

# Slope-Takers in Anonymous Markets\*

Daniel Quint and Marek Weretka<sup>†</sup>

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## Abstract

We present a learning-based selection argument for Linear Bayesian Nash equilibrium in a Walrasian auction. Endowments vary stochastically; traders model residual supply as linear, estimate its slope from past trade data, and periodically update these estimates. With quadratic preferences, this learning process converges to the unique LBN. In an example with non-quadratic preferences, it converges to a steady state close to a particular equilibrium of the corresponding deterministic setting; strategies played are not an equilibrium, but utility sacrificed is negligible. Anonymity and statistical learning therefore support use of LBN under quadratic utility, and motivate a related concept under non-quadratic utility.

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<sup>†</sup>University of Wisconsin-Madison, Department of Economics, 1180 Observatory Drive, Madison, WI 53706. E-mail: [dquint@ssc.wisc.edu](mailto:dquint@ssc.wisc.edu) and [weretka@wisc.edu](mailto:weretka@wisc.edu).

# 1 Introduction

Real-world financial markets are not perfectly competitive. Large traders can't trade arbitrary quantities at a fixed market price; the orders they place impact the terms they trade at. This price impact has been widely documented for institutional investors,<sup>1</sup> and traders must account for it to trade optimally.

The theoretical literature on imperfectly-competitive financial markets uses the stylized model of a Walrasian auctions, where traders submit downward-sloping demand schedules and trades occur at the resulting market-clearing price.<sup>2</sup> Since [Klemperer and Meyer \(1989\)](#), however, it is well-known that the set of (Bayesian) Nash equilibria in the Walrasian auction can be large, often infinite, and depends in a complex way on the distributions of traders' endowments. Without selection criteria, the predictive power of the theory is limited. For this reason, the financial literature restricts attention to quadratic preferences, and refines the equilibrium set by focusing on *Linear* Bayesian Nash equilibrium (LBN).<sup>3</sup> Within the quadratic framework, an LBN exists and is unique under fairly general conditions, so the LBN solution concept allows for tight predictions. This model has produced numerous insights regarding the design and regulation of modern financial markets that have become standard tools.

This quadratic/LBN approach, however, raises three significant concerns.

First, Bayesian Nash equilibrium supposes common knowledge among traders of the “game” being played – here, the set of traders and their payoff functions. Most real-world financial markets are not completely transparent, however, and many are fully anonymous, so traders often lack complete information about their counterparts. Instead, they estimate their price impact from available data on prices and individual trades, often relying on “market impact models” provided by Citigroup, EQ International, ITG, MCI Barra, OptiMark, and others.

Second, since Bayesian Nash outcomes can vary over a wide range, the predictive power of LBN comes not from the usual assumption of traders' rationality, but rather from the additional *ad hoc* assumption of bid linearity imposed by a modeler. The existing theory does not give convincing arguments for the LBN equilibrium being more plausible than other

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<sup>1</sup>See [Holthausen et al. \(1987\)](#), [Chan and Lakonishok \(1993, 1995\)](#), [Keim and Madhavan \(1995, 1996, 1998\)](#), and more recently [Frazzini et al. \(2018\)](#), among others.

<sup>2</sup>This framework originated with [Wilson \(1979\)](#), [Klemperer and Meyer \(1989\)](#), and [Kyle \(1989\)](#), and was further developed by [Vayanos \(1999\)](#), [Vives \(1999, 2011\)](#), [Weretka \(2011\)](#), [Rostek and Weretka \(2012, 2015\)](#), [Bergemann et al. \(2021\)](#), and too many others to cite all relevant papers; for an in-depth survey, see [Rostek and Yoon \(2020\)](#).

<sup>3</sup>Notable exceptions are [Klemperer and Meyer \(1989\)](#) and [Weretka \(2011\)](#), who study settings without asymmetry of information: the former considers producers with identical cost functions facing uncertain demand, the latter traders with heterogenous preferences and deterministic endowments.

Bayesian Nash equilibria.

And third, LBN is well-defined only when traders' marginal utilities are linear in their quantity traded, or when utility is quadratic. But quadratic preferences have been extensively tested empirically, and are considered unrealistic for several reasons,<sup>4</sup> and the profession therefore tends to be skeptical of results that rely on the assumption of quadratic utility (Browning and Lusardi (1996)).<sup>5</sup>

In this paper, we consider a model of large traders in anonymous markets that addresses these concerns. As is standard, we model a market as a Walrasian auction. Instead of *a priori* knowing the game, we assume that traders model their environment in a simple linear way, use statistical tools to estimate their price impacts from individual data, and trade optimally given their estimates. We call such traders *slope-takers*, because rather than taking the market price as given, they take as given the slope of the residual supply curve they face and best-respond to it.

We offer two sets of results. First, we show that if traders have quadratic preferences and periodically re-estimate their market impact based on recent trading data, market interactions converge to the LBN equilibrium. We therefore provide a behavioral foundation for the standard solution concept in anonymous markets with quadratic preferences.

Second, we extend our analysis to more natural, non-quadratic preferences. With deterministic endowments, one can still define the slope-taking equilibrium as a refinement of BNE, although the equilibrium need not be unique (Weretka (2011)). Utilizing numerical methods, we explore an example and show that when random endowments don't vary too much around a given level, bidding strategies of slope-taking traders converge fairly quickly to a steady state, which closely approximates the equilibrium of the corresponding deterministic setting. Of course, in this steady state with random endowments, traders are relying on a misspecified model of the market, and hence are not playing true best-responses for each realization of others' endowments.<sup>6</sup> However, we show that the loss in expected utility is negligible, even relative to the benchmark where they know other traders' exact strategies

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<sup>4</sup>Quadratic utility fails to generate precautionary behavior commonly observed in actual consumers (Blanchard and Mankiw (1988); Caballero (1990)). It also implies increasing absolute risk aversion, which would lead to predictions that less-wealthy consumers hold riskier assets and that the elasticity of intertemporal substitution decreases in consumption, contradicting empirical findings of Blundell et al. (1994), Attanasio and Browning (1993), and Atkeson and Ogaki (1996).

<sup>5</sup>A different rationalization for quadratic utility common in the finance literature is that consumers might have CARA utility and face normally-distributed asset payoffs, leading to mean-variance preferences; but asset returns tend to have much heavier tails than a normal distribution would suggest.

