcal{T} \) and convex (a union of three convex sets). The set is also compact; it is closed as it is a pre-image of a closed set by a continuous function, and it is bounded as all trades are bounded from below, and they all sum up to zero, therefore they are also bounded from above.

**Step 2. Continuous function \( V (\cdot) : \mathcal{T} \rightarrow \mathbb{R}_+^{N} \).**

(Argument for Step 2): Define \( V (\cdot) : \mathcal{T} \rightarrow \mathbb{R}_+^{N} \), as \( V (\cdot) \equiv (V_1 (\cdot), \ldots, V_N (\cdot)) \) where for each \( n = i, j \), \( V_i (t_i) \equiv -u_i' (e_i + t_i) \) and \( V_j (t_j) = c_j' (-t_j) \). Function \( V (\cdot) \) is not well defined for trades in which firm’s supply of some good is zero. This is because the second derivative of cost functions is not defined on the boundary of \( \mathbb{R}_+ \). For \( t_j = 0 \), function \( V_j (t_j) \) is then defined as a limit \( V_j (0) \equiv \lim_{t_j \to 0} c_j'' (-t_j) \). By interiority assumption, such limit is well defined, finite, and strictly.
greater than zero. Utility and cost functions are twice continuously differentiable; therefore, $V(t)$ is continuous on $T$.

Next we show that function $V(\cdot)$ maps $T$ into a subset $\mathcal{V}_2^\gamma \subset \mathbb{R}_+^N$ that is non-empty, convex, and compact. For any two non-negative scalars $\gamma$ and $\tilde{\gamma}$, satisfying $0 \leq \gamma \leq \tilde{\gamma} < \infty$, define a non-empty, convex, and compact box $\mathcal{V}_2^\gamma \subset \mathbb{R}_+^N$ as $\mathcal{V}_2^\gamma \equiv \{v \in \mathbb{R}_+^N | v \leq 1_{\gamma} \text{ and } v \geq 1_{\tilde{\gamma}}\}$, where $1 \in \mathbb{R}_+^N$ is a unit vector in $\mathbb{R}^N$. The two scalars are defined as follows

$$\tilde{\gamma} = \sup_{t \in T} \max_n V_n(t_n) \text{ and } \gamma = \inf_{t \in T} \min_n V_n(t_n).$$

(14)

Since $V(t)$ is continuous and $T$ is compact, both sup and inf are attained on $T$, therefore $0 < \gamma \leq \tilde{\gamma} < \infty$. By construction, the image of $T$ by function $V(\cdot)$ is contained in $\mathcal{V}_2^\gamma$.

**Step 3.** Continuous function $H(\cdot) : \mathcal{M}_0^\lambda \times \mathcal{V}_2^\gamma \rightarrow \mathcal{M}_0^\lambda$.

(Argument for Step 3): Given equilibrium convexity for all traders, $v \in \mathbb{R}_+^N$, the consistency condition on $\tilde{M}$ is equivalent to $\tilde{M}$ being a fixed point of function $H(\cdot, \cdot) \equiv (H_1(\cdot, \cdot), \ldots, H_N(\cdot, \cdot))$, where each component $H_n(\cdot, \cdot)$ is given by

$$H_n(M, v) = \frac{1}{N-1} H(M_{n'} + v_{n'} | n' \neq n),$$

(15)

Note that $H : \mathbb{R}_+^N \times \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$. Define $\bar{\lambda} \equiv \tilde{\lambda} / (N-2)$. By assumption $N > 2$, therefore $\bar{\lambda}$ exists, $0 < \bar{\lambda} < \infty$. Let $\mathcal{M}_0^{\bar{\lambda}} \equiv \{M \in \mathbb{R}_+^N | M \leq 1\bar{\lambda}\}$ be a non-empty convex and compact box. We now argue that $H(\cdot, \cdot)$ maps $\mathcal{V}_2^\gamma \times \mathcal{M}_0^{\bar{\lambda}}$ into a subset $\mathcal{M}_0^{\bar{\lambda}}$. Define $A(\cdot, \cdot) : \mathbb{R}_+^N \times \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$ as $A(\cdot, \cdot) = (A_1(\cdot, \cdot), \ldots, A_N(\cdot, \cdot))$, where $A_n(\cdot, \cdot)$ is given by

$$A_n(M, v) = \frac{1}{N-1} \left[ \frac{1}{N-1} \sum_{n' \neq n} (M_{n'} + v_{n'}) \right].$$

(16)

