Design of Divisible Good Markets*

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March 16, 2010

Abstract

Uniform price and discriminatory price auctions are the two most common formats for selling divisible goods. This paper establishes the revenue rankings for these formats and the Vickrey auction in markets with strategic buyers. Our analysis underscores the key role that bidder market power plays in design. We further rank the formats in terms of the criteria employed in the practical design of markets for divisible goods, such as encouraging bidder participation, fostering more aggressive bidding, and stabilizing prices. Our model accommodates small and large markets, as well as different risk preferences of the buyers and the seller.

JEL classification: D43, D53, G11, G12, L13

Keywords: Market Design, Price Impact, Demand Reduction, Divisible Good

1 Introduction

Markets for electricity, emission permits, gold and Treasury bonds all involve the sale of multiple units of a good. When placing orders for multiple units, large bidders reduce their demands in response to price impact. Mitigating bidder market power has been one of the central challenges in many divisible good markets. Indeed, nearly half of the issue in Treasury auctions is purchased by the top five bidders (*U.S. Treasury Report* (1998)). The

*We would like to thank Steven Durlauf, Ali Hortaçsu, Jakub Kastl, Paul Klemperer, Daniel Marszalec, Meg Meyer, Stephen Morris, James Wang and Jaime Zender for helpful comments and suggestions. We also benefitted from comments by seminar participants at MIT and Oxford (Nuffield), and conference participants at the Fall 2009 Midwest Theory Meetings at Penn State, the 2009 SAET Conference at Ischia and the 2009 SED Meeting in Istanbul. Rostek and Weretka gratefully acknowledge the financial support of the National Science Foundation (SES-0851876).

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concentration levels are even higher in electricity markets. Bidder price impact is of great practical importance, not only for market designers, but also for bidders in the construction of their optimal bidding strategy.

In practice, the two mechanisms most commonly used to sell divisible goods are *uniform price* and *discriminatory price* design. In both formats, bidders submit bidding schedules that specify quantities for various prices. The seller then aggregates these submitted schedules to determine the *stop-out price* (i.e., the price that clears the market), and bidders are allocated units for which their demands exceeded the market-clearing price. The formats differ in terms of payment; in a uniform price (*UPA*, "single price") auction, all winning bids are filled at the stop-out price, whereas in a discriminatory price (*DPA*, "pay-your-bid," or "multiple-price") auction, the marginal payment for different units is collected according to schedules submitted by the buyers. In a DPA, a buyer thus pays the area below his demand schedule for the quantities awarded.

The question regarding which of these two designs should be used has excited public interest at least since the report by Friedman (1960). Recent cross-country studies on Treasury practices reveal that out of 48 countries surveyed by Brenner, Galai and Sade (2009), 24 use a DPA to finance public debt, 9 use a UPA, and 9 employ both auction formats, depending on the type of security being issued.[1] Apart from Treasury securities, the two formats have also become standard designs when selling divisible goods in other financial markets, such as IPOs and repo,[2] markets for electricity,[3] and the national exchange for sulfur-dioxide emission permits, which uses a discriminatory price format.

Despite the importance of divisible good markets, providing theoretical results about the performance of the two designs has proven challenging. Important results are available on the revenue rankings for large, competitive markets (Swinkels (2001); Jackson and Kremer (2006)). Little is known, however, about the superiority of either format in markets that involve strategic bidders with market power. This paper provides those results and examines the design of divisible good markets more generally in a simple, symmetric-information setting.

[1] The remaining 6 countries use a different mechanism. In the United States, the discriminatory price auction was the traditional format used by the U.S Treasury to sell securities. The Treasury has been using this auction format to issue notes and bonds since the 1970s, and Treasury bills have been sold using multiple-price auctions since 1929. In September 1992, the Treasury began experimenting with the uniform-price format, encouraged by Milton Friedman as early as 1960 (Friedman (1960)), and in November 1998 adopted this design, which it still uses today, for all marketable securities. Empirical evidence regarding the superiority of either auction format in the Treasury experiment was inconclusive. (*U.S. Treasury Reports* (1995, 1998) provide a detailed overview of the findings.)

[2] The European Central Bank uses auctions in refinancing (repo) operations on a weekly and monthly basis. Since July 2000, these auctions have been discriminatory (see Bindseil, Nyborg and Strebulaev (2006)).

[3] U.K. electricity generators sell their products via daily auctions. A uniform price format was adopted in 1990, but in 2000 the U.K. electricity auctions switched to the discriminatory price format.

