An Ordinal Theorem of the Maximum

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Abstract

This paper extends the notion of equivalent variation, Hicks (1939) to an abstract decision problem. It also provides a modern, ordinal variant of the maximum theorem, Berge (1963) that formulates the assumptions in terms of underlying preferences and demonstrates the continuity of the classic preference-based welfare indices (i.e., the equivalent and compensating variations) as well as the upper hemicontinuity of the choice correspondence. We then apply the theorem to the relevant economic problems.

Key words: Maximum Theorem, Equivalent Variation, Ordinal Convergence
JEL classification numbers: D43, D53, G11, G12, L13

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1 Introduction

An important comparative statics result in mathematical economics, the maximum theorem Berge (1963),[1] shows that, in a maximization problem, whenever the utility function is jointly continuous (in alternatives and parameters), and a budget correspondence is continuous, then the value function of the program is continuous as well. This powerful result derives the properties of a value function from the primitive structure of an optimization problem. However, its usefulness in normative analyses is limited by the fact that it characterizes a cardinal object that does not have any welfare interpretation in contemporary economic theory.

This paper offers a modern, ordinal variant of the maximum theorem that reformulates these assumptions in terms of underlying preferences; furthermore, it demonstrates the continuity of the classic welfare index that is measurable with respect to preferences, namely the equivalent variations introduced by Hicks (1939), as well as, the choice correspondence. Therefore, we call our result an *ordinal theorem of the maximum*. Utilizing examples we further demonstrate that the Berge theorem's assumptions, such as representation by a jointly continuous utility or continuity of budget correspondence, by themselves, are insufficient for our result to hold. Finally, we discuss the applications of the theorem to important economic problems.

The paper contributes to the literature that extends the Berge theorem (Walker (1979); Leininger (1984); Ausubel and Deneckere (1993)). These papers relax some of the continuity requirements of the objective function and demonstrate upper hemicontinuity of choice correspondence. The contribution of this paper is twofold. First, we identify the conditions on a family of preferences, under which the *ordinal welfare index* is continuous. Second, in the Berge Theorem, the argument for the upper hemicontinuity of a choice correspondence, an ordinal object itself, is "tainted" by the assumption on a cardinal utility function. Our preference-based formulation "purifies" this part of the argument, clarifying the sufficient conditions in terms of preferences.

The paper proceeds as follows. Section 2 provides applications that illustrate the key ideas. Section 3 defines an abstract problem, develops mathematical tools for the ordinal continuity, and states the ordinal maximum theorem. Section 4 presents important examples of applications of the theorem. Finally, Section 5 demonstrates that the assumptions of the ordinal maximum theorem are tight.

[1] See, e.g., Ok (2007)

2 Motivating Examples

In this section, we introduce two applications that illustrate the problem studied in the paper. The applications share the following structure. A decision-maker chooses alternatives x from two feasible subsets of space $X \subset \mathbb{R}^2$, determined in a factual (*status quo*) and a counterfactual scenario. Following the literature, we use the terms scenarios and policies interchangeably. We are interested in the impact of the counterfactual policy on welfare, measured as an equivalent variation. The set(s) of feasible alternatives and preferences are parametrized by θ. As a result, the equivalent variation depends on the parameter value.

When θ approaches some limit point, the respective indifference curves become perfectly aligned with the limit ones. Similarly, the sets of feasible alternatives acquire the limit shapes. The alignment of upper-contour and feasible sets results in the convergence of the preference-based welfare. The existing economic theory lacks appropriate mathematical tools to demonstrate such continuity of equivalent variation. The goal of this paper is to fill this gap.

2.1 Consumer choice

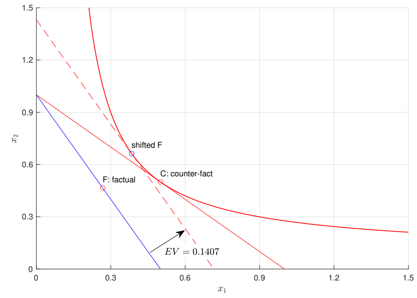
We start with the classic problem of a consumer choosing a bundle of two commodities. The consumer is effectively choosing among consumption profiles $x \in X = [0, 100]^2$ to maximize preferences parametrized by $\theta \in \Theta = [0, 2]$, represented by an additively separable utility $U(x, \theta) = u(x_1, \theta) + u(x_2, \theta)$, where individual utility function is iso-elastic,

$$u(x_n, \theta) = \begin{cases} \frac{x_n^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \\ \ln(x_n) & \text{otherwise} \end{cases} . \tag{1}$$

The set of feasible alternatives x is determined by the budget constraint $\sum_{n=1,2} \zeta_n(x_n - e_n) \leq 0$, where $\zeta = (\zeta_1, \zeta_2) = \mathbb{R}^2_{++}$ are prices of commodities and $e = (e_1, e_2)$ is the initial endowment of the two goods. Under factual policy p, prices are given by $\zeta = (1, 0.5)$ and endowment is $e = (0, 1)$. The counterfactual policy p' increases the price for the second commodity to $\zeta'_2 = 1$.

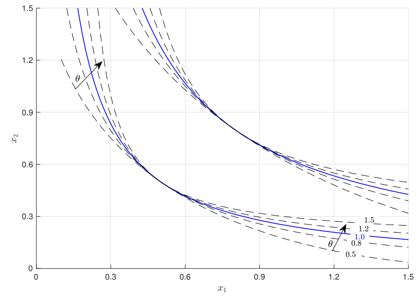
Figure 1. Portfolio Choice

A)



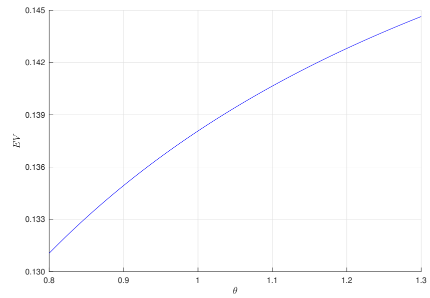
The figure demonstrates the equivalent variation for $\theta = 0.5$ in the consumer choice problem. Equivalent variation is given by the distance between the factual budget line (solid line), and the parallel line that is tangent to the indifference curve attained in the counterfactual scenario, along the 45^o line ($d = (1, 1)$).

B)



Note: The figure demonstrates the indifference curves passing through points $(0.5, 0.5)$ and (0.8,0.8) for different values of $\theta = 0.5, 0.8, 1, 1.2, 1.5$. In the neighborhood of $\theta = 1$ the indifference curves transform continuously in a parameter value.

C)



Note: The figure demonstrates the evolution of equivalent variation in parameter value θ, in the neighborhood of one.

Consider the behavior of the welfare index and choices in the neighborhood of $\theta = 1$. The utility function $U : X \times \Theta \to \mathbb{R}$ is (jointly) continuous on its domain. Also, for each policy, the set of feasible alternatives, (i.e., the budget set) is non-empty, compact, and independent of θ. As a result, by the Maximum Theorem, the optimal choices and the value functions are continuous in θ. Unfortunately, this result is not very useful in welfare comparisons. The core tenet of modern economic theory is that satisfaction from the consumption of goods cannot be measured effectively in cardinal units. For this reason, in an ordinal framework, the welfare effect brought about by the counterfactual scenario is often measured in terms of *equivalent variation*, Hicks (1939), a preference-based index defined as follows: Fix some consumption bundle $d \in X$ that gives a *welfare numeraire*. Equivalent variation, $EV_{p,p'}$, is a sufficient transfer of d, making the factual policy equally as attractive as the counterfactual one. Geometrically, equivalent variation is the smallest distance between the factual budget set and the counterfactual upper contour set, along direction d. In Figure 1.A we depict the equivalent variation for the riskless numeraire, $d = (1, 1)$.

As we show in Figure 1.B., in the neighborhood of $\theta = 1$, the consumer's indifference curves transform continuously into the ones derived from the logarithmic utility. The alignment of the upper contour sets, in turn, gives rise to the continuity of equivalent variation, reported in the last column of Table 1 and depicted in Figure 1.C. The Maximum Theorem characterizes cardinal values, and it does not apply to preference-based welfare. Moreover, the theorem cannot be easily reformulated to establish the continuity of equivalent variation. In Example 2 in the next section, we show that the central assumption of the theorem (i.e., the joint continuity of the utility representation) is too weak to guarantee the desired continuity of the preference-based welfare.