Since for any $(M, v) \in \mathcal{M}_0^{\lambda} \times \mathcal{V}_2^\gamma$, coefficients satisfy $M_{n'} \leq \bar{\lambda}$ and $v_{n'} \leq \tilde{\gamma}$

$$A_n(M, v) \leq \frac{1}{N-1} (\bar{\lambda} + \tilde{\gamma}) = \frac{1}{N-1} \bar{\lambda} + \frac{N-2}{N-1} \tilde{\gamma} = \bar{\lambda},$$

(17)

where the last step holds by definition of $\bar{\lambda}$. The $n^{th}$ component of function $H(M, v) (A_n(M, v))$ is a harmonic (arithmetic) mean of elements $M_{n'} + v_{n'}$ for all $n' \in N / n$, discounted by factor $1 / (N - 1) < 1$. Therefore, by the standard harmonic-arithmetic mean inequality $H_n(M, v) \leq A_n(M, v) \leq \bar{\lambda}$, and hence $H : \mathcal{M}_0^{\lambda} \times \mathcal{V}_2^\gamma \rightarrow \mathcal{M}_0^{\lambda}$.

**Step 4.** Continuous function $T(\cdot) : \mathcal{M}_0^{\lambda} \rightarrow T$.

(Argument for Step 4): Allocation function $T(\cdot)$ is defined as follows

$$T(M) \equiv \arg \max_{t \in T} \left[ \sum_i (u_i(t_i + e_i) - \frac{1}{2} M_i (t_i)^2) - \sum_j (c_j(-t_j) + \frac{1}{2} M_j (t_j)^2) \right].$$

(18)
The function in (18) is continuous, strictly concave, and set $T$ is compact, therefore, the allocation function $T(\cdot)$ is well defined. In addition, $T(\cdot)$ is continuous by a maximum principle.

**Step 5. Existence of a fixed point**

(Argument for Step 5): Define $F : T \times V_2^x \times M_0^\lambda \to T \times V_2^x \times M_0^\lambda$ as $F(\cdot, \cdot, \cdot) = (T(\cdot), V(\cdot), H(\cdot))$. In Steps 2 - 4 we argued that all three components of the function are continuous; therefore, $F$ is also continuous. In addition, $T$, $V_2^x$ and $M_0^\lambda$ are non-empty, convex, and compact; therefore, their Cartesian product also has the three properties. Consequently, the Brouwer fixed point theorem applies, and there exists a point $(t^*, v^*, M^*)$ satisfying $F(t^*, v^*, M^*) = (t^*, v^*, M^*)$.

**Step 6. Trade $t^*$ is in the interior of $T$**

(Argument for Step 6): We now show that $t^*$ is in the interior of $T$,

1. $t^*_j \leq 0$ is not binding

The partial derivative of the maximized function in (18) with respect to $t_j$ is positive for all $t_j < 0$ and equal to zero for $t_j = 0$. If for some $j$, $t^*_j = 0$, then $t^*_j = 0$ must hold for all $j' \in J$. Otherwise, one could increase the value of the function by reducing the supply of $j'$ and offset this change by increased supply by firm $j$. With all firms producing zero, there must exist at least one consumer with non-positive trade $t^*_i$ (as all trades sum up to zero) and the partial derivative of (18) with respect to $t_i$ evaluated at $t^*_i$, is $u'_i(t^*_i + e_i) - M^*_i t^*_i > 0$ as $M^*_i \geq 0$. Consequently, marginally reducing $t^*_j$ (hence increasing the supply of $j$) and at the same time increasing $t^*_i$ by the same amount increases the value of the objective function and does not violate the constraints defining $T$. This contradicts the optimality of $t^*$ on $T$.

2. $t^*_j \geq -y^\text{max}_j$ is not binding.

Suppose for some $j$, $t^*_j = -y^\text{max}_j$. Given the definition of $y^\text{max}_j$, the partial derivative of the objective function (18) with respect to $t_j$ evaluated at $t^*_j$ satisfies $c_j(-t^*_j) - M^*_j t^*_j \geq \bar{u}$. In addition, by the fact that all trades sum up to zero, there must be at least one consumer $i$ with strictly positive trade $t^*_i > 0$. For such a consumers derivative satisfies $u'_i(t^*_i + e_i) - M^*_i t^*_i < u'_i(e_i) \leq \bar{u}$. The two inequalities imply that marginally increasing $t^*_j$ (cutting on the supply) and reducing the demand $t^*_j$ by the same amount increases value of (18) and such change does not violate the constraints. This contradicts optimality of $t^*$.

3. $t^*_i \geq -e_i + \frac{1}{2}x^\text{min}_i$ is not binding.