Our model of a divisible good market most closely resembles the classical uniform-price setting by Klemperer and Meyer (1989). A given quantity of a perfectly divisible good is sold to I symmetric strategic buyers who have (weakly) decreasing linear marginal utility (e.g., mean-variance preferences). Supply is uncertain, which in the U.S. and other Treasury markets, for instance, captures the presence of the so-called *non-competitive bids* that are guaranteed to be filled. We examine the comparative design of three auction formats for divisible goods: The commonly used uniform and discriminatory price auctions, and—as a theoretical benchmark—the Vickrey auction (*VA*).[4] By analyzing linear Nash equilibria, which are unique for all designs, we are able to characterize bidder equilibrium price impact and deliver sharp comparisons. We study markets with an arbitrary, possibly small, number of strategic bidders, and both risk averse and risk neutral preferences of the bidders and the seller. One novel aspect of our analysis is a characterization of market power in the considered formats, which is instrumental in explaining bidder incentives, rankings of designs according to a number of criteria employed in practice, and market size effects on the relative performance of the three auction formats.

Results. We now describe the main predictions.

When bidders have price impact, there is a strong expected-revenue ranking of all designs. Most notably, the DPA dominates the UPA for all environments considered, and the UPA generates less revenue than the Vickrey auction in the strong *ex-post* sense. Since bidders have downward-sloping demands, even when there is no private information, none of the designs extract all of surplus. Since each format fails to extract all the surplus for a different reason (e.g., payments are less than bids in the Vickrey auction, bids are less than values in the discriminatory price auction, etc.), one expects the formats to generate different expected revenue. The unexpected result is that the ranking of expected revenues is always the same. Our main result can be explained through the following link between the two designs we uncover: Suppose that bidders had the same equilibrium price impact (i.e., faced residual supply with the same slope) in the UPA and the DPA. We show that the reduction of expected revenue due to optimal order shading in the DPA will then exactly match the loss of expected revenue due to the uniform payment structure in the UPA with price-taking bidders. The revenue superiority of the DPA can then be attributed to two effects: In equilibrium in the UPA, bidders do shade their bids in response to their market power, and also, bidder equilibrium price impact is higher in the UPA than in the DPA.

In the practical design of divisible good markets, considerations that are separate from

[4] In our setting, the (sealed-bid) Vickrey auction is equivalent to the ascending-bid clinging auction proposed by Ausubel (2004).

revenue are important in evaluating market performance. Motivated by the criteria used, for example, by the U.S. Department of the Treasury (1995, 1998), we further examine additional criteria of design for divisible good markets—encouraging bidder participation, viewed as an enhancement of market liquidity (UPA bests DPA); stabilizing the stop-out price, which is important in practice if other contracts are pegged to the stop-out price (DPA outperforms UPA); fostering more aggressive bidding (UPA outruns DPA); and increasing the transparency of an auction (UPA dominates DPA). That analysis highlights three results.

Examination of how the non-competitiveness of market structure—a change in market size, along with the induced change of bidder market power—affects the expected-revenue ranking of the considered formats indicates that the expected-revenue advantage of the DPA weakens in more competitive markets. As the market grows and the bidder price impact vanishes, the three designs bring the same expected revenue. Hence, in large markets, a risk neutral seller should be indifferent to auction format. One insight from the setting that we consider is that the competitive-market result does not follow from the standard revenue equivalence theorem, as the revenue obtained in the three designs is strictly lower than in the revenue-maximizing mechanism, despite the efficient allocation in all formats.

For small markets with strategic bidders, our model reveals a trade-off between expected revenue and riskiness for the seller. Precisely, for any distribution of market supply, one can find (risk averse) preferences for which either auction format is preferred by the seller. In turn, in large markets, while equivalent in expected revenues, the UPA, DPA, and VA are clearly not equivalent *ex post*. However, the revenue in the UPA and the VA second-order stochastically dominates that of the DPA for all distributions of market supply. As a result, the uniform price and Vickrey formats should be strictly preferred by any risk averse seller.

Further, our analysis draws attention to the critical role of entry in the assessment of design performance. The view shared by practitioners and theorists alike is that market design should encourage more competitive market structures. For markets of single objects, a compelling formalization of this point was offered by Bulow and Klemperer (1996), who demonstrated that attracting a new bidder is more important for revenue than is the choice of a particular selling mechanism. We argue that this recommendation is even more important when selling divisible goods. When bidders have decreasing marginal utility, even when they are identical, a new bidder increases the total available surplus in the market—an effect not present in the markets for single objects. One key lesson is that, while entry is greater in the UPA, from a social-welfare standpoint, entry is excessive in all of the designs. Thus, a seller concerned about social welfare would prefer the DPA.