Table 1. Consumer choice of and equivalent variation.

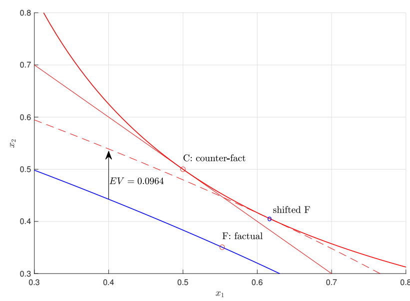
	$\theta = -0.3$	$\theta = -0.1$	$\theta = -0.01$	ln	$\theta = 0.01$	$\theta = 0.1$	$\theta = 0.3$
x_p	$(0.21, 0.57)$	$(0.24, 0.52)$	$(0.25, 0.50)$	$(0.25, 0.50)$	$(0.25, 0.50)$	$(0.26, 0.48)$	$(0.27, 0.46)$
$x_{p'}$	$(0.50, 0.50)$	$(0.50, 0.50)$	$(0.50, 0.50)$	$(0.50, 0.50)$	$(0.50, 0.50)$	$(0.50, 0.50)$	$(0.50, 0.50)$
$EV_{p,p'}$	0.136	0.135	0.138	0.138	0.138	0.140	0.145

2.2 Labor-leisure choice with taxes

In the previous application, the trader's preferences vary in θ, while the sets of feasible alternatives are fixed polytopes. We now consider an example in which the set of feasible alternatives is non-linear and parametrized by θ.

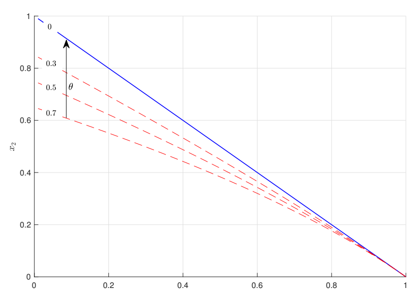
Figure 2. Labor-Leisure Choice

A)



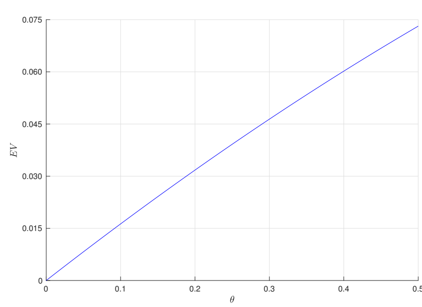
Note: The figure demonstrates the equivalent variation for $\theta = 0.7$ in the portfolio choice application. Equivalent variation is given by the distance between the factual budget curve (solid curve) and the shifter curve that is tangent to the indifference curve attained in the counterfactual scenario, along the vertical line ($d = (0, 1)$).

B)



Note: The figure demonstrates the frontier of the feasible set for different values of $\theta = 0.7, 0.5, 0.3, 0$. The frontier transforms continuously in a parameter value.

C)



Note: The figure demonstrates the evolution of equivalent variation in parameter value θ.

Consider a problem of a consumer choosing leisure x_1 and consumption good x_2 from a compact box $X = [0, 1]^2$. Preferences are represented by utility $U(x) = \ln x_1 + \ln x_2$. The consumer is endowed with one unit of time, used for leisure and labor supply. Gross labor income is given by $i = w(1 - x_1)$, where the real wage rate is equal to one, $w = 1$. Under counterfactual policy labor income is subject to taxation. Consumption is equal to net income $x_2 = (1 - \tau) \times i$. As a result, the set of feasible alternatives is given by $x \in X$ that satisfy $(1-\tau)x_1 + x_2 \leq 1 - \tau$. The factual tax rate is progressive, $\tau = \theta i/(1+i)$, where $\theta \in \Theta = [0, 0.5]$ is the maximal tax bracket. The counterfactual tax rate is zero: $\tau = 0$. Equivalent variation is measured in terms of the consumption good, $d = (0, 1)$, which is the minimal distance between the frontier of the factual feasible set and the shifted variant that is tangent to the counterfactual indifference curve (see Figure 2.1).

For each policy, the correspondence that gives the collection of feasible bundles x is non-empty, compact-valued, and continuous in θ. As a result, by the Maximum Theorem, the corresponding optimal choices and value functions are continuous. In the next section, we show that the properties of a feasible correspondence required in the theorem are also insufficient for the continuity (Example 3) or the existence (Example 4) of equivalent variation. Still, as evident from Figure 1.B., small variations in the parameter value in the labor-leisure choice problem generate only negligible perturbations of the feasible set of alternatives. As a result, the minimal distance between the factual budget set and counterfactual upper counter set, which gives equivalent variation, is continuous in θ; see Table 2 and Figure 2.C.

Table 2. Labor-leisure choice and equivalent variation.

	$\theta = 0$	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$	$\theta = 0.5$
x_p	$(0.5, 0.5)$	$(0.51, 0.48)$	$(0.51, 0.46)$	$(0.52, 043)$	$(0.53, 0.41)$	$(0.53, 0.39)$
$x_{p'}$	$(0.50, 0.50)$	$(0.50, 0.50)$	$(0.50, 0.50)$	$(0.50, 0.50)$	$(0.50, 0.50)$	$(0.50, 0.50)$
$EV_{p,p'}$	0	0.016	0.031	0.046	0.060	0.073

3 An Ordinal Theorem of the Maximum

3.1 Equivalent variation

Consider an agent characterized by a parametric family of preferences $\{\succeq_\theta\}_{\theta\in\Theta}$ over the compact set of alternatives $X \subset \mathbb{R}^N$ where parametric space $\Theta \in \mathbb{R}^M$ is compact. Policy p determines a set of feasible alternatives that may depend on parameters θ. Mathematically, a policy is represented by the feasibility correspondence $B_p : \Theta \rightrightarrows \mathbb{R}^N$. For p, a choice correspondence $x_p : \Theta \rightrightarrows X$ is defined in the standard way, namely, as[2]

$$x_p(\theta) \equiv \{x \in B_p(\theta) \cap X | x \succeq_\theta y \text{ for all } y \in B_p(\theta) \cap X\},$$

and the maximal upper contour correspondence, $\bar{\Psi}_p : \Theta \rightrightarrows X$, is

$$\bar{\Psi}_p(\theta) \equiv \{y \in X | y \succeq_\theta x \text{ for some } x \in x_p(\theta)\}.$$

Fix welfare numeraire $d \in \mathbb{R}^N_+$, such that $d \neq 0$ and the parameter value θ. Consider a factual policy p and the counterfactual one p'. Equivalent variation is a minimal (possibly negative) transfer of numeraire, which makes the set of feasible alternatives under factual policy $B_p(\theta)$ equally attractive as the counterfactual set $B_{p'}(\theta)$. In an abstract decision problem, this notion can be formalized as follows. Equivalent variation is given by the solution to the following program

$$EV_{p,p'}(\theta) \equiv \min_{x \in X, \tau \in \mathbb{R}} \tau, \tag{2}$$

subject to

$$x \in \bar{\Psi}_{p'}(\theta) \text{ and } x \in B_p(\theta) + \tau d.$$

In the general framework, equivalent variation is geometrically represented by a minimal (signed) distance between the factual feasible set $B_p(\theta)$ and the counterfactual upper contour set $\bar{\Psi}_{p'}(\theta)$, along vector d.[3]

Definition (2) naturally extends the Hicksian notion of equivalent variation for a consumer problem to an abstract decision problem. It also reformulates the index in real terms. Our definition highlights the fact that welfare is measurable with respect to preferences and sets of feasible alternatives. The index is not affected by the normalization of utility or prices. For d that coincides with the price numeraire (for the factual policy), our notion is equivalent to the standard money metric index.

An alternative index, *compensating variation*, informs how much of flow d a consumer is willing to sacrifice to not return to the factual policy once the counterfactual policy is implemented. Formally, in terms of equivalent variation, compensating variation can be written as $CV_{p,p'} \equiv -EV_{p',p}$. Consequently, the continuity of equivalent variation demonstrated in the next section is straightforwardly applicable to the compensating variation.