Suppose for some $i$, $t^*_i = -e_i + \frac{1}{2}x^\text{max}_i$. By construction $x^\text{max}_i < e_i$, therefore $t^*_i < 0$. In addition, $u'_i(t^*_i + e_i) - M^*_i t^*_i > \bar{u}$. Since the trades sum up to zero, there must exist some other consumer $i'$ with $t^*_i > 0$ and for such consumer $u'_i(t^*_i' + e_{i'}) - M^*_i t^*_i' < \bar{u}$. The two inequalities imply that increasing the demand of $i$ and offsetting this change by the reduction of trade $i'$ does not violate the constraints and increases the value of (18). This contradicts optimality of $t^*$ on $T$. 

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Step 7. Fixed point \((t^*, v^*, M^*)\) defines a slope-taking equilibrium \((p^*, t^*, M^*)\) in \(E\).

(Argument for Step 7): The objective function in (18) is strictly concave, which, together with the interiority demonstrated in Step 6, implies that \(t^*\) maximizes (18) on a larger set of trades defined by equality constraints \(\sum_n t_n = 0\). It follows that there exists a Lagrangian multiplier, \(p^*\), such that \(t^*\) solves the unconstrained maximization problem, with the objective function (18) augmented by the additional term (constraints) \(-p^*\sum_n t_n\). The first-order optimality conditions for this program are as follows: for any consumer, \(u_i'(t_i^* + e_i) - M_i^* t_i^* = p^*\), and for any firm \(j\) the condition becomes \(c_j''(-t_j^*) - M_j^* t_j^* = p^*\). For each \(i\), define \(m_i^* = -p^* t_i^*\) and for each firm \(m_j^* = c_j(-t_j^*)\). The two conditions and two definitions are necessary and sufficient for \(t_n^*\) to be optimal at prices \(p^*\) and \(M_n^*\) (see (12)). Since \(t^* \in T\), the markets clear, \(\sum_n t_n^* = 0\). Finally, by interiority of trades \(v_i^* = -u_i''(t_i^* + e_i)\) and \(v_j^* = c_j''(-t_j^*)\), therefore \(M^* = H(M^*, v^*)\), at \(t^*\), which is sufficient for \(M^*\) to be mutually consistent. This proves the existence of a slope-taking equilibrium \((p^*, t^*, M^*)\) when \(N > 2\). To see that \(p^* > 0\), observe that, if there exists some firm for which \(t_j^* < 0\), then \(\bar{p} = c_j'(-t_j^*) - M_j^* t_j^* > 0\). Otherwise, there must exist at least one consumer with \(t_i^* \leq 0\) and hence \(\bar{p} = u_i'(t_i^* + e_i) - M_i^* t_i^* > 0\).

Step 8. Nonexistence for \(N = 2\).

(Argument for Claim 8): In proving the existence of equilibrium, when defining the upper bound on the price impacts, \(\lambda\), we used the fact that \(N > 2\). In this step, we show that this condition is also necessary for the existence of equilibrium. To see it, observe that, in the case of two traders the consistency conditions are \(\bar{M}_1 = \bar{M}_2 = u_2''\) and \(\bar{M}_2 = \bar{M}_1 - u_1''\). The two equations imply \(u_1'' = -u_2''\), which is impossible, since utility functions are strictly convex.

**Proof:** Theorem 3. For any \(n\), let \(\delta_n', \delta_n'' \in \mathbb{R}_{++}\) be the positive scalars, \(\delta_n \equiv (\delta_n', \delta_n'')\) and \(\delta \equiv \{\delta_n\}_n\). Fix economy \(E = (u, e, c)\). A vector \(\delta\) uniquely defines a perturbed economy \(E_\delta = (\hat{u}, e, \hat{c})\), the economy with preferences and technology modified as follows

\[
\hat{u}_i(x_i) = u_i(x_i) + \delta_i' x_i + \delta_i'' (x_i)^2, \tag{19}
\]

and for firm \(j\)

\[
\hat{c}_j(y_j) = c_j(y_j) + \delta_j' y_j + \delta_j'' (y_j)^2. \tag{20}
\]

Consider any open ball \(P \equiv B(0, r) \subset \mathbb{R}^{2N}\) with radius \(r > 0\) sufficiently small to guarantee that, for any perturbation \(\delta \in P\) preserves strict monotonicity and strict concavity (convexity) of the utility (cost) functions on the set of relevant trades \(T_\delta\) (see (13)). By continuity of utility and cost functions, such radius exists. Ball \(P\) spans a family of perturbed economies around original economy \(E\). Observe that \(P\) is a connected set with a positive Lebesgue measure in \(\mathbb{R}^{2N}\). For any perturbed economy, \(E_\delta\), with \(\delta \in P\), the set of all slope-taking equilibria is defined as a set of critical points of the system of equations