Other Related Literature. To put our model and results in perspective, the reasons behind the lack of general theoretical guidelines for the design of divisible good markets are the following. In the seminal analysis of divisible good ("share") auctions, Wilson (1979) demonstrated that the uniform price and discriminatory price auctions have a continuum of (Bayesian) Nash equilibria, which significantly differ in the revenues predicted. More recently, beginning with Ausubel and Cramton (2002) and Wang and Zender (2002), several studies constructed theoretical examples showing that the sets of equilibria of a DPA and a UPA cannot be unambiguously ranked in terms of revenue. Given the lack of dominance of equilibrium sets for the two auction formats, it is hard to compare the revenue or other criteria in the uniform and discriminatory price auctions without a plausible equilibrium selection argument. For the uniform price design, the equilibrium selection that has become the workhorse model in the financial microstructure and industrial organization literature is the linear (Bayesian) Nash equilibrium (Kyle (1989), Vives (2008)). For the discriminatory price format, despite there being a number of insightful characterizations of optimal bids, revenue rankings have yet to be provided for markets with strategic bidders.[5]

Working with a linear Nash equilibrium allows us to overcome both problems discussed above: This paper provides an analytical characterization of the unique symmetric equilibrium bidding strategies in the discriminatory price auction with an arbitrary number of bidders in all settings that admit linear equilibria. Thus, the paper offers a complete characterization of the comparative design of the considered formats for linear Nash equilibria, in the following sense. In divisible good markets, by making choices contingent on prices, bidders can tailor their demands to each state of the world. In the discriminatory price (but not in the uniform price or Vickrey format), bidding aggressively in one state affects payments in other states, and the importance of such cross-state externality depends on the uncertainty about the residual supply. The dependence of the optimal bid on the type of distribution raises the question about which distributions admit a linear equilibrium in a DPA. We fully characterize such distributions and establish rankings for all environments that admit linear equilibria.

[5] Federico and Rahman (2003) examined competitive and monopolistic market structures. The present paper instead studies strategic interactions in markets with I bidders. Hortaçsu (2002a) derived a symmetric linear Bayesian equilibrium for private-value auctions with two bidders and exponentially distributed private signals. Unfortunately, Hortaçsu's result cannot be used to rank the revenues between the UPA and the DPA, even for markets with two bidders, because there exists no linear equilibrium with two bidders in the UPA. Working with step bidding functions in a private-value setting with I bidders, Kastl (2008a) demonstrated that, as the number of bid-points one is allowed to submit increases, the first-order condition obtained by Hortaçsu (2002a) for two bidders obtains in the limit. In response to the lack of theoretical results for strategic settings, the question of revenue ranking has been studied in the growing empirical literature that seeks structural methods to compare auction mechanisms, for a given data set of individual bids, by constructing policy counterfactuals (Hortaçsu (2002b); Wolak (2003, 2007); Février, Préget and Visser (2004); Armantier and Sbai (2006); Hortaçsu and Puller (2008); Kastl (2008b)).

Structure of the Paper. Section 2 presents the model of divisible good auctions. Section 3 derives the linear equilibria in the Vickery, uniform price, and discriminatory price auctions and examines the relative role of market power and uncertainty when bidding in the three formats. Section 4 reports our central results concerning the ranking of revenues and other criteria that have been used in design. Section 5 examines which results extend to settings with private information among other characteristics. Finally, Section 6 offers conclusions. All proofs are presented in the Appendix.

2 Model Set-up

2.1 General Assumptions

In a market with $I \geq 2$ bidders,[6] the seller offers for sale $I \times \bar{Q}$ units of a perfectly divisible good, where $\bar{Q}$ is the quantity *per capita*. Each bidder i derives utility from the amount of the divisible good received in an auction, q_i, and money. His utility is given by a stochastic quasilinear utility function $u(q_i) + money_i$. The utility function $u(\cdot)$ is quadratic in the good being auctioned and, thus, in terms of money, the marginal utility is linear in q_i,

$$v(\cdot) \equiv \frac{\partial u(\cdot)}{\partial q_i} = v - \rho q_i, \tag{1}$$