<sup>6</sup>In this way, our exercise is analogous to Esponda and Pouzo (2016) who study learning in games when players have misspecified models, as well as the macroeconomic literature on adaptive OLS learning in the DSG model (Marcet and Sargent (1989a,b); Evans and Honkapohja (1995, 2012))) and consumers who use linear equations to approximate nonlinear relationships (Hommes and Sorger (1998) and Branch and McGough (2005)).

and realized endowments. In a cost-benefit sense, then, if the variation in endowments isn't too large, it doesn't appear "worth it" for traders to develop a more complex market model, since best-responding to an easily-estimated linear model allows them to achieve very nearly the same level of expected utility.

Thus, within the quadratic framework, we offer a learning-based argument that justifies the linear refinement criterion for Bayesian Nash equilibrium; and we show that the same learning argument motivates an extension of an analogous solution concept to markets with non-quadratic preferences.

## 2 Quadratic Utility

### 2.1 Environment

The environment is standard. There are  $I > 2$  agents with utility which is quasilinear in money  $m_i$  and quadratic in a tradable good  $x_i$ ,

$$U_i(x_i, m_i) = u_i(x_i) - m_i = \beta_i x_i - \frac{v_i}{2} x_i^2 + m_i \quad (1)$$

where  $\beta_i$  and  $v_i$  are parameters. Agents have endowments  $e_i$  of the traded good; endowments of money are normalized to zero. We model trade as a Walrasian auction: after learning their endowments, each agent submits a demand schedule  $d_i : \mathbb{R} \rightarrow \mathbb{R}$  giving their demand as a function of price. Given the profile  $d(\cdot) = \{d_i(\cdot)\}_{i \in I}$  of demand schedules submitted, the market-clearing price  $\bar{p}$  is determined by  $\sum_{i \in I} d_i(\bar{p}) = 0$ ; each trader  $i$  buys  $d_i(\bar{p})$  units of the good at that price, earning utility  $u_i(e_i + d_i(\bar{p})) - \bar{p}d_i(\bar{p})$ .

### 2.2 Solution Concept

We first review two standard solution concepts. Walrasian, or competitive, equilibrium is defined in this environment by assuming that traders take price as given, choosing demand  $d_i(p) = \arg \max_q \{u_i(e_i + q) - qp\}$  at each price. This gives a natural benchmark, suitable for large markets, and does not require agents to know anything about the other traders or their strategies.

Walrasian equilibrium is not a suitable solution in thin markets, where traders face elastic residual supply and can gain by accounting for their impact on price. Beginning with [Wilson \(1979\)](#), [Klemperer and Meyer \(1989\)](#), and [Kyle \(1989\)](#), people have modeled traders as acting strategically, looking at profiles of demand schedules  $\{d_i(\cdot)\}_i$  that constitute Bayesian Nash equilibria. This approach accommodates strategically sophisticated traders in thin markets,

but has two substantial shortcomings. First, it assumes that players know the primitives of the game, i.e., the identities and strategies of the other traders – unlikely in anonymous markets. And second, this approach is known to lead to extreme multiplicity of equilibrium. What has become standard, then, is to focus on the unique equilibrium in which players’ demand schedules are linear. While expedient, this assumption is typically made without justification.

The solution concept we consider is *slope taking equilibrium*. Rather than knowing all the primitives of the environment, each trader  $i$  operates under the hypothesis that he faces a linear residual supply curve

$$p_i(q_i) = \alpha_i + \lambda_i q_i + \varepsilon_i \quad (2)$$

where  $\lambda_i$  is trader  $i$ ’s *price impact*. Thus, instead of assuming traders are price-takers (as in Walrasian equilibrium) or infer their price impact from a complete understanding of the game (as in LBN), we assume traders are strategic but account for the impact of their trades in a parsimonious and reduced-form way. “Slope taking” refers to the fact that rather than taking price as given, traders take as given the slope of the residual supply curve they face. (We will consider how traders learn their price impacts in the next section.)

Given linear beliefs (2) and objective function  $u_i(e_i + q_i) - q_i p_i(q_i)$ , trader  $i$  best-responds with the demand schedule

$$d_i^{ST}(p, \lambda_i) = \frac{\beta_i - v_i e_i - p}{v_i + \lambda_i} \quad (3)$$

which we call a slope-taking strategy. Note that trader  $i$ ’s beliefs about the distribution of  $\varepsilon_i$  (or the value of  $\alpha_i$ ) in (2) don’t matter, as the demand profile  $d_i^{ST}$  allows the trader to simultaneously best-respond to all possible values of  $\alpha_i + \varepsilon_i$ .<sup>7</sup>

The slope-taking strategy  $d_i^{ST}$  is linear in price. Thus, if traders  $j \neq i$  play these strategies, the residual supply trader  $i$  faces is indeed linear, and accurately captured by (2). If traders  $j \neq i$  play slope-taking strategies based on beliefs  $\{\lambda_j\}_{j \neq i}$ , the true price impact of trader  $i$ , derived from (3) and the market-clearing condition, is

$$\bar{\lambda}_i = \frac{1}{\sum_{j \neq i} \frac{1}{v_j + \lambda_j}} \quad (4)$$

We therefore refer to a profile of price impacts  $\{\lambda_i\}_{i \in I}$  as *consistent* if  $\lambda_i = \left( \sum_{j \neq i} \frac{1}{v_j + \lambda_j} \right)^{-1}$  for each  $i$ , i.e., if each trader knows his true price impact given the others’ strategies.

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<sup>7</sup>Maximizing  $\beta_i(e_i + q_i) - \frac{v_i}{2}(e_i + q_i)^2 - q_i p_i(q_i)$  over  $q_i$  when  $p'_i = \lambda_i$  gives the first-order condition  $q_i = \frac{\beta_i - v_i e_i - p}{\lambda_i + v_i}$ , so the Walrasian auction allows the trader to simultaneously satisfy the first-order condition, and therefore simultaneously best-respond, for each realization of  $\varepsilon_i$ .

**Definition 1.** A profile of demand schedules  $\{d_i(\cdot)\}_{i \in I}$  is a **slope-taking equilibrium** if  $d_i(\cdot) = d_i^{ST}(\cdot, \lambda_i)$  for each  $i$  and the profile  $\{\lambda_i\}_{i \in I}$  is consistent.

Since slope-taking strategies are linear, in a slope-taking equilibrium, trader  $i$ 's beliefs (2) accurately capture all payoff-relevant details of the environment and the other traders' strategies, so (3) is a true best-response. In this environment, then, any slope-taking equilibrium is a Bayesian Nash equilibrium, coinciding with the unique LBN.

**Remark 1.** In the environment with quadratic preferences, a slope-taking equilibrium exists, is unique, and coincides with the unique Linear Bayesian Nash equilibrium.