\[
\Psi_\delta(\bar{p}, \bar{t}, \bar{M}) = 0, \tag{21}
\]
where
\[ \Psi_\delta(\bar{p}, \bar{t}, \bar{M}) \equiv \begin{pmatrix}
\hat{u}_i' - \bar{M}_i \bar{t}_i - \bar{p} & \text{for all } i \in I \\
\hat{c}_j'(-\bar{t}_j) - \bar{M}_j \bar{t}_j - \bar{p} & \text{for all } j \in J \\
\bar{M}_n - (\sum_{n' \neq n}(\bar{M}_{n'} + \hat{V}_{n'}(\bar{t}_{n'}))^{-1})^{-1} & \text{for all } n \in N
\end{pmatrix} , \quad (22) \]
and \( \hat{V}_i(\bar{t}_i) \equiv -\hat{u}_i''(\bar{t}_i + e_i) \), and \( \hat{V}_j(\bar{t}_j) = \hat{c}_j''(-\bar{t}_j) \). Given \( \delta \) fixed equation (21) constitutes a system of \( 2N + 1 \) equations and the same number of unknowns. In economy \( \mathcal{E}_\delta \), all slope-taking equilibria are locally unique if \( \Psi_\delta(\cdot) \) is transverse to 0. Unfortunately, this is not necessarily true for arbitrary \( \mathcal{E}_\delta \). We will argue, however, that transversality holds for almost all economies in \( \mathcal{P} \), and hence the lack of local uniqueness occurs only as a degenerate case. We first extend function \( \Psi_\delta(\cdot) \) to a domain that includes perturbation parameters \( \delta \). Next we show that the extended function \( \Psi(\cdot, \cdot) \) is transverse to zero, and, finally, using a transversality theorem, we establish the transversality of \( \Psi_\delta(\cdot) \) for almost all \( \delta \), and hence we demonstrate the generic determinacy of an equilibrium.

We prove the transversality of \( \Psi(\cdot, \cdot) \) using a method of perturbation. First observe that any of the first-order conditions (\( N \) first equations) can be perturbed by varying corresponding \( \delta_n' \). Since \( \delta_n' \) enters exactly one equation in the whole system, such perturbation does not affect any other equation. Each of the equations in the second group, defining consistent price impacts, can be perturbed by changing \( \bar{M}_n \) to \( \bar{M}_n \) and offsetting this change by the adjustment of \( \delta_n'' \) so that the sum of the two elements stays constant \( (\bar{M}_n + \bar{V}_n(t_n)) = \hat{M}_n + \bar{V}_n(t_n) + \delta_n'' \) and simultaneously by changing \( \delta_n' \) to compensate the effects of the change in \( \delta_n'' \) in the first group of equations. Finally, the market clearing condition can be perturbed by varying \( \bar{t}_n \) for some \( n \) and compensating this change by adjusting \( \delta_n \) to keep the values of equations in the first two groups unchanged. This proves the transversality of \( \Psi(\bar{p}, \bar{t}, M, \delta) \) to 0. Applying the transversality theorem, one can establish that there exists an open subset \( \hat{\mathcal{P}} \subset \mathcal{P} \) with full Lebesgue measure of \( \mathcal{P} \), such that, for any \( \delta \in \hat{\mathcal{P}} \) restriction \( \Psi_\delta(\bar{p}, \bar{t}, M, \delta) \) is transverse to zero. Consequentially, in any perturbed economy \( \mathcal{E}_\delta \) from \( \hat{\mathcal{P}} \), all equilibria are locally unique.

**PROOF: THEOREM 4.** Let \( (\bar{p}, \bar{t}, \bar{M}) \) be a slope-taking equilibrium. For any \( i \in I \), the necessary and sufficient condition of optimality of \( \bar{t}_i \) given \( \bar{p} \) and \( \bar{M}_i \) is \( u_i'(\bar{t}_i + e_i) = \bar{p} + \bar{M}_i \bar{t}_i \) and similarly for firm \( j \in J \) the condition becomes \( c_j'(-\bar{t}_j) = \bar{p} + \bar{M}_j \bar{t}_j \). In a slope-taking equilibrium, allocation is Pareto efficient if, for any \( n = i, j \), the marginal utilities and costs of all traders coincide, \( u_i'(\bar{t}_i + e_i) = c_j'(-\bar{t}_j) \). Also, since the consistency condition implies that price impacts are discounted harmonic averages of strictly positive scalars, the equilibrium price impacts are strictly positive.