where the stochastic intercept v is the same for all bidders, and the parameter $\rho \in \mathbb{R}_+$ measures the convexity of the utility function.[7] The joint c.d.f. of intercept v and *per capita* supply, $F(v, \bar{Q})$, is common knowledge. We show that the existence of a linear equilibrium (in a DPA) requires that $F(\bar{Q}|v)$ be from the Generalized Pareto class with the support of $\bar{Q}$ starting at 0. It then follows in our comparative analysis that the realizations of $\bar{Q}$ are non-negative. Our model permits correlation between the valuation v and the supply $\bar{Q}$, which is important for the analysis of price volatility. We study the information structure in which bidders know v, but are uncertain about the supply being auctioned (e.g., Klemperer and Meyer (1989)), and the seller does not know v. We later show that assumptions about the bidders' knowledge of the environment can be considerably weakened in our model. It is

[6] In the UPA (but not the DPA), there exists no linear equilibrium with two bidders. Our characterization of the DPA does include this case. The non-existence of equilibrium in the UPA with two traders is also present in closely related models by Klemperer and Meyer (1989) with a vertical demand and Kyle (1989).

[7] When the auctioned good is a risky asset with a normally distributed payoff, and the bidders have CARA utility functions, then ρ measures risk aversion.

useful to introduce an index of *market size*,

$$\gamma \equiv 1 - \frac{1}{I-1}, \tag{2}$$

which is monotone in I and ranges between 0 and 1. The closer γ is to one, the more endogenously competitive the market interactions will be. Throughout, endogenous variables with a bar "$^-$" denote equilibrium.

2.2 Auction Formats

We study the comparative design of three auction formats in markets for divisible goods, the commonly used uniform price and discriminatory price designs, as well as the theoretical benchmark of the Vickrey auction. Bidders submit weakly downward-sloping bid schedules, which specify the quantity demanded for any price, $q_i(p) : \mathbb{R} \to \mathbb{R}_+$. The seller then determines the aggregate demand, $Q(p) \equiv \sum_{i=1}^{I} q_i(p)$, and finds the maximal price $\bar{p}$ for which the demand equalizes the supply, $Q(\bar{p}) = I\bar{Q}$. This price is called a *stop-out price*.[8] In a uniform price, discriminatory price, or Vickrey auction, the quantity obtained by each bidder corresponds to the bid evaluated at the stop-out price, $\bar{q}_i = q_i(\bar{p})$. The three auction formats differ in the payments made by the bidders (in terms of numeraire). In a uniform price auction, bidder i pays the stop-out price for each unit obtained, and the total payment is equal to $\bar{q}_i \times \bar{p}$. In a discriminatory price auction, for each unit, a bidder pays the valuation revealed in his submitted bid, and the total payment corresponds to the area below the submitted bid schedule, up to the quantity awarded, $\int_{\bar{p}}^{p^0} q_i(p) \times p dp$, where p^0 is the price at which the bid quantity attains the value of zero or, otherwise, p^0 equals infinity. Lastly, in the Vickrey auction, for the q^{th} unit, bidder i is charged the reported marginal value of the object by other bidders if the remaining $\bar{Q} - q$ units are reallocated efficiently to other bidders; i.e., the opportunity cost to others, which is uniquely defined, given efficiency. Thus the total payment of bidder i corresponds to the area below the residual supply faced by bidder i.

Throughout, we evaluate the revenue loss resulting from market power and decreasing marginal utility in the three formats, relative to the total surplus *per capita* (TS) equal to

$$TS = v\bar{Q} - \frac{1}{2}\rho\bar{Q}^2. \tag{3}$$

In the symmetric information setting, total surplus can be fully extracted;[9] it coincides with

[8] To complete the definition of the game, if there is no such price or if multiple prices exist, then no trade takes place. As we show, there exists a unique stop-out price.

[9] An example of the optimal mechanism, which extracts the total surplus in the unique Nash equilibrium, is the following: Bidders report their types $\bar{v}_1, \bar{v}_2, \ldots \bar{v}_I$. All bidders with the highest reports,

the revenue in the benchmark optimal mechanism. This allows us to demonstrate that the revenue equivalence does not hold when bidders' marginal utility is decreasing, even in large auctions.

2.3 Linear Equilibrium

It is well known that, in games in which strategies are demand (or supply) functions, the (Nash) equilibrium is not determinate; that is, there exists a continuum of equilibria the revenues of which differ significantly (see, e.g., Klemperer and Meyer (1989)). This paper analyzes linear equilibria, in which agents submit bid schedules that are linear in prices. The strategy space is not restricted to the class of linear bids; rather, in a linear equilibrium, it is optimal for a bidder to submit a linear bid, given that the other bidders play linear strategies. Our treatment of bidder behavior is closest to that in the model by Klemperer and Meyer (1989),[10] based on uniform price, in that introducing uncertainty in the residual supply makes bidders bid optimally for all prices, not just for the equilibrium price, and this refines the set of equilibria.