[2]Note that the domain of correspondence B_p is $\mathbb{R}^N$. Correspondence $B_p(\theta) \cap X \equiv \{x \in B_p(\theta) | x \in X\}$ for any $\theta \in \Theta$ gives a set of available alternatives that are affordable.

[3]Note that, in the program that defines equivalent variation, optimization is over a tuple x, τ. This is because equivalent variation is a minimal distance between the two sets, and x is an endpoint of the "arrow" that determines this distance.

3.2 Joint continuity of a family of preferences

We first formalize ordinal continuity in terms of a parameter in an abstract decision problem. Intuitively, a family of preferences is continuous in θ whenever the associated upper counter sets do not vary too much with the small perturbations of the parameter. More precisely, for any convergent sequence of the parameter, contour sets do not implode or explode in the limit. This property is captured by the continuity of the associated weakly-better-than-x correspondence,

$$\Psi(x,\theta) \equiv \{y \in X | y \succeq_\theta x\},$$

that for any pair (x,θ), gives a collection of all alternatives that are at least as good as x with respect to preferences $\succeq_\theta$.

Definition 1. *The family of preferences $\{\succeq_\theta\}_{\theta\in\Theta}$ is jointly continuous on $X \times \Theta$ whenever associated correspondence $\Psi : X \times \Theta \rightrightarrows X$ is continuous (i.e., upper and lower hemicontinuous.)*

Before we derive the implications of ordinal continuity for the comparative statics of choice and welfare, we find it insightful to relate our concept to the continuity of a utility representation. Suppose a jointly continuous function, $U : X \times \Theta \to \mathbb{R}$, represents a family of preferences. It is then straightforward to show that the associated correspondence Ψ is upper hemicontinuous (see proof of Lemma 1, Step 1). However, this correspondence may fail to be lower hemicontinuous.

Example 1. *Consider a set of alternatives $X = [0,4]^2$ and parametric space $\Theta = [0,1]$. For a jointly continuous utility function*

$$U(x,\theta) = (1-\theta) \times \min(x_1,x_2) \tag{3}$$

at $\bar{x} = (2,2)$ and $\theta \in [0,1)$ the upper contour set is $\Psi(\bar{x},\theta) = [2,4]^2$. At $\theta = 1$, this set discontinuously expands to the entire box X and correspondence Ψ is not lower hemicontinuous.

The example shows that a representation of a family of preferences by a jointly continuous utility function does not suffice for the joint continuity of preferences in the sense of Definition 1. How about the implication in the other direction? Note that the joint continuity of family $\{\succeq_\theta\}_{\theta\in\Theta}$ implies a continuity of preferences $\succeq_\theta$ in x for each fixed value $\theta \in \Theta$.[4]

[4]For the singleton set $\Theta = \{\theta\}$, our notion of continuity is stronger than the standard definition of the continuous preferences, according to which the weakly-better sets are closed. The latter condition is implied by the upper hemincontinuity of $\Psi(x) \equiv \{y \in X | y \succsim_\theta x\}$. The lower hemicontinuity imposes some additional restrictions that, for example, rule out thick indifference curves. One can show that with the additional structure on the set, X, the two notions are equivalent for strictly monotone preferences (see Lemma 1).

It then follows from Debreu's theorem that any individual member of a family necessarily admits a utility representation that is continuous in x. In fact, it can be shown that the entire family of preferences that satisfies Definition 1 admits a representation that is *jointly* continuous.[5] Therefore, our notion of joint continuity of preferences is stronger than the representation by a jointly continuous utility function.

We conclude this section with a simple test for the joint continuity of preferences. We say that a set of alternatives X has a maximal element, if there exists $\overline{x} \in X$ such that $x \leq \overline{x}$ for all $x \in X$. The next lemma states a sufficiency condition for the continuity of a weakly-better-than x correspondence.

Lemma 1. *Suppose a convex set X has a maximal element, a parametric family of preferences defined over $X \times \Theta$ admits utility representation $U : X \times \Theta \to \mathbb{R}$ that is jointly continuous and that, for each $\theta \in \Theta$, the preferences are strictly monotone. Then, the correspondence $\Psi : X \times \Theta \rightrightarrows X$ is thereby continuous.*

One can verify continuity of the upper contour correspondence on e.g., a compact box $X = [0, 1]^N$, by providing a strictly increasing utility representation that is jointly continuous in (x, θ) on the domain. For a family of preferences that can be represented by a jointly continuous utility function, such as (3), correspondence Ψ may fail to be lower hemicontinuous due to thick indifference sets that potentially appear in a limit. Such sets are ruled out by the assumption of the strict monotonicity of preferences.

3.3 The main theorem

An outstanding result in mathematical economics, the Maximum Theorem, Berge (1963), shows that, whenever the utility function is jointly continuous, and the budget correspondence is continuous, the value function of the program itself is continuous while, the choice is upper hemicontinuous. This section offers a modern, ordinal variant of the theorem that demonstrates the continuity of an equivalent variation. To this end, we make the following assumption regarding a family of preferences and budget sets:

Assumption 1. *Family $\{\succeq_\theta\}_{\theta \in \Theta}$ is jointly continuous, and correspondences $B_p \cap X : \Theta \rightrightarrows X$ and $B_{p'} \cap X : \Theta \rightrightarrows X$ are continuous and non-empty valued.*

Furthermore, our theorem requires the budget set to have a well-behaved boundary. For this purpose, we assume that feasible alternatives correspondence is derived from a

[5] The argument that supports the representation result is as follows. Define a preference relation $\succeq$ on $X \times \Theta$ by $(x, \theta) \succeq (y, \theta')$ if $\theta = \theta'$ and $x \succeq_\theta y$. By joint continuity of $\{\succeq_\theta\}_{\theta \in \Theta}$ preference $\succeq$ is a close subset of $(X \times \Theta) \times (X \times \Theta)$. Such a relation is known to admit a Richter-Peleg representation, that is a continuous function u such that $(x, \theta) \succeq (y, \theta')$ implies $u(x, \theta) \geq u(y, \theta')$ and $(x, \theta) \succ (y, \theta')$. I am grateful to an anonymous referee for formulating this argument.

constraint, $B_p(\theta) \equiv \left\{x \in \mathbb{R}^N | b_p(x,\theta) \le 0\right\}$, where the function $b_p : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ satisfies the following assumption:[6]

Assumption 2. *The following conditions hold:*

1. *Function b_p is (jointly) continuous and strictly increasing,*

2. *For each $\theta \in \Theta$ there exists $\tau^+, \tau^- \in \mathbb{R}$ such that*

$$X \subset (B_p(\theta) + \tau^+ d) \text{ and } X \cap (B_p(\theta) + \tau^- d) = \emptyset$$

.

Condition (2) in the assumption is technical. It requires that one can translate the budget set along the vector d so that it includes all and none of the alternatives from the set X, respectively. This is a joint condition on the function b_p and the vector d. Note that the condition can be easily verified in both problems considered in Section 2.

Importantly, Assumption 2 permits types of policies that define budget sets with kinks. Consider, for example, the consumer problem from Section 2.1, in which a price of commodity $n = 1, 2$ differs, depending on whether the consumer is buying or selling the good. In that case, the corresponding function is

$$b_p(x) = \sum_{n=1,2} \zeta_n^b \max(x_n - e_n, 0) + \sum_{n=1,2} \zeta_n^s \min(x_n - e_n, 0),$$

where $\zeta_n^b, \zeta_n^s > 0$ is a buying and a selling price, respectively. Function b_p is not differentiable, and the slope of the budget set is not well defined at the endowment point. Still, the function b_p as well as the boundary of the budget set are continuous. As a result, the minimal distance between the factual budget set and the counterfactual upper contour set is well defined, and it changes continuously in θ. Other policies associated with non-smooth budget frontiers include sales taxes with several tax brackets and rationing, among others.

The next lemma provides a test to verify Assumption 1 for feasible correspondences $B_p \cap X : \Theta \rightrightarrows X$ that satisfy Assumption 2. We say that a set of alternatives X has a minimal element, if there exists $\underline{x} \in X$ such that $x \geq \underline{x}$ for all $x \in X$.