**Step 1.** Pareto efficiency of equilibrium implies \( J = 0 \) and efficiency of endowments.

(Argument for Step 1): In equilibrium, trades are optimal for all \( n \in N \), and hence marginal utilities and costs are equalized with marginal expenditure. This with Pareto efficiency implies that \( \bar{M}_i \bar{t}_i = \bar{M}_j \bar{t}_j = x \) for all \( i, j \). Since price impacts are positive, for any \( n = i, j \), the trade can be written as \( \bar{t}_n = x/\bar{M}_n \). Then by the market clearing condition, \( 0 = \sum_n x/\bar{M}_n \), and hence \( x = (\sum_n (\bar{M}_n)^{-1})^{-1}0 = 0 \). Consequently, for any \( n \) equilibrium trades are given by \( \bar{t}_n = x/\bar{M}_n = 0 \).
Consider an economy with $J > 0$. From the optimality conditions for a firm, it follows that $\bar{p} = c'_j(0) - \bar{M}_j0 = 0$ which, in turn by the optimality condition for a consumer implies $u'_i(e_i) = 0$, thus contradicting strict monotonicity of preferences. It follows that, only pure exchange economy, $J = 0$, might be consistent with Pareto efficiency. In such economy, given $\bar{t}_i = 0$ for all $i \in I$, the optimality condition implies that $u'_i(e_i) = \bar{p}$, and hence with no trade marginal utilities for all consumers coincide. It follows that initial allocation is Pareto efficient.

**Step 2.** $J = 0$ and efficient endowments imply efficient equilibrium allocation.

(Argument for Step 1): Fix an arbitrary equilibrium $(\bar{p}, \bar{t}, \bar{M})$. The first-order conditions imply that $\bar{t}$ also solves

$$\bar{t} = \arg \max_t \sum_i \left\{ u_i(t_i + e_i) - \frac{1}{2} \bar{M}_i(t_i)^2 \right\} - \bar{p} \sum_i t_i$$

(23)

The objective function is strictly concave, and hence there exists at most one $\bar{t}$ that is optimal. However, since $u'_i(e_i) = u'_i(e_i)$, no trade is such a solution and hence it is a unique equilibrium trade consistent with $\bar{p}, \bar{M}$. It follows that allocation coincides with endowments and hence equilibrium allocation is Pareto efficient.

**Proof: Theorem 5.** In this proof we use notation from the proof of Theorem 2. By similar arguments as in Step 6 of this proof, the only trades consistent with slope-taking equilibrium are within set $T \subset \mathbb{R}_+^N$. Given $T$, the second derivatives of the utility and cost functions possibly observed in equilibrium are within a compact set $V^\gamma_{2^l}$, where the set and two scalars are defined in Step 3. In the following arguments, I focus on a $k$–replica of the economy, and therefore I assume that there are $k$ traders of each type $n$.

**Step 1.** In a $k$-replica economy, $\bar{M}_n \leq \frac{\gamma}{N \times k - 2}$, for any $n$.

(Argument for Step 1): In $k$-replica, each agent trades with $k$ trades of each other type (there are $N - 1$, other types) and $k - 1$ traders of its own type. Given $v \in V^\gamma_{2^l}$, price impacts $\bar{M}$ are consistent if, and only if, they are a fixed point of map $H$ where the $n^\text{th}$ element, $H_n(\cdot)$, is given by

$$H_n(\bar{M}, v) = \left[ k \sum_{n' \neq n} (\bar{M}_{n'} + v_{n'})^{-1} + (k - 1)(\bar{M}_n + v_n)^{-1} \right]^{-1}.$$  

(24)

For each type $n$ define map

$$A_n(\bar{M}, v) = \frac{1}{N \times k - 1} \left( \frac{1}{N \times k - 1} \left( k \sum_{n' \neq n} (\bar{M}_{n'} + v_{n'}) + (k - 1)(\bar{M}_n + v_n) \right) \right).$$

(25)

Since in equilibrium $v \in V^\gamma_{2^l}$, equation (25) implies the following inequality $A_n(\bar{M}, v) \leq \frac{\max_n \bar{M}_n + \gamma}{N \times k - 1}$, and by the harmonic-arithmetic mean inequality $H_n(\bar{M}, v) \leq \frac{\max_n \bar{M}_n + \gamma}{N \times k - 1}$. Since vector $\bar{M}$ is consistent if, and only if, $\bar{M} = H(\bar{M}, v)$, therefore the expression is also an upper bound on $\max_n \bar{M}_n \leq \frac{\max_n \bar{M}_n + \gamma}{N \times k - 1}$. Solving for $\max_n \bar{M}_n$ gives an upper bound on equilibrium price impacts $\bar{M}_n \leq \frac{\gamma}{N \times k - 2}$ for any $n$. 

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Step 2. Convergence to a competitive equilibrium.