For the uniform price mechanism, the linear (Bayesian) Nash equilibrium has been widely used in modeling financial, electricity and other oligopolistic markets. In our comparative design problem, the benefits of focusing on the linear equilibrium are threefold. Much of the difficulty associated with the study and design of divisible good markets stems from the large size of the bidders' strategy space. Examining linear equilibria allows us to reduce that space to two dimensions—the slopes and the intercepts of the bidders' own schedules. In regard to the uniform price format, we show that the linearity property also implies that the complexity of the bidders' problem, which is measured by the information sufficient for the bidders to bid optimally, is in fact one-dimensional and concerns their individual price impact (Section 4.4). Finally, the equilibria we derive for all three designs are unique, which allows us to compare the performance of the auction formats in a consistent way.

The selection of a linear equilibrium finds empirical support in Hortaçsu's (2002a) study of the Turkish Treasury auction. Taking a nonparametric approach, Hortaçsu demonstrates that a straight interpolation through bid points can, on average, explain 92% of the observed variation and argues that a divisible good auction that generates linear equilibrium bidding strategies can provide a good description of the data. Similarly, Hortaçsu and Puller (2008) find that linear bids provide an excellent fit in the spot market for electricity in Texas.

$\bar{v}_i = \max(\bar{v}_1, \bar{v}_{2,\ldots}\, \bar{v}_I)$, receive $I\bar{Q}/\bar{I}$ in terms of the good, where $\bar{I} \geq 1$ is the number of the bidders with the maximal bid, and they pay their reported surplus $\bar{v}_i I\bar{Q}/\bar{I} - \frac{1}{2}\rho\left(I\bar{Q}/\bar{I}\right)^2$. Bidders with lower reports receive and pay zero. By the Bertrand-type argument, this mechanism implements the outcome in which the seller extracts the entire surplus from bidders, who bid truthfully, in a unique Nash equilibrium.

[10] Klemperer and Meyer (1989) did not restrict attention to linear strategies.

Furthermore, the linear equilibrium aptly describes financial, electricity, and other divisible good markets in which bidders collect information about their residual market by estimating their price impact from price-quantity data.[11] The evidence suggests that such estimates are independent of price levels, which can be shown to imply a linear equilibrium. (This will be clarified in our analysis.)

Our analysis of uniform and discriminatory price auctions exploits the following feature of the linear equilibrium. Fix a profile of demands that constitute a linear equilibrium. Applying market clearing, each bidder i can then be viewed as trading against an upward-sloping, linear residual supply with a deterministic slope μ_i and a stochastic intercept x. The residual supply is found by solving $q_i = I \times \bar{Q} - \sum_{j \neq i} q_j(p)$ for p. The slope of the residual supply μ_i measures the *price impact* of trader i. Since the intercept x is a deterministic function of v and the *per capita* quantity $\bar{Q}$, its c.d.f. $G_i(\cdot)$ can be derived from $F(\cdot|v)$. Observe crucially that, from the perspective of bidder i, his price impact μ_i and distribution $G_i(\cdot)$ contain all of the payoff-relevant information about the strategies of other bidders; that is, the payoff function of bidder i is the same for all strategy profiles of other bidders that induce the same μ_i and $G_i(\cdot)$. Looking at the bidding problem in this way allows us to investigate the relative role of market power (μ_i) and uncertainty ($G_i(\cdot)$) on bidding. Moreover, we can find equilibrium in all auction formats in three steps: We first find the best response of bidder i to a residual supply function, taking μ_i and $G_i(\cdot)$ and, thus, the strategies of others as given; next, we find equilibrium price impacts μ_i, and then derive $G_i(\cdot)$ from $F(\cdot)$.

A realization of the supply function defines a *state*. It is convenient to refer to the realizations of the residual supply that, for a fixed price, are associated with smaller and greater quantities as *lower states* and *higher states*, respectively.