## 2.3 Learning from Data on Past Trades

Next, we give a simple learning argument to demonstrate how the interactions among agents in an anonymous market naturally converge to the slope-taking equilibrium. We consider a model in which the Walrasian auction is repeated infinitely many times. Time is discrete and divided into periods indexed by  $T$ , interpreted as years; each period consists of subperiods, indexed by  $t$ , interpreted as days.

Trader  $i$ 's utility parameter  $\beta_i$  and endowment  $e_i$  vary randomly each subperiod. The endowment consists of a “common” and an idiosyncratic component,  $e_i^t = \bar{e}_i^t + \tilde{e}_i^t$ .  $(\beta_i, \bar{e}_i)$  may be arbitrarily correlated across traders, but  $\tilde{e}_i$  is independent of  $\{\beta_j, \bar{e}_j, \tilde{e}_j\}_{j \neq i}$ , which will be important for estimation. Each trader  $i$  privately observes the realization of  $(\beta_i, \bar{e}_i, \tilde{e}_i)$  at the start of each subperiod. Traders' convexity parameters  $v_i > 0$  are deterministic and fixed.

Each trader  $i$  enters period  $T$  with some estimate of his price impact  $\lambda_i^T$ , and acts optimally given that estimate throughout the period. At the end of period  $T$ , each trader uses the time series of trade data collected within period  $T$  to re-estimate his price impact. Since traders  $j \neq i$  submit demand schedules  $d_j(p) = \frac{\beta_j - v_j e_j - p}{v_j + \lambda_j}$ , the residual supply curve facing trader  $i$  (the negative sum of the other traders' demand schedules) shifts horizontally with  $\beta_j$  and  $e_j$ . By assumption, this shift is independent of  $\tilde{e}_i$ , the idiosyncratic portion of trader  $i$ 's endowment, which therefore serves as an instrument for trader  $i$ 's realized trades  $q_i$ , allowing each trader to consistently estimate his price impact:<sup>8</sup>

**Remark 2.** A trader can consistently estimate his price impact from the time series  $(\bar{p}^t, q_i^t, \tilde{e}_i^t)$  by regressing  $\bar{p}$  on  $q_i$  using  $\tilde{e}_i$  as an instrument.

The main result of our paper is that as the number of subperiods in each period grows, this learning process converges to the slope-taking equilibrium. For simplicity, we define

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<sup>8</sup>We explore consistency of estimation in a more general environment in the next section.

estimation as being “perfect” if in the limit, periods are long enough for price impacts to be estimated without error (“years have many days”). Remark 2 implies that this holds as long as period length increases without bound as  $T$  increases.

**Theorem 1.** *Let  $\{v_i\}_i \in \mathbb{R}_{++}^I$ , and  $\{\lambda_i^0\}_i \in \mathbb{R}_+^I$ . Under perfect estimation,  $\{\lambda_i^T\}_i$  converges to the unique set of consistent price impacts, and strategies converge (pointwise) to the slope-taking equilibrium.*

## 2.4 Learning Dynamics

Next, we use a simple example to study learning dynamics. Fix  $I > 2$  and let  $\beta_i = v_i = 1$  for all traders, so that  $u_i(x_i) = x_i - \frac{1}{2}x_i^2$ . The consistency condition is then

$$\lambda_i = \frac{1}{\sum_{j \neq i} \frac{1}{1+\lambda_j}}$$

for each  $i$ . This system of  $I$  equations has a unique solution,  $\bar{\lambda}_i = \frac{1}{I-2}$  for all  $i$ , and the unique slope-taking equilibrium has demand schedules

$$d_i(p) = d_i^{ST}(p, \bar{\lambda}_i) = \frac{I-2}{I-1}(1 - e_i - p)$$

In the limiting case where periods are very long, so that traders’ estimates at the end of each period exactly match their actual price impact from that period, estimates follow the deterministic path

$$\lambda_i^{T+1} = \frac{1}{\sum_{j \neq i} \frac{1}{1+\lambda_j^T}}$$

If traders have homogeneous initial estimates ( $\lambda_i^0 = \lambda_j^0$  for all  $i, j$ ), this further simplifies to  $\lambda_i^{T+1} = \frac{1}{I-1}(\lambda_i^T + 1)$ ; with  $I > 2$ , this sequence monotonically approaches the unique steady state  $\bar{\lambda}_i = \frac{1}{I-2}$ . While traders learn their actual price impacts exactly after each period, equilibrium is not reached after the first round of estimation; this is because after each period, all traders update their strategies based on their new price impact estimates, making price impact a moving target.<sup>9</sup>

The speed of convergence to the steady state depends positively on the number of traders, because in more competitive markets, the mutual reinforcement of price impacts is smaller. With symmetric initial estimates  $\lambda_i^0$ , the fraction of the gap to the steady state closed in each round of estimation,  $\frac{\lambda_i^{T+1} - \lambda_i^T}{\lambda_i - \lambda_i^T}$ , is  $\frac{I-2}{I-1} \times 100\%$ . Thus, with four traders, two-thirds of the

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<sup>9</sup>This effect is particular to games based on Nash in demands: it is not present in Cournot competition, in which one round of perfect estimation suffices for convergence to an equilibrium.

gap to the slope-taking equilibrium is closed in each round of estimation; with ten traders, 89% of the gap is closed in each round.

Since Theorem 1 is a limit result, we use simulations to illustrate the speed of convergence. Figures 1 and 2 show the results of simulations of the learning model. For panes (a) and (b) of Figure 1, simulations start with perfectly competitive beliefs, that is,  $\lambda_i = 0$ ; traders re-estimate price impacts after each period, and each period has twice as many observations as the previous one, with period 1 having 100 observations, period 2 having 200, period 3 having 400, and so on. In each subperiod, trader endowments are drawn independently from a standard normal distribution. Figure 1 panel (a) shows that with four traders, price impact estimates are both dispersed and far from the equilibrium level  $\bar{\lambda}_i = \frac{1}{I-2} = 0.5$  after one period (“year 1”), and move toward equilibrium in periods 2 and 3; by year 4, estimates are centered close to equilibrium, and become more precise with each subsequent period. Panel (b) shows that with ten traders, estimates approach equilibrium much faster – estimates are centered very close to the equilibrium value of 0.125 after two periods, and become less dispersed quickly.

Figure 2 shows a similar simulation for a three-trader market with heterogenous starting estimates of  $\lambda_1^0 = 0$ ,  $\lambda_2^0 = 0.5$ , and  $\lambda_3^0 = 1.5$ . Note that the ranking of trader estimates flips in each period – trader 3 starts with the highest initial belief, has the lowest (in expectation) after one period, the highest after 2 periods, and so on. Moreover, beliefs homogenize rapidly: the traders are basically indistinguishable from each other, and their price impact estimates are reasonably dispersed but centered close to the equilibrium value of 1, after 3 periods, getting less dispersed as additional periods go by.