(Argument for Step 2): For any $\varepsilon > 0$, we define $k(\varepsilon) \equiv \left(\frac{2}{\gamma} + 2\right) \frac{1}{N}$ (ignore the integer problem). Given the result from the previous step, for such $k(\varepsilon)$ in any slope-taking equilibrium consistent price impacts $\tilde{M}_n$ satisfy $\tilde{M}_n \leq \frac{1}{N \times k - 2} \gamma = \varepsilon$ for any $n$. In addition, for any $\tilde{M}$ there exists a unique $\tilde{t}$ and $\tilde{p}$ such that $(\tilde{p}, \tilde{t}, \tilde{M})$ is a slope-taking equilibrium. To see it, observe that optimality for all $i$ and $j$ requires that

$$u_i'(\tilde{t}_i + e_i) - \tilde{M}_i \tilde{t}_i = c_j'(-\tilde{t}_j) - \tilde{M}_j \tilde{t}_j. \quad (26)$$

Since for all traders augmented marginal utility or cost is decreasing in $\tilde{t}_n$ and $\sum_n \tilde{t}_n = 0$, there is a unique $\tilde{t}$ that solves (26) subject to market clearing, given fixed value $\tilde{M}$. The equilibrium price is then given by $\tilde{p} = u_i'(\tilde{t}_i + e_i) - \tilde{M}_i \tilde{t}_i$ for arbitrary $i$. This shows that there exist functions, $\tilde{t}_n(\tilde{M})$ for any $n \in \mathcal{N}$ and $\tilde{p}(\tilde{M})$, mapping the vectors of price impacts into equilibrium trades and prices. Because (26) is continuous, by the standard argument $\tilde{t}(\tilde{M})$ and $\tilde{p}(\tilde{M})$ are also continuous in $\tilde{M}$. Therefore, for any $\varepsilon$ there exists $\varepsilon'$ such that for any $\tilde{M} \leq \varepsilon'$, one obtains $\|\tilde{t}_n(\tilde{M}) - \tilde{t}_n(0)\| \leq \varepsilon$ and $\|\tilde{p}(\tilde{M}) - \tilde{p}(0)\| \leq \varepsilon$. However, $\tilde{t}_n(0)$ and $\tilde{p}(0)$ correspond to the competitive trade and price. To conclude for any $\varepsilon$, define $k(\varepsilon) \equiv \max \left( \left(\frac{2}{\gamma} + 2\right) / N, \left(\frac{2}{\gamma} + 2\right) / N \right)$. It follows that for any $k \geq k(\varepsilon)$, in a $k$-replica economy in any slope-taking equilibrium is in $\varepsilon$-distance from a competitive equilibrium.

PROOF. PROPOSITION 1. The proof consists of the following steps: First, I prove two useful facts: In Step 1, we show that the convexity of buyers’ utility function is always greater than those of the sellers, for any arbitrary level of price impact. Then in Step 2, we argue that, in the considered slope-taking equilibrium, the price impacts of a seller exceed those of a buyer. The two results are used in the proof of the main result in Step 3.

For any (fixed) equilibrium $(\tilde{p}, \tilde{t}, \tilde{M})$, the set of all consumers is partitioned into two groups: buyers, $\mathcal{I}_b = \{ i \in \mathcal{I} | \tilde{t}_i \geq 0 \}$, and sellers, $\mathcal{I}_s = \{ i \in \mathcal{I} | \tilde{t}_i < 0 \}$. Nonactive traders $\mathcal{I}_{na} = \{ i \in \mathcal{I} | \tilde{t}_i = 0 \}$ are “buyers” for which trades are equal to zero. Since the initial allocation is not Pareto efficient, the sets $\mathcal{I}_s$ and $\mathcal{I}_b$ are non-empty. For any $p \geq 0$ and $M_i \geq 0$, we define a slope-taking convexity of trader $i$ as a second derivative evaluated at the slope-taking trade $\tilde{t}_i(p, M_i)$,

$$v_i(p, M_i) \equiv |u_i''(\tilde{t}_i(p, M_i) + e_i)|. \quad (27)$$

In Step 1 we show that, at equilibrium price $\tilde{p}$, and positive but otherwise arbitrary price impacts $M_b$, $M_{na}$, $M_s$, buyers always have higher slope-taking convexity, given $u'' > 0$.