Lemma 2. *Suppose convex set X has a minimal element, $B_p(\theta) \cap X$ is non-empty for any $\theta \in \Theta$, and function $b_p : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ satisfies Assumption 2. Correspondence $B_p \cap X : \Theta \rightrightarrows X$ is thereby continuous.*

We now state the ordinal maximum theorem. In this result, we assume a family of preferences $\{\succeq_\theta\}_{\theta \in \Theta}$ over a compact set of alternatives $X \subset \mathbb{R}^N$ and compact parametric space $\Theta \in \mathbb{R}^M$.

[6] I am grateful to an anonymous referee for suggesting the reformulation of the assumption in terms of the translation condition.

Theorem 1. *(An Ordinal Theorem of the Maximum): Suppose Assumptions 1-2 hold. Equivalent variation* $EV_{p,p'} : \Theta \to \mathbb{R}$ *is a well-defined continuous function, while choice correspondences* $x_p : \Theta \rightrightarrows X$ *and* $x_{p'} : \Theta \rightrightarrows X$ *are non-empty and upper hemicontinuous.*

At a high level of abstraction, the continuity of equivalent variation can be understood in terms of the geometry of the contour sets $\bar{\Psi}_{p'}(\theta)$ and of the function $b_p(\cdot, \theta)$. Given the continuous weakly-better-than-x correspondence, for parameter values of θ close to $\bar{\theta}$, the sets $\bar{\Psi}_p(\theta)$ cannot be too different in terms of shape from their limit at $\bar{\theta}$. Analogously, budget sets are very similar to the limit set. It then follows that, for $\theta \simeq \bar{\theta}$, the minimal distance between these two sets, measured along direction d, should not be too different from the same statistic evaluated at $\bar{\theta}$.

In many settings functions b_p and hence the boundaries of budget sets are smooth. This is, for example, the case in both problems considered in Section 2. For the applications with smooth boundaries, the conditions of the theorem can be further simplified as follows:

Assumption 2'. *Function* b_p *is differentiable, and partial derivatives are bounded away from zero, i.e.,* $\partial b_p / \partial x_n \geq \underline{b} > 0$ *for all* $n = 1, ..., N$ *and* $(x, \theta) \in \mathbb{R}^N \times \Theta$.

Under Assumption 2', the function b_p is (jointly) continuous and strictly increasing. In the appendix, we also verify translation condition (2) in Assumption 2. This implies the following:

Corollary 1. *(Smooth boundry): Suppose Assumptions 1 and 2' are satisfied. Theorem 1 holds.*

4 Applications

In this section, we establish the existence and continuity of preference-based welfare in four relevant economic applications. We first consider the general problems of a consumer's and producer's choice. The third application considers a portfolio choice problem while the fourth applications generalize labor-leisure example presented in Section 2.

4.1 Consumer's problem

We now consider the classic consumer's problem with $N < \infty$ commodities. Commodity space is a compact box $X = [0, \bar{x}]^N$ where $\bar{x} > 0$ is large. The consumer's preferences are represented by the continuous utility function $U(x)$ that does not depend on the parameter values. Feasible consumption profiles satisfy $\sum_{n=1}^{N} \theta_n x_n \leq \theta_{N+1}$ where θ_n for all $n \leq N$ gives price of commodity n and θ_{N+1} is income. Parametric space is a compact box $\Theta = \{\theta \in$

$\mathbb{R}^{N+1}|\underline{\alpha} \leq \theta_n \leq \overline{\alpha}$ for $n = 1, ..., N+1\}$ for some $0 < \underline{\alpha} < \overline{\alpha}$. In the factual scenario, prices and income are fixed at some $\overline{\theta} \in \Theta$. In the counterfactual scenario they are given by $\theta \in \Theta$. Welfare numeraire is $d \in \mathbb{R}^N_+/\{0\}$.

Corollary 2. *In the consumer's problem, the equivalent (compensating) variation is a well-defined and continuous function.*

Proof: We verify assumptions of Corollary 1. Observe that X and Θ are non-empty compact boxes. Feasible correspondence can be written as $B_p(\theta) \equiv \left\{x \in \mathbb{R}^N | b_p(x, \theta) \leq 0\right\}$ where $b_p(x, \theta) = \sum_{n=1}^N \theta_n x_n - \theta_{N+1}$. Note that $\partial b_p / \partial x_n = \theta_n \geq \underline{\alpha} > 0$ for all $n = 1, ..., N$ and $(x, \theta) \in \mathbb{R}^N \times \Theta$. Consequently, Assumption 2' is verified. Functions, b_p and $b_{p'}$ are continuous in θ and, hence, by Lemma 2, correspondences $B_p \cap X : \Theta \rightrightarrows X$ and $B_{p'} \cap X : \Theta \rightrightarrows X$ are continuous. Also correspondences are non-empty valued as $0 \in B_p(\theta) \cap X$. Finally, the family of preferences admits a representation that is continuous in x and independent from θ. Hence, this representation is jointly continuous in x, θ. By Lemma 1 the correspondence $\Psi : X \times \Theta \rightrightarrows X$ is continuous—namely, the family of preferences is jointly continuous. Thus, Assumption 1 holds. The result then follows from Corollary 1. □

4.2 Producer's problem

We next look at the textbook problem of a producer choosing production plans from N-dimensional input-output space, $X \subset \mathbb{R}^N$ that is a large compact box. Production sets differ in two scenarios. The factual set is given by $Y_p = \{x \in \mathbb{R}^N | b_p(x) \leq 0\}$ and the counterfactual one is $Y_{p'} = \{x \in \mathbb{R}^N | b_{p'}(x) \leq 0\}$. Functions $b_p : X \to \mathbb{R}$ and $b_{p'} : X \to \mathbb{R}$ satisfy Assumption 2' and $Y_p \cap X$ and $Y_{p'} \cap X$ are non-empty. The producer maximizes profit $U(x, \theta) = \sum_{n=1}^N \theta_n x_n$ on X, where θ_n denotes the price of input x_n. The parametric space is $\Theta = \{\theta \in \mathbb{R}^N | \underline{\alpha} \leq \theta_n \leq \overline{\alpha}$ for $n = 1, ..., N\}$ for some $0 < \underline{\alpha} < \overline{\alpha}$. Welfare numeraire is $d \in \mathbb{R}^N_+/\{0\}$.

Corollary 3. *In the producer's problem, the equivalent (compensating) variation is a well-defined and continuous function.*

Proof: As in the case of the consumer's problem, X and Θ are non-empty compact boxes. For each policy, the feasible correspondence can be written as $B_p(\theta) \equiv \left\{x \in \mathbb{R}^N | b_p(x) \leq 0\right\}$ and it satisfies Assumption 2'. Functions b_p and $b_{p'}$ are independent from θ and they are continous. By Lemma 2, correspondences $B_p \cap X : \Theta \rightrightarrows X$ and $B_{p'} \cap X : \Theta \rightrightarrows X$ are continuous. Finally, a family of preferences admits a representation that is jointly continuous in x, θ and is strictly monotone. By Lemma 1, correspondence $\Psi : X \times \Theta \rightrightarrows X$ is continuous, meaning the family of preferences is jointly continuous. Thus, Assumption 1 holds. The result then follows from Corollary 1. □

4.3 Portfolio choice

We next formalize the continuity argument in the problem of portfolio choice. This application is an extension of the example from Section 2.1. The trader is choosing random consumption in $N < \infty$ states of the world. The preferences over consumption profiles $x \in X = [0, \bar{x}]^N$. The trader has CRRA preferences represented by $U(x, \theta) = Eu(x, \theta)$, where the instantaneous utility function is given by (1). Policies determine Arrow prices and endowment for which feasible consumption profiles satisfy a standard budget constraint $\sum_{n=1}^{N} \zeta_n (x_n - e_n) \leq 0$. Arrow prices and endowments in factual and counterfactual scenarios are denoted by (ζ, e) and (ζ', e'), respectively. The fundamentals satisfy $\zeta, \zeta' \in \mathbb{R}^N_{++}$ and $e, e', d \in \mathbb{R}^N_{+}/\{0\}$. The portfolio choice problem is parametrized by relative risk aversion $\theta \in \Theta \subset \mathbb{R}_{++}$, where Θ is a compact interval.