## 3 Non-Quadratic Preferences

### 3.1 Slope-Taking Equilibrium with Fixed Endowments

Linear Bayesian Nash equilibrium is well-defined only in environments with quadratic preferences; the existing literature therefore focuses almost exclusively on such settings. Quadratic preferences, however, are considered unrealistic in the context of financial markets. In this section, we demonstrate that slope-taking equilibrium can be extended to make predictions for more natural preferences as well.

We continue to model traders as believing they face a linear residual supply curve (2). With non-quadratic utility, a trader’s best-response demand schedule  $d_i^{ST}$  is now nonlinear,<sup>10</sup>

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<sup>10</sup>The optimal demand schedule solves the familiar first-order condition equating marginal utility with marginal payment,  $u'_i(e_i + q_i) = p + \lambda_i q_i$ . For non-quadratic preferences, the derivative  $u'_i(\cdot)$  is nonlinear, and hence the implicitly defined demand function is not linear in  $p$ .



so traders will face nonlinear residual supply curves based on other traders' demands. By best-responding to a linear model, traders rely on a misspecified model.

When trader endowments are fixed and the appropriate second-order conditions hold, this model misspecification is inconsequential for optimal trading. By best-responding to linear supply (2) with  $\lambda_i$  equal to the correct slope at the market-clearing price, players submit an actual best-response to others' bids. As a result, slope-taking with appropriate price impact beliefs is still a Nash equilibrium.

For an illustrative example, consider a setting with 4 traders with identical utility functions  $u_i(x_i) = \log(x_i)$  and fixed endowments  $(e_1, e_2, e_3, e_4) = (3, 3, 1, 1)$ . (We choose asymmetric endowments so that even without randomness, there is trade in equilibrium.) A trader with log utility and linear beliefs  $\lambda_i$  best-responds by submitting the demand schedule<sup>11</sup>

$$d_i^{ST}(p) = \frac{1}{2\lambda_i} \left( -p - \lambda_i e_i + \sqrt{(p - \lambda_i e_i)^2 + 4\lambda_i} \right) \quad (5)$$

This demand schedule is nonlinear, so when traders play slope-taking strategies, they create residual supply curves for each other which are not accurately described by the linear model (2). However, we can still calculate their slopes and find a trader's actual price impact at a given price level given his opponents' strategies, which is

$$\bar{\lambda}_i = \frac{1}{\sum_{j \neq i} \left( \frac{1}{2\lambda_j} - \frac{1}{2\lambda_j} \frac{p - \lambda_j e_j}{\sqrt{(p - \lambda_j e_j)^2 + 4\lambda_j}} \right)}$$

This equation plays the role of the consistency condition (4) above. We can iterate this consistency condition (calculating the market-clearing price at each step) and find that it converges to a unique steady state of  $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \approx (0.1374, 0.1374, 0.1203, 0.1203)$ , along with the market-clearing price  $p \approx 0.5200$ , at which traders 1 and 2 each sell 0.6662 units of the good and traders 3 and 4 each buy that quantity.

With fixed endowments, the slope-taking strategies (5) with these steady-state price impacts form a Bayesian Nash equilibrium. The top panel of Figure 3 illustrates this equilibrium, showing trader 1's perspective: the orange curve shows the actual residual supply curve faced, the gray line the linear residual supply curve trader 1 perceives, and the blue curve the demand schedule the trader submits as a best-response. As noted above, these strategies are a Bayesian Nash equilibrium – traders are playing actual best-responses at the equilibrium market-clearing price, although not at off-equilibrium prices. We define this

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<sup>11</sup>Solving  $\max_{q_i} \{\log(e_i + q_i) - q_i p(q_i)\}$  under the belief that  $p'(q_i) = \lambda_i$  gives FOC  $\frac{1}{e_i + q_i} - p - q_i \lambda_i = 0$ ; solving for  $q_i$  via the quadratic formula (and selecting the root where  $q_i \geq -e_i$ ) gives (5).

equilibrium as a slope-taking equilibrium: players are playing nonlinear strategies as best-responses to linear beliefs, which are “locally correct” at the equilibrium price, so strategies are true best-responses. In this example, slope-taking equilibrium is unique; [Weretka \(2011\)](#) shows that with general  $u$ , slope-taking equilibrium exists, but need not be globally unique.

### 3.2 Learning with Random Endowments

With fixed endowments, traders have no variation from which to learn their price impact. Once we introduce stochastic endowments to model statistical learning, however, trade could occur at various points on a trader’s residual supply curve; linear beliefs are no longer correct at all possible market-clearing prices on the equilibrium path, and the slope-taking strategy is no longer an actual best-response. However, by continuing to explore the example above with stochastic endowments, we can illustrate three ideas that we conjecture hold more generally:

1. With non-quadratic utility and stochastic endowments, the learning model considered above converges to a steady state.
2. As the support of the random endowments shrinks, the steady state of the learning model converges to the slope-taking equilibrium of the corresponding deterministic setting.
3. Even with substantial randomness in endowments and slope-taking traders therefore playing meaningfully suboptimal strategies, their utility loss – the utility they sacrifice by relying on a simple but misspecified model of supply – is negligible.

In light of the second and third points, we can think of slope-taking strategies in non-quadratic settings in either of two ways: as an equilibrium selection argument for the fixed-endowments case by thinking of it as the limit as randomness vanishes, or as an appealing heuristic-based solution concept for the case with nonvanishing randomness in endowments.

As before, we assume the game is repeated many times, with traders trading optimally in each subperiod given their current beliefs and re-estimating their price impact after each period. With non-quadratic utility, traders’ slope-taking strategies create a residual supply curve that is nonlinear and changes from subperiod to subperiod in a way not described by a parallel shift. Nonetheless, IV estimation continues to “work,” now recovering a weighted average of trader  $i$ ’s price impact over the relevant range of endowments. Formally:

**Remark 3.** Suppose traders play slope-taking strategies based on a profile of price impacts  $\{\lambda_i\}_{i \in I}$  which need not be consistent. Let  $\bar{\lambda}_i(e)$  denote trader  $i$ ’s price impact on the margin,

as a function of all traders’ realized endowments. As the number of subperiods in a period increases, trader  $i$ ’s IV estimate of his price impact converges in probability to a particular weighted expectation of  $\bar{\lambda}_i(e)$ .