Step 1. For any $b \in \mathcal{I}_b$, $s \in \mathcal{I}_s$, and na $\in \mathcal{I}_{na}$, at equilibrium price $\tilde{p}$, and arbitrary positive $M_b$, $M_{na}$, $M_s$ slope-taking convexities satisfy

$$v_b(\tilde{p}, M_b) \geq v_{na}(\tilde{p}, M_{na}) > v_s(\tilde{p}, M_s) > 0. \quad (28)$$

(Argument for Step 1): slope-taking demand $\tilde{t}_i(\tilde{p}, M_i)$ is defined by the equality of marginal utility and the marginal revenue

$$u_i'(\tilde{t}_i + e_i) = \tilde{p} + M_i \tilde{t}_i. \quad (29)$$
By definition, \( \tilde{t}_b(\bar{p}, \bar{M}_b) \geq 0 \), and it follows that \( u'_b(e_b) \geq \bar{p} \). Therefore, from (29) at equilibrium price, buyers’ slope-taking trade is non-negative \( \tilde{t}_b(\bar{p}, M_b) \geq 0 \) for any \( M_b \geq 0 \). Similarly, for a seller it is strictly negative. The derivative of a slope-taking demand function with respect to \( M_i \) is given by

\[
\frac{\partial \tilde{t}_i(\bar{p}, M_i)}{\partial M_i} = -\frac{\tilde{t}_i(\bar{p}, M_i)}{M_i + v_i(\bar{p}, M_i)}.
\] (30)

Since the denominator is always strictly positive, the slope-taking trade is non-increasing on the whole domain in \( M_i \) for the buyers (constant for non-active traders) and increasing for the sellers. Consequently, for arbitrary price impact \( M_b > 0 \), the slope-taking demand of a buyer is bracketed by a competitive and zero trade \( \tilde{t}_b(\bar{p}, 0) > \tilde{t}_b(\bar{p}, M_b) \geq 0 \). By analogous argument, for each seller, the trade is bracketed by \( \tilde{t}_s(\bar{p}, 0) < \tilde{t}_s(\bar{p}, M_s) < 0 \). With identical quasilinear preferences, the competitive consumption at \( \bar{p} \) is the same for all traders

\[
\tilde{t}_b(\bar{p}, 0) + e_b = e_{na} = \tilde{t}_s(\bar{p}, 0) + e_s,
\] (31)

For any strictly positive (but otherwise arbitrary) price impacts \( M_b, M_{na}, M_s \), this equation and trade bracketing conditions imply the following inequality

\[
\tilde{t}_b(\bar{p}, M_b) + e_b < e_{na} < \tilde{t}_s(\bar{p}, M_s) + e_s.
\] (32)

With the second derivative of the utility function increasing in consumption, \( (u'' > 0) \), the relation between slope-taking consumptions (32) implies the following ranking of convexities:

\[
v_b(\bar{p}, M_b) \geq v_{na}(\bar{p}, M_{na}) > v_s(\bar{p}, M_s),
\] (33)

for arbitrary, strictly positive price impacts \( M_b, M_{na}, M_s \).

**Step 2.** With \( u'' > 0 \), for any \( b \in \mathcal{I}_b, s \in \mathcal{I}_s \), and \( na \in \mathcal{I}_a \), the equilibrium price impacts satisfy

\[
\bar{M}_s > \bar{M}_{na} \geq \bar{M}_b.
\] (34)

(Argument for Step 2): The system of consistent \( \bar{M} \) for any \( i \) satisfies

\[
\bar{M}_i = \frac{1}{I-1} \mathcal{H}(\bar{M}_{i'} + v(\bar{p}, \bar{M}_{i'})|i' \neq i).
\] (35)

\( I \) such conditions imply that ranking of \( \bar{M}_i \) coincides with a reversed ranking of equilibrium convexities. Suppose for two traders \( i \) and \( i' \), that \( v_i(\bar{p}, \bar{M}_i) > v_{i'}(\bar{p}, \bar{M}_{i'}) \) and \( \bar{M}_i \geq \bar{M}_{i'} \). Condition (35) shows that \( \bar{M}_i \) is a harmonic average of \( I-2 \) elements that also define \( \bar{M}_{i'} \) and one element that is strictly smaller as by assumption \( \bar{M}_{i'} + v_{i'}(\bar{p}, \bar{M}_{i'}) < \bar{M}_i + v_i(\bar{p}, \bar{M}_i) \) and hence \( \bar{M}_{i'} > \bar{M}_i \). Since harmonic average is monotone, this implies \( \bar{M}_i < \bar{M}_{i'} \), which is a contradiction. This, together with the result from Step 1, is sufficient for the result of Step 2.

**Step 3.** \( u'' > 0 \) implies \( \bar{p} > \bar{p}^{Walras} \).
(Argument for Step 3): Let \( \tilde{M} \) be any scalar satisfying \( \tilde{M}_a > \tilde{M} > \tilde{M}_b \), for all \( b \in \mathcal{I}_b \) and \( s \in \mathcal{I}_s \).