Corollary 4. *In the portfolio choice problem, the equivalent (compensating) variation is a well-defined and continuous function.*

Proof: X and Θ are non-empty compact boxes. Feasible correspondence can be written as $B_p(\theta) \equiv \left\{ x \in \mathbb{R}^N | b_p(x,) \leq 0 \right\}$ where $b_p(x) = \sum_{n=1}^{N} \zeta_n (x_n - e_n)$. It follows that $\partial b_p / \partial x_n = \zeta_n \geq \min_n \zeta_n > 0$ for all $n = 1, ..., N$ and $(x, \theta) \in \mathbb{R}^N \times \Theta$. Consequently, Assumption 2' is verified. Functions, b_p and $b_{p'}$ are independent from θ, hence continuous. By Lemma 2, correspondences $B_p \cap X : \Theta \rightrightarrows X$ and $B_{p'} \cap X : \Theta \rightrightarrows X$ are continuous. Also correspondences are non-empty valued as $e \in B_p(\theta) \cap X$. Finally, the family of preferences admits a representation that is continuous in x, θ. (For $\theta = 1$, continuity can be easily established using De L'Hopital rule.) The preferences are also strictly monotone for all θ. By Lemma 1, the correspondence $\Psi : X \times \Theta \rightrightarrows X$ is continuous. The result then follows from Corollary 1. □

The choice problem with Arrow securities is outcome-equivalent to the more general framework with arbitrary securities traded in complete financial markets, in which asset prices satisfy the no-arbitrage condition. Consequently, the corollary directly applies to complete market settings with assets. Moreover, the result can be extended to incomplete market problems.

4.4 Labor-leisure problem

Consider the labor-leisure problem from Section 2.2 generalized as follows. Preferences over $x \in X$ are represented by utility $U(x) = u_1(x_1) + u_2(x_2)$ where the individual components u_1 and u_2 are strictly increasing, are differentiable, and satisfy Inada conditions. The wage rate can take arbitrary value $w > 0$. The counterfactual tax rate $\tau : \mathbb{R}_+ \times \Theta \to [0, 1]$ is continuous and satisfies

$$\underline{\alpha} < 1 - \tau(i, \theta) - \partial \tau'(i, \theta) / \partial i < \overline{\alpha},$$

on its domain, for some positive constants $\underline{\alpha} < \overline{\alpha}$. This assumption ensures that consumption $x_2 = [1-\tau] \times i$ is increasing in gross income i and its derivative is bounded away from zero. The parametric space is given by $\Theta \subseteq [0,1]$.

Corollary 5. *In the labor-leisure problem, the equivalent (compensating) variation is a well-defined and continuous function.*

Proof: X and Θ are non-empty compact boxes. The feasible correspondence can be written as $B_p(\theta) \equiv \left\{x \in \mathbb{R}^N | b_p(x,\theta) \leq 0\right\}$ where $b_p(x,\theta) = x_2 - [1-\tau(i,\theta)] \times i$ and labor income is an implicit function of leisure $i = w(1-x_1)$.[7] Then, $\partial b_p/\partial x_1 = [1-\tau(i,\theta) - \partial\tau'(i,\theta)/\partial i]w$ and, hence, $\partial b_p/\partial x_1 \geq \underline{\alpha} w > 0$ and $\partial b_p/\partial x_2 = 1 > 0$ for all $(x,\theta) \in X \times \Theta$. Consequently, Assumption 2' is verified. Tax rates, and hence functions b_p and $b_{p'}$, are continuous in θ and by Lemma 2 correspondences $B_p \cap X : \Theta \rightrightarrows X$ and $B_{p'} \cap X : \Theta \rightrightarrows X$ are continuous. Also, the correspondences are not empty as endowment point $x = (1,0)$ is feasible for any tax rule. Finally, preferences are independent of θ and are continuous in x. Consequently, a family of preferences admits a representation that is continuous in x, θ. Preferences are strictly monotone for all θ. By Lemma 1 correspondence $\Psi : X \times \Theta \rightrightarrows X$ is continuous, meaning a family of preferences is jointly continuous and Assumption 1 holds. The result then follows from Corollary 1. □

5 Tightness of the Assumptions

We next give three examples to demonstrate that the assumptions of the Berge Maximum Theorem are insufficient for the continuity of equivalent variation and that our assumptions are tight.[8] In all the examples, we assume $X = [0,4]^2$ and $\Theta = [0,1]$ in Figure 3. We first show that our result does not hold when the joint continuity of preferences is replaced by the representation of a jointly continuous utility.

Example 2. *Consider a family of preferences represented by* (3)*. In the previous section, we have demonstrated that this jointly continuous utility function defines correspondence* Ψ *that fails to be lower hemicontinuous. Consider constant budget correspondences,* $B_p(\theta) = \{x \in \mathbb{R}^2 | x_1 + x_2 \leq 0\}$ *and* $B_{p'}(\theta) = \{x \in \mathbb{R}^2 | x_1 + x_2 - 4 \leq 0\}$ *that satisfy Assumptions 1-2. Suppose welfare numeraire is given by commodity one,* $d = (1,0)$*, so that the equivalent variation is*

[7] The counterfactual tax rate is defined for $i \geq 0$. However, one can extend the rate function to negative values $i < 0$ so that the derivative of the function is continuous and bounded. The extension does not affect normative predictions because, by Inada conditions, the consumption-leisure choice is interior.

[8] As a by-product, our proof also clarifies the implicit ordinal assumptions of the Maximum Theorem that underlie the upper hemicontinuity of the choice correspondence. In the Appendix, we show this result under the assumption that the weakly-better-than-x correspondence is upper but not necessarily lower hemicontinuous (see proof of Lemma 3, Step 2).

measured along horizontal axis. For all values $\theta \in [0,1)$, counterfactual choice is $(2,2)$; the upper contour set is $\bar{\Psi}_{p'}(\theta) = [2,4]^2$, and the minimal horizontal distance between this set and factual budget set $B_p(\theta)$ is $EV_{p,p'}(\theta) = 4$. For $\bar{\theta} = 1$ the upper contour set is given by box $\bar{\Psi}_{p'}(\bar{\theta}) = X$, and the distance discontinuously drops to $EV_{p,p'}(\bar{\theta}) = 0$. The corresponding sets are depicted in Figure 3.A.

The continuity of the budget correspondence assumed by the maximum theorem is also by itself insufficient for the continuity of the welfare index, even with jointly continuous preferences.

Example 3. *In this example consider a continuous factual budget correspondence $B_p(\theta) = \{x \in X | x_1 = 1 \text{ or } x_2 = 2\theta\}$ that defines cross-shaped budget sets. Let the counterfactual correspondence be given by a singleton $B_{p'}(\theta) = \{2,2\}$ and the family of the preferences be represented by utility function $U(x,\theta) = \min(x_1, x_2)$ for which the counterfactual upper contour set $\bar{\Psi}_{p'}(\theta) = [2,4]^2$ does not depend on θ. As it is clear from Figure 3.B., for any $\theta \in [0,1)$ the minimal signed horizontal distance between the upper contour set and the factual budget set is $EV_{p,p'}(\theta) = 1$ and the limit equivalent variation at $\bar{\theta} = 1$ discontinuously decreases to $EV_{p,p'}(\bar{\theta}) = -2$. Assumption 2 eliminates such discontinuities of the welfare index.*

Finally, the equivalent variation may fail to exist in settings with jointly continuous preferences and a continuous boundary function, when the translation condition (2) in Assumption 2 is not satisfied. Note, that the example also violates alternative Assumption 2' as partial derivatives of the function b_p are not bounded away from zero.