Consistency in the case of quadratic utility (Remark 2) is the special case where  $\bar{\lambda}_i$  is fixed, since traders play linear strategies whose slopes don’t vary with endowments. The proof for the non-quadratic case (available on request) is mechanical, but the main intuition can be understood by analogy with the Local Average Treatment Effect of [Imbens and Angrist \(1994\)](#). We want the causal effect of a trader’s quantity traded on market-clearing price. Think of each unit a trader bids for as a separate observation, an “individual” who might receive a treatment (get purchased) or no treatment (not get purchased). The instrument – the idiosyncratic part of endowment – shifts the demand curve the trader submits, which shifts the probability of treatment for each “individual.” The key is that since slope-taking strategies (5) are decreasing pointwise in  $e_i$ , the instrument satisfies Imbens and Angrist’s monotonicity requirement – an increase in the instrument decreases demand at every price, thus decreasing the likelihood of “treatment” for every individual (the likelihood each unit is bought). This allows us to essentially estimate a local average treatment effect – the average marginal effect a unit traded has on price.<sup>12</sup>

To illustrate converge, we work with the same log-utility example as before, now with endowments which are random in each subperiod and are distributed uniformly on the interval  $[2.5, 3.5]$  for traders 1 and 2 and on  $[0.5, 1.5]$  for traders 3 and 4. Within each period, traders submit demand schedules (5) in each subperiod based on their current price impact estimate; at the end of each period, traders re-estimate price impacts by regressing realized prices on their realized trades, using their endowment as an instrument. We simulated this learning process, beginning with homogeneous initial beliefs  $\lambda_i^0 = 0.01$ ; the first period used 100 observations, with the number of observations (subperiods) doubling in each subsequent period. Figure 1 panes (c) and (d) show the results of this simulation, displaying the distribution of traders’ price impact estimates at the end of each period. The figure shows price impact estimates converging toward a point near 0.14 for traders 1 and 2, and near 0.12 for traders 3 and 4. (After ten “years,” about 99% of estimates for traders 1 and 2 are between 0.138 and 0.144, and 99% of estimates for traders 3 and 4 are between 0.121 and 0.125.) They seem to be converging to a steady state at  $\lambda_1 = \lambda_2 \approx 0.1405$  and  $\lambda_3 = \lambda_4 \approx 0.1227$ . Additional simulations suggest that price impact estimates converge here from any starting

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<sup>12</sup>We don’t argue our framework is a direct application of LATE, just that the analogy helps to understand why IV estimation works. If what varied from subperiod to subperiod was not endowment but a feature of preferences causing a twist rather than a shift in a trader’s demand, monotonicity would fail; we have a simple example like this in which a trader using IV would estimate a *negative* price impact even though he actually faced an upward-sloping residual supply curve in each subperiod.

point.<sup>13</sup>

### 3.3 Convergence to Equilibrium as Randomness Shrinks

Unsurprisingly, as the variation in endowments gets smaller and the model more closely approximates the deterministic case, the steady state approaches the slope-taking equilibrium of the deterministic model. We repeated the same simulation, varying the amount of “noise” in endowments, that is, the width of the interval endowments are drawn from. We ran separate simulations varying bidder endowments independently and uniformly over the intervals  $3 \pm \epsilon$  and  $1 \pm \epsilon$  for  $\epsilon$  equal to 1, 0.5 (the case shown above), 0.1, and 0.01. For each simulation, the graphs analogous to those in Figure 1 panes (c) and (d) looked virtually identical, with the peak of the distributions shifted slightly. Table 1 shows, for each set of simulations, the average price impact estimate for each trader at the end of period 10, with the slope-taking equilibrium with fixed endowments shown for comparison.

Table 1: Average simulated price impact estimates after 10 periods

$e_1$ and $e_2$	$U[2, 4]$	$U[2.5, 3.5]$	$U[2.9, 3.1]$	$U[2.99, 3.01]$	3
$e_3$ and $e_4$	$U[0, 2]$	$U[0.5, 1.5]$	$U[0.9, 1.1]$	$U[0.99, 1.01]$	1
$\lambda_1$	0.1508	0.1405	0.1375	0.1374	0.1374
$\lambda_2$	0.1508	0.1405	0.1375	0.1374	0.1374
$\lambda_3$	0.1308	0.1227	0.1204	0.1203	0.1203
$\lambda_4$	0.1308	0.1227	0.1204	0.1203	0.1203

### 3.4 Utility Foregone by Sticking To a Simple Model

So in a non-quadratic setting, slope-taking equilibrium exists – as an equilibrium – when endowments are fixed; when endowments are random, our simple learning process converges to a steady state, and when endowments vary only a little, this steady state is very close to the deterministic slope-taking equilibrium.

However, in the latter case, traders are not behaving optimally: in the non-deterministic setting, traders are best-responding to a misspecified optimization problem, and are not playing actual best-responses even on the equilibrium path.

<sup>13</sup>When a trader starts with a high initial belief about price impact (greater than 0.5), this sometimes leads them to estimate a negative price impact in the first period. We assume that traders believe price impacts to be non-negative, and therefore that if they get a negative estimate, they instead believe they have zero price impact, which we approximate as  $\lambda_i = 0.001$  since (5) is undefined at  $\lambda_i = 0$ .

An important question, then, is how close to optimal these strategies are. If slope-taking strategies are far from optimal in non-deterministic non-quadratic settings, players would have a strong incentive to better understand the actual environment they’re in and best-respond appropriately. If slope-taking strategies are nearly optimal, however, then this is an appealing solution concept – players use a simple, parsimonious model of the game that delivers nearly the same payoff as a more complete understanding.

To understand how much utility traders “sacrifice” by following a slope-taking strategy, we focus on trader 1. We compare his average realized utility when everyone plays their steady state slope-taking strategy, to the utility he could achieve if he knew the actual residual supply curve he faced in each period (based on the other traders’ slope-taking strategies and realized endowments) and optimized based on that. Thus, when we measure the utility trader 1 leaves on the table by playing a slope-taking strategy, we are allowing him two different “improvements” – replacing a misspecified linear model with knowledge of the true environment he faces *on average, and* knowing the other traders’ realized endowments *in each subperiod*. This therefore gives an upper bound on how much trader 1 could gain by switching from a linear to a more complex nonlinear model of the environment he’s in, or how much he’s “sacrificing” by maintaining only a simple model of the world. Table 2 shows the gap, i.e., how much utility, on average, trader 1 could gain by switching from his slope-taking strategy to the true *ex post* best-response in each subperiod. Since utility has no natural scale, the second row divides the utility loss by the marginal utility of consumption in that period, indicating the amount of additional endowment of the traded good that would give trader 1 the same incremental utility as exact knowledge of the residual supply curve. These results are based on simulations, so we also look at the “worst-case” utility gaps over the 1,000,000 realizations of each environment generated for the simulations; the last two rows show this worst-in-a-million-simulations gap between utility earned from the slope-taking strategy and utility available given exact knowledge of the true residual supply curve, in utility terms and in terms of the equivalent amount of the traded good.