By the result from Step 2, such scalar exists. The strict monotonicty of \( \tilde{t}_i(p, \cdot) \) in \( M_i \) shown in (30) and ranking of price impacts imply that aggregate excess demand evaluated at \( \tilde{M} \) is negative \( D(\tilde{M}) = \sum_i \tilde{t}_i(p, \tilde{M}) < 0 \). In addition, for arbitrary common scalar \( M \), a partial derivative of \( D(\cdot) \) with respect to \( M \) is given by

\[
\frac{\partial D(M)}{\partial M} = -\sum_i \tilde{t}_i(p, M) \frac{M + v_i(p, M)}{M + \tilde{v}}.
\]

By \( \tilde{v} > 0 \), I denote a convexity of the utility function evaluated at the competitive consumption given \( \tilde{p} \), \( (\tilde{p} = u'(x_i)) \). By the result from Step 1, \( v_s(\tilde{p}, M) < \tilde{v} < v_b(\tilde{p}, M) \) for any \( M > 0 \).

Replacing the convexities with a common value \( \tilde{v} \) in the derivative of \( D(M) \) gives a lower bound of this derivative

\[
\frac{\partial D(M)}{\partial M} > -\sum_i \tilde{t}_i(M) \frac{M + \tilde{v}}{M + \tilde{v}} = -\frac{D(M)}{M + \tilde{v}}.
\]

The inequality results from the fact that, for the buyers (positive trades), replacing \( v_i(\tilde{p}, M) \) with \( \tilde{v} \) decreases the denominator, while for sellers (negative trade), the denominator goes up. Inequality (37) defines a lower bound on the slope of \( D(M) \). Observe that \( D(M) \) is upward sloping in \( M \) as long as \( D(M) < 0 \) and \( M \geq 0 \). Since excess demand evaluated at the intermediate level \( \tilde{M} \) is negative; \( D(\tilde{M}) < 0 \), therefore \( D(M) \) is increasing in \( M \) around this price impact. Now we show that it is nondecreasing on the whole interval \([0, \tilde{M}]\). To see it, consider a subset of all \( M \) from the interval, for which the excess demand is greater than zero \( D(M) > 0 \). This subset is a pre-image of a closed set by a continuous function and hence is closed. Since it is also bounded, it is also compact. Therefore, it must contain an upper bound \( \bar{M} \). By construction, on the interval \( [\bar{M}, \tilde{M}] \), function \( D(M) < 0 \), and hence its derivative is strictly positive – the function is increasing. This implies that, for any \( M \) from this interval, \( D(M) < D(\tilde{M}) < 0 \) and hence by continuity, at \( \bar{M} \), function \( D(\bar{M}) \leq D(\tilde{M}) < 0 \); however, this contradicts the assumption that \( D(\bar{M}) \geq 0 \). Consequently, for all \( M \in (0, \tilde{M}] \), excess demand \( D(M) \) is strictly negative and hence is increasing in \( M \). It follows that \( D(0) < D(\tilde{M}) < 0 \). But this shows that at the slope-taking equilibrium price we observe excess supply \( D(0) = \sum_i \tilde{t}_i(\bar{p}, 0) < 0 \). The derivative of \( D(p, 0) \) with respect to \( p \) (for any price) is \( \frac{\partial D(p, 0)}{\partial p} = -\sum_i \frac{1}{v_i(p, 0)} < 0 \); hence, it is decreasing in price. The unique competitive price solves \( D(p^{Walras}, 0) = 0 \); therefore, it satisfies \( \bar{p} > p^{Walras} \). Consequently, we observe a positive price bias in a slope-taking equilibrium. The proofs for \( u_i^{\prime\prime} < 0 \) and \( u_i^{\prime\prime} = 0 \) are analogous to the presented case, and hence we leave it to the reader.

**Proof: Proposition 2.** Consider an economy with \( J \) firms with technology characterized by a constant marginal cost. The consistency condition for the price impacts and symmetry across firms imply \( \tilde{M}_j = ((J - 1)(M_j)^{-1} + \sum_i(M_i + v_j)^{-1})^{-1} \), which is equivalent to \( (M_j)^{-1}(2 - J) = \sum_i(\tilde{M}_i + v_i)^{-1} \). If \( J \geq 2 \), then the left-hand side of the equation is less or equal to zero. But this is a contradiction, since the right-hand side is strictly positive.
Let \((10,0)\) be endowment of numéraire and non-numéraire of a buyer, and \((0,1)\) is the seller’s endowment of the two commodities. In the slope-taking equilibrium, slopes of two budget lines, determined by marginal payments, differ for both types of trades and so do marginal rates of substitution. Thus the allocation is Pareto inefficient.