Example 4. *In the previous example replace the factual budget set with the set derived from the budget constraint function*

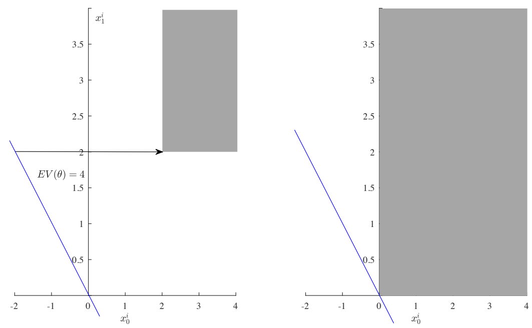
$$b_p(x,\theta) = \begin{cases} -(x_1-3)(x_2-2)+1 & \text{if } x \leq (3,2) \\ \tilde{b}_p & \text{otherwise} \end{cases} .$$

where $\tilde{b}_p$ is an arbitrary strictly increasing and differentiable extension of the function $-(x_1 - 3)(x_2-2)+1$ for $x \notin \{y \in \mathbb{R}^2 | y \leq (3,2)\}$.[9] As we show in Figure 3.C., even though the budget set has a smooth boundary, with counterfactual upper contour set $\bar{\Psi}_{p'}(\theta) = [2,4]^2$ an equivalent variation is not well-defined for any θ.

[9] Note that the extension takes values above one and, hence, is inconsequential for the shape of the budget set.

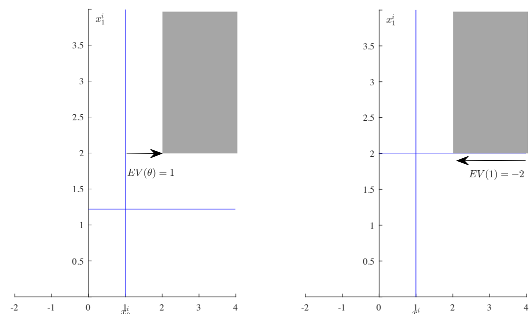
Figure 3. Failures of the Ordinal Maximum Theorem

A)



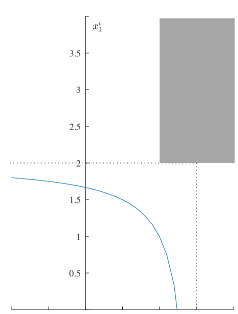
Note: The figure demonstrates the failure of the ordinal maximum theorem due to a lack of lower hemicontinuity of correspondence Ψ. The left panel shows the counterfactual upper contour set $\bar{\Psi}_{p'}(\theta)$ (shaded area) and the factual budget set $B_p(\theta)$ (the area south-west of the solid blue line) for $\theta \in [0, 1)$. The minimal horizontal distance between the sets is 4. The panel on the right shows the analogous sets for $\bar{\theta} = 1$. Due to the discontinuous "explosion" of the upper contour set, the distance between sets drops to zero.

B)



Note: The figure demonstrates the necessity of a smooth downward-slopping boundary of budget sets. The cross-shaped budget set consists of the horizontal and vertical lines. For each $\theta \in [0, 1)$ the horizontal part of the budget set is below set $\bar{\Psi}_{p'}(\theta)$ and, thus the minimal horizontal distance between the two sets is determined by the vertical part and is equal to one (left panel). In the limit $\bar{\theta} = 1$, the horizontal part of the budget set becomes relevant, and the signed minimal horizontal distance is -2.

C)



Note: The figure demonstrates the non-existence of the equivalent variation due to the unbounded derivatives of $b_p(\cdot, \theta)$. As it is clear from the picture, no horizontal shift of a budget set allows for the attainment of a point in $\bar{\Psi}_{p'}(\theta)$ and equivalent variation is not well-defined.

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Appendices

A Tests of continuity

We first offer the proofs of the cardinal/nominal tests of continuity for the relevant correspondences.

Proof of Lemma 1: The proof consists of two steps: showing upper and lower hemicontinuity.

Step 1. In this step we show upper hemicontinuity. Consider an arbitrary convergent sequence in $X \times \Theta \times X$, denoted by $z^h, \theta^h, y^h \to \bar{z}, \bar{\theta}, \bar{y}$, such that $y^h \in \Psi(z^h, \theta^h)$ for each $h = 1, 2, \ldots$. Preferences admit a utility representation, for which $U(y^h, \theta^h) \geq U(z^h, \theta^h)$. By joint continuity of U, the inequality is preserved in the limit and $U(\bar{y}, \bar{\theta}) \geq U(\bar{z}, \bar{\theta})$, which implies $\bar{y} \in \Psi(\bar{z}, \bar{\theta})$.

Step 2. In this step we show lower hemicontinuity. Consider convergent sequence $z^h, \theta^h \to \bar{z}, \bar{\theta}$ in $X \times \Theta$ and $\bar{y} \in \Psi(\bar{z}, \bar{\theta})$. Note that utility function satisfies $U(\bar{y}, \bar{\theta}) \geq U(\bar{z}, \bar{\theta})$. We consider two cases. Suppose first that the limit point is the maximal element, i.e., $\bar{y} = \overline{x}$. Define subsequence $y^k = \overline{x}$ for all $k = 1, 2, \ldots$ and $z^k, \theta^k \equiv z^{h_{(k)}}, \theta^{h_{(k)}}$ where $h_{(k)} = k$. Since by strict monotonicity $U(y^k, \theta) = U(\overline{x}, \theta) \geq U(x, \theta)$ for all $(x, \theta) \in X \times \Theta$, so $y^k \in \Psi(z^k, \theta^k)$, and $y^k \to \overline{x} = \bar{y}$, which completes the argument.

Suppose now that $\bar{y} < \overline{x}$. Define sequence of y^k as follows. For any k let $y^k \equiv \alpha^k \bar{y} + \left(1 - \alpha^k\right) \overline{x} > \bar{y}$ where α^k is large enough so that $\|y^k - \bar{y}\| \leq 1/k$ and hence $y^k \to \bar{y}$. By strict monotonicity, $U(y^k, \bar{\theta}) > U(\bar{y}, \bar{\theta}) \geq U(\bar{z}, \bar{\theta})$. Define the subsequence $z^k, \theta^k \equiv z^{h_{(k)}}, \theta^{h_{(k)}}$ as follows. For $k = 1$ choose index $h_{(1)}$ such that $U(y^1, \theta^{h_{(1)}}) > U(z^h, \theta^{h_{(1)}})$ is satisfied. Note that such an element exists since $z^h, \theta^h \to \bar{z}, \bar{\theta}$, and the strict inequality is preserved by the fact utility function is jointly continuous. Call it $z^{h(1)}, \theta^{h(1)}$. Repeat this step for $k = 2, 3, ...$, each time selecting an element $h(k)$, given by $z^{h_{(k)}}, \theta^{h_{(k)}}$, from the sequence truncated to $h = h_{(k-1)} + 1, ...$ elements. By construction, $y^k \to \bar{y}$ and $y^k \in \Psi(z^k, \theta^k)$. □

Proof of Lemma 2 : The proof consists of two steps: showing upper and lower hemicontinuity.

Step 1. In this step we show upper hemicontinuity. Consider arbitrary convergent sequence in $\Theta \times X$, denoted by $\theta^h, y^h \to \bar{\theta}, \bar{y}$, such that $y^h \in B_p(\theta^h) \cap X$ for each $h = 1, 2, ...$ By definition of feasible correspondence $b_p(y^h, \theta^h) \leq 0$. By joint continuity of b_p, the inequality is preserved in the limit and $b_p(\bar{y}, \bar{\theta}) \leq 0$, which implies $\bar{y} \in B_p(\bar{\theta}) \cap X$.

Step 2. In this step we show lower hemicontinuity. Consider a convergent sequence $\theta^h \to \bar{\theta}$ in Θ and $\bar{y} \in B_p(\bar{\theta}) \cap X$. By definition of feasible correspondence, $b_p(\bar{y}, \bar{\theta}) \leq 0$. We consider two cases. Suppose first the minimal element $\bar{y} = \underline{x}$. Define constant sequence $y^k = \underline{x}$ for all $k = 1, 2, ...$ Since $B_p(\theta^k) \cap X$ is non-empty, there exists $x \in X$ for which $b_p(x, \theta^h) \leq 0$. Since $\underline{x} \leq x$, $b_p(\underline{x}) \leq b_p(x) \leq 0$ and hence $y^k \in B_p(\theta^k) \cap X$ and $y^k \to \underline{x} = \bar{y}$, which completes the argument.