Table 2: Trader 1’s average gain from switching to true best response

$e_1$ and $e_2$	$U[2, 4]$	$U[2.5, 3.5]$	$U[2.9, 3.1]$	$U[2.99, 3.01]$
$e_3$ and $e_4$	$U[0, 2]$	$U[0.5, 1.5]$	$U[0.9, 1.1]$	$U[0.99, 1.01]$
Average utility loss	0.000571	0.000103	0.000004	$3.7 \times 10^{-8}$
Scaled to endowment	0.001268	0.000237	0.000009	$8.5 \times 10^{-8}$
“Worst in a million” utility loss	0.031089	0.002265	0.000045	$3.9 \times 10^{-7}$
Scaled to endowment	0.067917	0.004974	0.000100	$9.2 \times 10^{-7}$

In all cases examined, this utility gap is extremely small. In an environment where trader 1’s endowment varies over the interval  $[2.5, 3.5]$ , having a perfect model of residual supply (including knowledge of the other traders’ endowments in each subperiod) would give the same expected utility increase as an additional 0.0002 units of endowment. In the one-in-a-million realization where knowledge of the true residual supply was the most valuable, this knowledge would be worth the same as about 0.005 additional units of the consumption good. Both the average and “one-in-a-million” utility gap decrease roughly with the square of the size of the intervals, so with less variation, the utility loss is significantly smaller.

To see why the utility loss is so small, it’s useful to examine a particular “worst-case” example. In this environment (with endowments uniform on  $[2.5, 3.5]$  and  $[0.5, 1.5]$ ), out of a million simulated realizations of endowments, the one where trader 1 could have gained the most by knowing the true residual supply curve was when all three of the other traders received their smallest possible endowments: endowments were  $(e_1, e_2, e_3, e_4) \approx (3.33, 2.50, 0.51, 0.50)$ . The bottom pane of Figure 3 shows the actual residual supply curve trader 1 faced that subperiod (in blue), and the one he believed he was facing (in orange).<sup>14</sup> (Quantities on the  $x$  axis are negative because trader 1 is always a seller.) With the other traders having low endowments, the residual supply curve trader 1 faced was steeper than he believed, so his actual price impact (0.1825 at the realized trade) was greater than he thought (0.1405).

The orange dot shows the trade (and price) realized under the slope-taking strategy, which is optimal given the orange residual supply curve. The blue dot shows the actual utility-maximizing trade that was possible given the other traders’ demand schedules. Had he known the true residual supply he faced, trader 1 would have chosen to sell less of the good, since this would have increased the price he sold the rest at by more than he anticipated.

The actual optimal trade looks significantly different from what happened at the slope-taking strategy – selling 1.04 units at price 0.633 instead of 1.14 units at 0.615 – but the impact on *utility* would have been extremely small. The blue dot is only very slightly away from the orange supply curve, and the orange dot gives more utility than any other point on the orange supply curve, so the utility of the blue dot can’t be much more than that. Knowing the actual residual supply curve he faced would have increased trader 1’s utility from 1.4866 to 1.4888. And to emphasize again, this would have been the single *biggest* utility gain available to trader 1, out of a million simulated realizations of the environment; and would have been the utility equivalent of receiving 0.005 more endowment of the traded

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<sup>14</sup>With a slope-taking strategy, traders submit a demand schedule that is a best-response to all linear residual supply curves with the same slope. Thus, trader 1 believed he faced the curve shown in orange, or any other with the same slope; after learning the realized trade, this trade remained the *ex post* best response to the supply curve shown in orange.

good in a setting where his endowment was varying uniformly on  $[2.5, 3.5]$ .

Our takeaway from this exercise, then, is that when the variation in endowments is not too large, the utility “sacrificed” by players who maintain only a linear model of price impact is negligible; so slope-taking strategies, while not literally an equilibrium, are nonetheless an appealing solution concept. Players in an anonymous market can easily learn, from internal data, all they need to know to play optimally in response to a linear model; and the utility gain from learning more than that, even if it included full knowledge of their opponents’ private information, is extremely small.

## 4 Conclusion

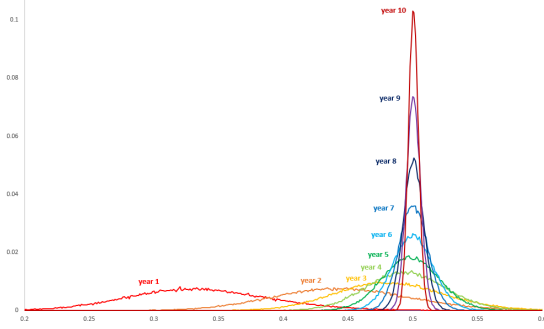
In this paper, we seek to understand behavior in anonymous markets where traders have price impact but do not know all the details of the other traders in the market. We consider a simple “slope-taking” heuristic, where traders model the residual supply they face as linear (whether or not it truly is) and periodically re-estimate its slope from internally-observable trading data.

In a quadratic environment, we’ve shown that this learning process converges to the unique Linear Bayesian Nash equilibrium, giving a powerful argument for an equilibrium selection choice largely accepted in the financial literature. In a non-quadratic environment, the slope-taking equilibrium is only a Nash equilibrium when endowments are deterministic. When endowments are random, the same learning process still converges to a steady state close to the corresponding deterministic slope-taking equilibrium; and the utility sacrificed by traders who follow the slope-taking strategy, even relative to the benchmark where they could observe other traders’ endowments and bids and best-respond exactly, is negligible. The incentives to “go beyond” a linear model to a more complex model of the environment are therefore quite weak. We therefore see slope-taking equilibrium as a plausible heuristic-based solution concept in non-quadratic settings if uncertainty in endowments is not too large.

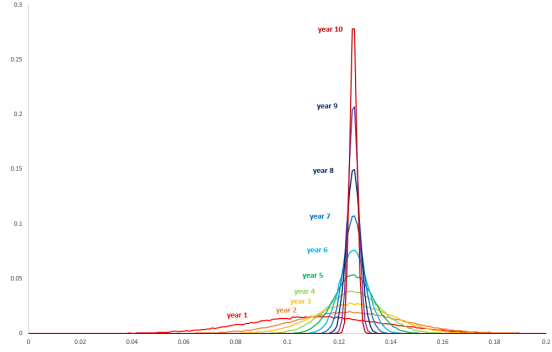
Figure 1: Simulation results, homogeneous initial estimates

*Quadratic utility, symmetric traders*

(a) *Four traders*

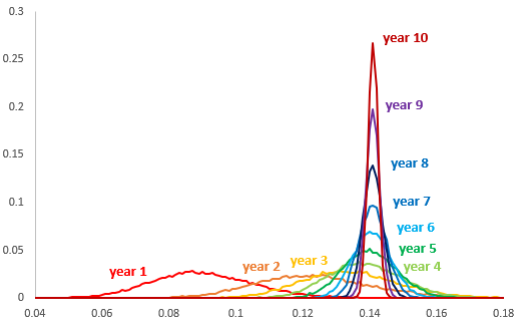


(b) *Ten traders*

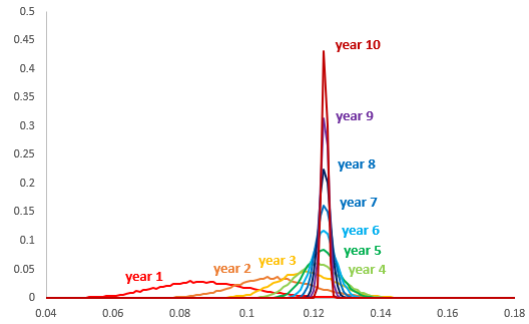


*Non-quadratic utility, asymmetric traders*

(c) *Traders 1 and 2*



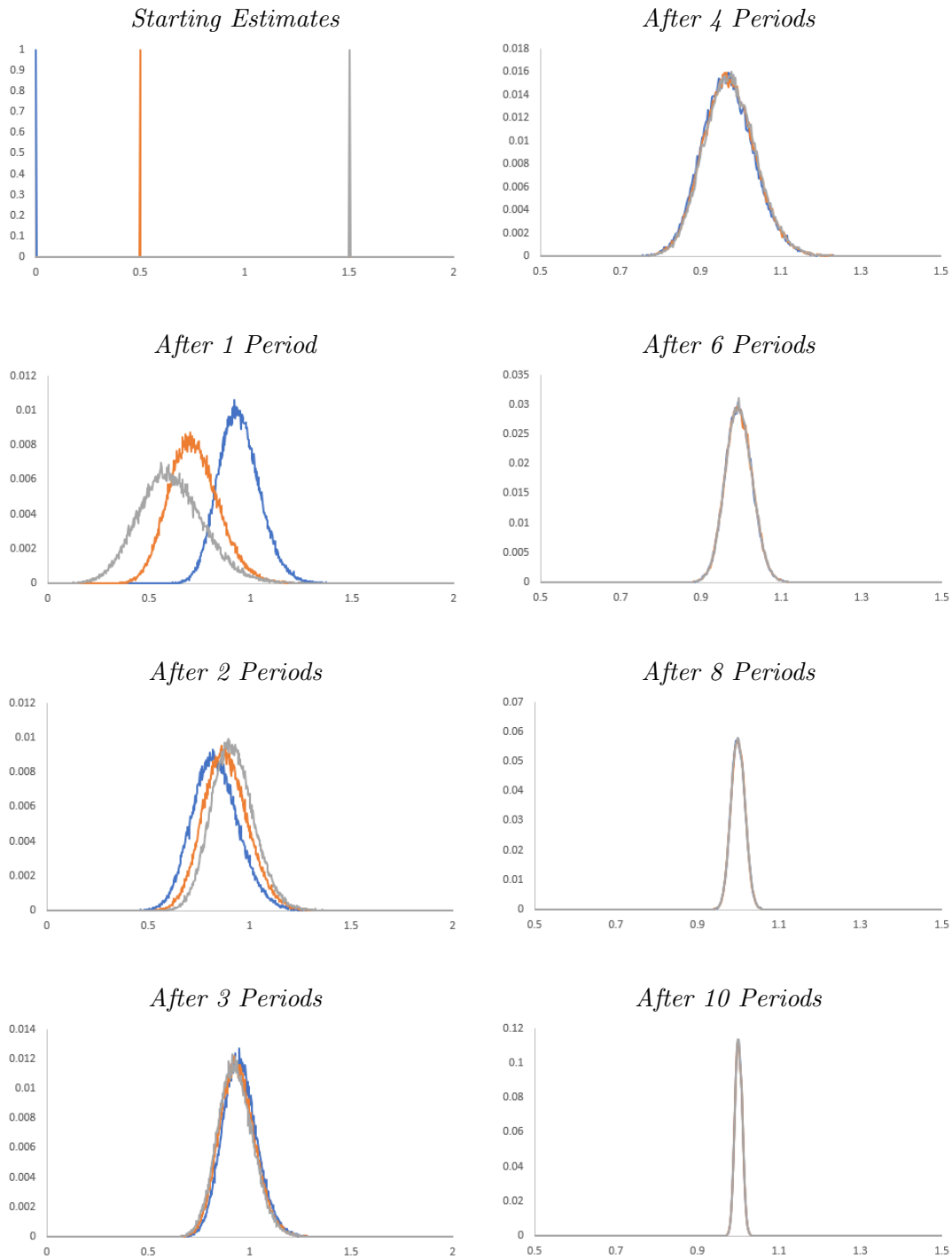
(d) *Traders 3 and 4*



Panels (a) and (b) based on 25,000 simulations of a four-trader market and 10,000 simulations of a ten-trader market with homogeneous initial price impact estimates  $M_i^0 = 0$ . Endowments are distributed i.i.d. standard normal; the first period had 100 observations, and the number of observations doubled in each period. Graphs show the distribution of price impact estimates after each period.  $y$  axes refer to the fraction of observations found within each “bucket” of size 0.001. Panels (c) and (d) based on 10,000 simulations with homogeneous initial price impact estimates  $\lambda_i^0 = 0.01$ . Endowments distributed uniformly on  $[2.5, 3.5]$  for traders 1 and 2 and  $[0.5, 1.5]$  for traders 3 and 4. First period had 100 observations, number of observations doubled in each period. Graphs show the distribution of price impact estimates after each period.  $y$  axes refer to the fraction of observations found within each “bucket” of size 0.001.