Suppose now that $\bar{y} > \underline{x}$. For $k = 1, 2, ...$, let $y^k \equiv \alpha^k \bar{y} + \left(1 - \alpha^k\right) \underline{x} < \bar{y}$ where α^k is large enough so that $\|y^k - \bar{y}\| \leq 1/k$. By construction, $y^k \to \bar{y}$. For any k, by strict monotonicity in the first argument, $b_p(y^k, \bar{\theta}) < b(\bar{y}, \bar{\theta}) \leq 0$. Define subsequence $\theta^k \equiv \theta^{h_{(k)}}$ as follows. For $k = 1$ choose index $h_{(1)}$ such that $b_p(y^k, \theta^{h_{(1)}}) < 0$ is satisfied. Note that such element exists since $\theta^h \to \bar{\theta}$, and the strict inequality is preserved by the fact that b_p function is jointly continuous. Call it $\theta^{h_{(1)}}$. Repeat this step for $k = 2, 3, ...$ each time selecting element $\theta^{h_{(k)}}$ from the sequence truncated to $h = h_{(k-1)} + 1, ...$ elements. By construction, $y^k \to \bar{y}$ and $y^k \in B_p(\theta^k) \cap X$.

B Main Theorem and Corollary

Proof of Theorem 1:

Throughout the proof, we fix policies and welfare numeraire p, p', d and use the shorthand notation $EV = EV_{p,p'}$. By $z : \Theta \rightrightarrows X$ we denote the correspondence that for each $\theta \in \Theta$ gives all the alternatives $z \in X$ for which tuple (z, τ) solves the program (2).

The proof of the theorem proceeds as follows. In Lemma 3 we demonstrate the existence and the continuity of the choice correspondence. Lemma 4 shows that the equivalent variation is well-defined for any parameter value and policy pair, and that it is attained on a compact set. Lemma 5 demonstrates the continuity of the equivalent variation on a parametric space.

Lemma 3. *For any p, the correspondence $x_p : \Theta \rightrightarrows X$ is non-empty valued and upper hemicontinuous.*

Proof of Lemma 3:

Step 1. Non-empty valuedness of the correspondence $x_p : \Theta \rightrightarrows X$. Fix $\theta \in \Theta$. For any $x \in X$, the sets $\{y \in X | y \succeq_\theta x\}$ and $\{y \in X | x \succeq_\theta y\}$ are closed; otherwise, there would exist convergent sequences in X, $x^h, y^h \to \bar{y}, \bar{x}$ such that $x^h \succeq_\theta y^h$ but $\bar{y} \succ_\theta \bar{x}$, contradicting upper hemicontinuity of Ψ (that is equivalent to the property of a closed graph, given compact X). Thus, preferences

$\succeq_\theta$ are continuous for all $\theta \in \Theta$. By the Debreu representation theorem, preferences $\succeq_\theta$ admit a utility representation continuous in x for each θ. Set $B_p(\theta)$ is a preimage of a closed interval by a continuous function $b_p(\cdot, \theta)$, and hence it is closed. Thus, $B_p(\theta) \cap X$ is compact. By Assumption 1, it is also non-empty. It then follows from the extreme value theorem that the optimal choice is attained on $B_p(\theta) \cap X$ and the choice correspondence $x_p : \Theta \rightrightarrows X$ is non-empty valued.

Step 2. Upper hemicontinuity of $x_p : \Theta \rightrightarrows X$: Consider a convergent sequence $x^h, \theta^h \to \bar{x}, \bar{\theta}$ for which $x^h \in x_p(\theta^h)$. Since $B_p(\theta) \cap X$ is upper hemicontinuous, it has a closed graph and therefore $\bar{x} \in B_p(\bar{\theta}) \cap X$. Consider any alternative $\bar{y} \in B_p\left(\bar{\theta}\right) \cap X$ and any arbitrary sequence $y^h, \theta^h \to \bar{y}, \bar{\theta}$. Since by Assumption 1 correspondence $B_p(\cdot) \cap X$ is lower hemicontinuous, there exists convergent subsequence $y^k, \theta^k \to (\bar{y}, \bar{\theta})$ such that $y^k \in B_p\left(\theta^k\right) \cap X$. By optimality, $x^k \in x_p(\theta^k)$ and $y^k \in B_p\left(\theta^k\right) \cap X$; therefore, one has $x^k \in \Psi\left(y^k, \theta^k\right)$. Since Ψ is upper hemicontinuous with compact range, it has closed graph and $\bar{x} \in \Psi\left(\bar{y}, \bar{\theta}\right)$. This in turn implies $\bar{x} \succeq_{\bar{\theta}} \bar{y}$. This is true for all $\bar{y} \in B_p\left(\bar{\theta}\right) \cap X$ and $\bar{x} \in x_p\left(\bar{\theta}\right)$. Note that continuity of $x_{p'} : \Theta \rightrightarrows X$ holds by identical arguments.

□

In the next lemma we show that equivalent variation is well-defined.

Lemma 4. *Equivalent variation exists and is uniformly bounded.*

Proof of Lemma 4: Fix $\bar{\theta} \in \Theta$. Define the closed interval $Q \equiv [\tau^-, \tau^+] \subset \mathbb{R}$, where the corresponding endpoints are from Assumption 2. By point (1) of this assumption, the set $B_p(\theta) + \tau d$ is increasing in τ. Therefore from point (2) of the same assumption it follows that $\tau^- < \tau^+$ and interval $Q \equiv [\tau^-, \tau^+] \subset \mathbb{R}$ is non-empty. We first show that Program 2, augmented by additional constraint $\tau \in Q$, has a solution. Then we demonstrate that the constraint is not binding, and so the solution to the restricted program defines equivalent variation.

Let

$$A \equiv \left\{(z, \tau) \in X \times Q | z \in B_p(\bar{\theta}) + \tau d\right\},$$

and

$$B \equiv \{(z, \tau) \in X \times Q | z \in \bar{\Psi}_{p'}(\bar{\theta})\}.$$

Program 2 augmented by constraint $\tau \in Q$ can be reformulated as $\min \tau$ subject to $(z, \tau) \in A \cap B \subset \mathbb{R}^{N+1}$. Set A is closed (by continuity of b_p and closeness of $X \times Q$) and bounded (by boundedness of $X \times Q$). Similarly, set B is closed (by closeness of X, Q and continuity of preferences) and bounded (by boundedness of $X \times Q$). It follows that $A \cap B \subset \mathbb{R}^{N+1}$ is compact. Pick arbitrary $z \in x_{p'}(\bar{\theta})$, that by Lemma 3 is well defined. Note that $z \in \bar{\Psi}_{p'}(\bar{\theta})$ and hence $z \in X$. It follows that $(z, \tau^+) \in B$. Moreover, by point (2) of Assumption 2, one has $z \in X \subset B_p(\bar{\theta}) + \tau^+ d$, hence $(z, \tau^+) \in A$. It follows that $(z, \tau^+) \in A \cap B$, i.e., the set $A \cap B$ is non-empty. By the extreme value theorem, a solution to the program exists and is attained on set $A \cap B$. Denote it by $\bar{z}, \bar{\tau}$. Note that the constraint $\tau \in Q$ is not binding since $A \cap B = \emptyset$ for all $\tau \leq \tau^-$ and $A \cap B = B$ is constant for all $\tau \geq \tau^+$.

The two observations imply that $(\bar{z}, \bar{\tau})$ is a solution to unrestricted Program 2. Since minimum takes at most one value, equivalent variation $EV(\cdot)$ is a well-defined function on Θ. □

We next prove continuity of equivalent variation.

Lemma 5. *Equivalent variation is continuous.*

Proof of Lemma 5:

Suppose that, for some $\bar{\theta} \in \Theta$, equivalent variation is not continuous. This implies that there exists $\varepsilon > 0$ such that for any $\delta > 0$ one can find θ satisfying $\left\|\theta - \bar{\theta}\right\| < \delta$ for which either $EV(\theta) \geq EV\left(\bar{\theta}\right) + \varepsilon$ or $EV(\theta) \leq EV\left(\bar{\theta}\right) - \varepsilon$. This in turn implies that there exists sequence $\theta^h \to \bar{\theta}$ in Θ for which either $EV\left(\theta^h\right) \geq EV(\bar{\theta}) + \varepsilon$ or $EV\left(\theta^h\right) \leq EV(\bar{\theta}) - \varepsilon$ for all $h = 1, 2, ...$ We argue that this is an impossibility.

Step 1. Upper bound. Suppose there exists a sequence $\theta^h \to \bar{\theta}$ and $\varepsilon > 0$ such that $EV\left(\theta^h\right) \geq EV\left(\bar{\theta}\right) + \varepsilon$ for all h. Pick an arbitrary $x^h \in x_{p'}\left(\theta^h\right)$. Note that by Lemma 3, the set $x_{p'}\left(\theta^h\right)$ is non-empty so such x^h exists. For any θ^h, the maximal upper contour set attainable under p' can be written as $\bar{\Psi}_{p'}\left(\theta^h\right) = \Psi\left(x^h, \theta^h\right)$. Since $x^h \in X$, and X is compact, there exists a convergent subsequence $\theta^k, x^k \to \bar{\theta}, \bar{x}$. By Lemma 3, correspondence $x_{p'}(\cdot)$ is upper hemicontinuous and by compactness of X it has a closed graph. This implies that $\bar{x} \in x_{p'}\left(\bar{\theta}\right)$. The maximal upper contour set at $\bar{\theta}$ is given by $\bar{\Psi}_{p'}\left(\bar{\theta}\right) = \Psi\left(\bar{x}, \bar{\theta}\right)$.

Consider the convergent sequence $\theta^k, x^k \to \bar{\theta}, \bar{x}$, defined in the previous paragraph. Let $\bar{z}, \bar{\tau} \in X \times Q$ be a solution to the program that defines equivalent variation at $\bar{\theta}$. By Lemma *4*, such solution exists. By definition of equivalent variation, $\bar{z} = \Psi\left(\bar{x}, \bar{\theta}\right)$ and $\bar{z} - d\bar{\tau} \in B_p\left(\bar{\theta}\right)$. Since by Assumption 1 correspondence Ψ is lower hemicontinuous, there exists a subsequence $x^l, \theta^l \to \bar{x}, \bar{\theta}$ and $z^l \to \bar{z}$ such that $z^l \in \Psi\left(x^l, \theta^l\right) = \bar{\Psi}_{p'}\left(\theta^l\right)$ for all l. For each $m = 1, 2, ...$ let $\tau^m \equiv \bar{\tau} + 1/m$. By condition (1) in Assumption 2 one has $b_p(\bar{z} - \tau^m d, \bar{\theta}) < 0$ for all m. By joint continuity of b_p there exists an increasing sequence of natural numbers L^m such that $b_p(z^l - \tau^m d, \theta^l) < 0$ for $L^m \leq l < L^{m+1}$. For any m let $(z^m, \theta^m) \equiv (z^l, \theta^l)$ where $L^m \leq l < L^{m+1}$. By construction $z^m \in B_p(\theta^m) + \tau^m d$ and $z^m = \bar{\Psi}_{p'}\left(\theta^m\right)$ for all m. Hence, $\tau^m \geq EV\left(\theta^m\right) \geq EV\left(\bar{\theta}\right) + \varepsilon$, which is a contradiction since $\tau^m \to \bar{\tau} = EV(\bar{\theta})$.

Step 2. Lower bound: Suppose there exists a sequence $\theta^h \to \bar{\theta}$ and $\varepsilon > 0$ such that $EV\left(\theta^h\right) \leq EV\left(\bar{\theta}\right) - \varepsilon$ for all h. For each h, pick arbitrary $x^h \in x_{p'}\left(\theta^h\right)$ and $z^h, \tau^h \in X \times Q$ that solves Program 2. By Lemmas 3 and 4, such sequences exist. Since $x^h, z^h \in X$ and $\tau^h \in Q$ for all h, and these sets are compact, sequence x^h, z^h, τ^h has a convergent subsequence $x^k, z^k, \tau^k \to \bar{x}, \bar{z}, \bar{\tau}$ where $\bar{x}, \bar{z} \in X$ and $\bar{\tau} \in Q$. By definition of equivalent variation, for every k, $z^k \in \bar{\Psi}_{p'}\left(\theta^k\right) = \Psi\left(x^k, \theta^k\right)$ and $z^k \in B_p\left(\theta^k\right) + \tau^k d$. By Assumption 1, $\bar{z} \in \Psi\left(\bar{x}, \bar{\theta}\right) = \bar{\Psi}_{p'}\left(\bar{\theta}\right)$ and $\bar{z} \in B_p\left(\bar{\theta}\right) + \bar{\tau} d$. Thus, pair $\bar{z}, \bar{\tau}$ satisfies constraints in Program 2 at $\bar{\theta}$. Hence,

$$\lim_{k\to\infty} EV(\theta^k) = \lim_{k\to\infty} \tau^k \equiv \bar{\tau} \geq EV\left(\bar{\theta}\right).$$

Therefore, equivalent variation evaluated at $\bar{\theta}$ is the lower bound for the limit of the subsequence $EV\left(\theta^k\right)$. This in turn contradicts proposition $EV\left(\theta^h\right) \leq EV\left(\bar{\theta}\right) - \varepsilon$ for all n given fixed $\varepsilon > 0$.□

Proof of Corollary 1 :

Under alternative Assumption 2', the function b_p is differentiable and hence is (jointly) continuous. Since the partial derivatives are strictly positive, the function is strictly increasing. We next verify the translation condition, (2), in Assumption 2. Fix $\theta \in \Theta$.

Step 1. Existence of τ^+: Define $\bar{x} = (\bar{x}_1, ..., \bar{x}_N) \in \mathbb{R}^N$ as $\bar{x}_n = \max_{x \in X} x_n$ for any n. Observe that $X \subset \{x \in \mathbb{R}^N | x \leq \bar{x}\}$. If $b_p(\bar{x}, \theta) \leq 0$ let $\tau^+ \equiv 0$. By the monotonicity of the function b_p, for any $x \in X$, $b_p(x, \theta) \leq b_p(\bar{x}, \theta) \leq 0$. Therefore $X \subset (B_p(\theta) + \tau^- d)$. Next suppose $b_p(\bar{x}, \theta) > 0$. Let $\tau^+ \equiv b_p(\bar{x}, \theta) / (\underline{b} \sum_{n=1}^N d_n) > 0$. By the mean value theorem,

$$b_p(\bar{x}, \theta) - b_p\left(\bar{x} - \tau^+ d, \theta\right) \geq (\underline{b} \sum_{n=1}^N d_n) \tau^+ = b_p(\bar{x}, \theta), \tag{4}$$

hence $b_p(\bar{x} - \tau^+ d, \theta) \leq 0$. By the monotonicity of the function b_p, for any $x \in X$, $b_p(x - \tau^+ d, \theta) \leq 0$ and hence $x \in B_p(\theta) + \tau^- d$. This implies $X \subset (B_p(\theta) + \tau^- d)$.

Step 2. Existence of τ^-: Define $\underline{x} = (\underline{x}_1, ..., \underline{x}_N) \in \mathbb{R}^N$ as $\underline{x}_n = -1 + \min_{x \in X} x_n$ for any n. Observe that $X \subset \{x \in \mathbb{R}^N | x >> \underline{x}\}$. If $b_p(\underline{x}, \theta) \geq 0$ let $\tau^- \equiv 0$. By the strict monotonicity of the function b_p, for any $x \in X$, $b_p(x, \theta) > b_p(\underline{x}, \theta) \geq 0$. Therefore $X \cap (B_p(\theta) + \tau^- d) = \emptyset$.

Next suppose $b_p(\underline{x}, \theta) < 0$. Let $\tau^- \equiv b_p(\underline{x}, \theta) / (\underline{b} \sum_{n=1}^N d_n) < 0$. By the mean value theorem,

$$b_p(\bar{x}, \theta) - b_p\left(\bar{x} - \tau^- d, \theta\right) \leq (\underline{b} \sum_{n=1}^N d_n) \tau^- = b_p(\bar{x}, \theta), \tag{5}$$

hence $b_p(\bar{x} - \tau^- d, \theta) \geq 0$. By monotonicity of function b_p, for any $x \in X$, $b_p(x - \tau^- d, \theta) > 0$ and, hence, $x \notin B_p(\theta) + \tau^- d$. This implies $X \cap (B_p(\theta) + \tau^- d) = \emptyset$. □