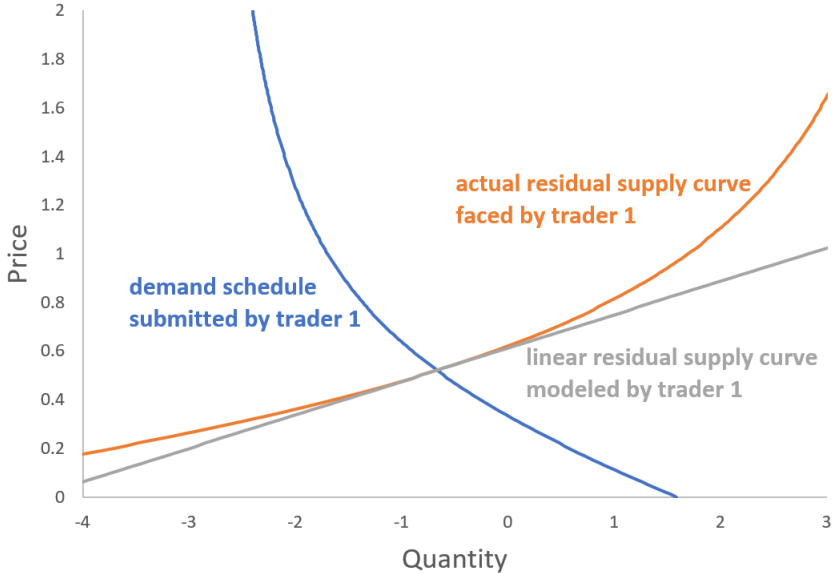
Figure 2: Simulation results, heterogeneous initial estimates.



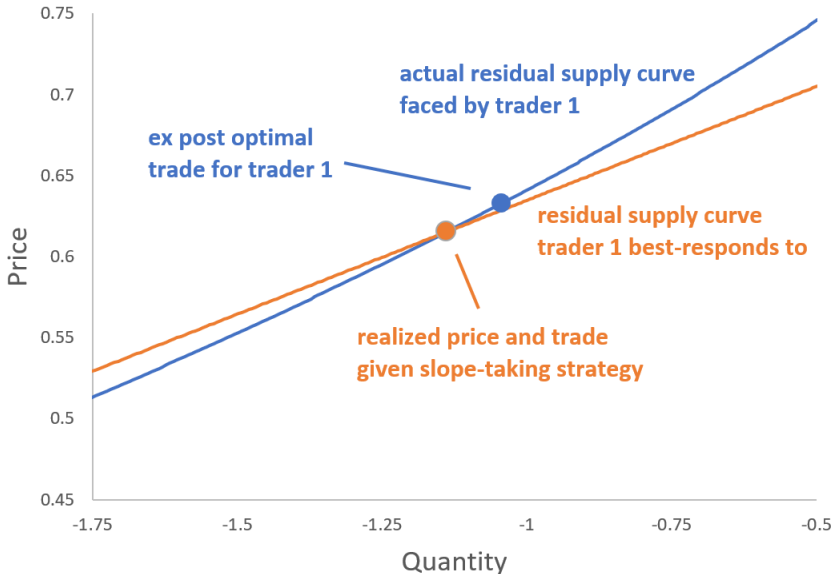
Based on 100,000 simulations of a three-trader market with initial price impact estimates  $M_1^0 = 0$ ,  $M_2^0 = 0.5$ ,  $M_3^0 = 1.5$ . Endowments are distributed i.i.d. standard normal; the first period had 100 observations, and the number of observations doubled in each period. Graphs show the distribution of price impact estimates for each trader (trader 1 in blue, 2 in orange, 3 in gray) after select periods (“years”). Note that x axes are fixed for each column, but vary across the columns; y axes refer to the fraction of observations found within a “bucket” of size 0.0025.

Figure 3: Actual versus modelled residual supply, Trader 1's point of view

*Slope-taking equilibrium, deterministic endowments*



*“Worst-in-a-million” utility gap, random endowments*



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## 5 Appendix

PROOF OF THEOREM 1. Let  $\lambda^T = \{\lambda_1^T, \dots, \lambda_I^T\} \in \mathbb{R}_+^I$ . Under perfect estimation, the learning process is fully described by the first-order difference equation  $\lambda^{T+1} = H(\lambda^T) = \{h_1(\lambda^T), \dots, h_I(\lambda^T)\}$ , where  $h_i(\lambda^T) \equiv \left(\sum_{k \neq i} \frac{1}{\lambda_k^T + v_k}\right)^{-1}$ .

Fix  $\lambda^0 \in \mathbb{R}_+^I$ , and let  $\lambda^* = \frac{I-1}{I-2} \max_{i \in I} \lambda_i^0 + \frac{1}{I-2} \max_{i \in I} v_i$ . It's straightforward to show that if  $\lambda_i^T \leq \lambda^*$  for all  $i$  then  $\lambda_i^{T+1} \leq \lambda^*$  for all  $i$ ; since by construction  $\lambda_i^0 \leq \lambda^*$  for all  $i$ , we can restrict attention to  $\lambda \in [0, \lambda^*]^I$ .

Let  $J(\cdot)$  be the Jacobian of  $H(\cdot)$ , which is an  $I \times I$  matrix with zeros on the diagonal and off-diagonal terms

$$\alpha_{ij} = \frac{\partial}{\partial \lambda_j} \left( \left( \sum_{k \neq i} (\lambda_k + v_k)^{-1} \right)^{-1} \right) = \frac{((\lambda_j + v_j)^{-1})^2}{\left( \sum_{k \neq i} (\lambda_k + v_k)^{-1} \right)^2} \quad (6)$$

Note that  $\alpha_{ij} > 0$  ( $j \neq i$ ) and  $\sum_j \alpha_{i,j} < 1$  for any  $\lambda$ . For any two vectors  $\lambda', \lambda'' \in [0, \lambda^*]^I$ , letting  $\lambda^t \equiv t\lambda'' + (1-t)\lambda'$ , we can calculate  $h_i(\lambda'') - h_i(\lambda')$  as

$$h_i(\lambda'') - h_i(\lambda') = \int_0^1 \frac{\partial h_i}{\partial t}(\lambda^t) dt = \int_0^1 \sum_{k=1}^I (\lambda_k'' - \lambda_k') \alpha_{ik}(\lambda^t) dt$$

From here,

$$|h_i(\lambda'') - h_i(\lambda')| \leq \int_0^1 \left( \max_{k \in I} |\lambda_k'' - \lambda_k'| \right) \sum_{k \in I} \alpha_{ik}(\lambda^t) dt$$

Defining

$$\beta \equiv \max_i \sup_{\lambda \in [0, \lambda^*]^I} \sum_j \alpha_{i,j}(\lambda) < 1$$

and letting  $\|\cdot\|^*$  denote the L-infinity norm  $\|x\|^* = \max_i |x_i|$ ,

$$\|H(\lambda'') - H(\lambda')\|^* \leq \beta \|\lambda'' - \lambda'\|^*$$

so  $H$  is a contraction mapping on  $[0, \lambda^*]$ , and the learning process therefore converges to the unique profile of consistent price impacts  $\bar{\lambda}$ .  $\square$