3 Bidding Behavior and Order Shading

We now characterize and contrast the optimal bidding behavior in the uniform price, discriminatory price and Vickrey auctions. In doing so, we highlight the importance of market non-competitiveness, and the relative role of market power and uncertainty in bidding and the structure of order shading. For any quantity q, *order shading* (OS) is defined as the difference between the marginal utility $v - \rho q$ and the bidding price p. The details of the

[11] In his report for the Federal Energy Regulatory Commission (2003), Cramton wrote: "... in my experience advising dozens of bidders in electricity and other markets, I have found that bidders, either explicitly or implicitly, develop their bid curves by taking into account the price-quantity tradeoff from incremental increases in bid prices. In some cases, I have observed power companies explicitly compute the residual demand curves in order to determine their optimal price-quantity bids in power markets." (p. 26)

analysis are given in the Appendix, which presents derivations of best responses.

3.1 Vickrey Auction

In a Vickrey auction, the bidder's payment in each state corresponds to the area below his residual supply. The familiar truthful-bidding property obtains, that is, bidders do not shade their bids below their marginal utility. In particular, neither market power nor uncertainty has any impact on bidding. In the optimum, for any quantity q, the bidding price p coincides with the marginal utility $v - \rho q$. The stop-out price in the Vickrey auction is equal to

$$\bar{p} = v - \rho \bar{Q}. \tag{4}$$

The payment of bidder i for each unit is determined as the opportunity cost measured by its hypothetical value to the other bidders had he not participated. The total payment of bidder i can then be found by using the residual supply faced by bidder i, $p(q_i) = v + (1 - \gamma)\rho q_i - (2 - \gamma)\rho \bar{Q}$.

3.2 Uniform Price Auction

The first-order condition we derive for a UPA ((30) in the Appendix) characterizes the structure of order shading,

$$OS \equiv v - \rho q - p = \mu_i \times q. \tag{5}$$

In any given state, the bidder shades his bid just enough to balance the negative price impact externality that results from increasing the price for all units being purchased in that state. Since price impact externality is linear and increasing in q, so is the order shading in the uniform price auction, with no order shading present at $q = 0$. The optimal bid for a bidder with price impact μ_i is equal to

$$q_i(p) = \frac{v - p}{\mu_i + \rho}. \tag{6}$$

The bid schedule (6) is the best response to a random residual supply with the deterministic slope μ_i and an arbitrary distribution $G_i(\cdot)$. As long as price impact μ_i is positive, the inverse bid becomes steeper than the marginal utility, since the bidder reduces his order more for higher quantities.

To pin down the equilibrium price impact for bidder i, fix equilibrium price impacts for all other bidders, and assume that they adopt the optimal strategy (6). Aggregating the bids of bidders other than i gives i's residual supply, $q_i = I \times \bar{Q} - \sum_{j \neq i} q_j(p)$, the slope of which can be characterized as

$$\bar{\mu}_i = \left(\sum_{j \neq i} (\bar{\mu}_j + \rho)^{-1} \right)^{-1}. \tag{7}$$

Condition (7) captures that price impacts mutually reinforce among strategic bidders in a uniform price auction. That feedback effect increases the overall level of non-competitiveness in the market. It can be shown that the solution to I non-linear conditions (7) exists if, and only if, $I > 2$. Moreover, the solution is unique and symmetric.[12] Given the symmetry, condition (7) becomes $\bar{\mu}_i = (1-\gamma)(\bar{\mu}_i + \rho)$. The bidders' equilibrium price impact in a UPA is given by

$$\bar{\mu}_i = \frac{1-\gamma}{\gamma}\rho. \tag{8}$$

Having endogenized the price impacts, we can determine the equilibrium bidding strategies.

Proposition 1 (Equilibrium in the UPA) *In a unique linear Nash equilibrium, the strategy of each bidder with valuation v is equal to*

$$q_i(p) = \frac{\gamma}{\rho}(v - p). \tag{9}$$

Equilibrium exists if, and only if, $\gamma > 0$.

Proposition 1 and market clearing give the equilibrium stop-out price in the UPA

$$\bar{p} = v - \frac{\rho}{\gamma}\bar{Q}. \tag{10}$$

In contrast to the Vickrey auction, the equilibrium price in the uniform price design increases with market size γ. Intuitively, in a larger uniform price auction, the price impact is smaller, which reduces the incentives to shade orders.

The derived equilibria in a UPA and a VA have an attractive property that they hold for any non-degenerate distribution of market supply $F(\bar{Q}|v)$. For the *ex post* property, two features of the design are crucial. First, by making choices contingent on prices, a bid can be conditional on the state of the world, thereby allowing a bidder to hedge away any uncertainty about residual supply; furthermore, there is no cross-state externality in the payments. While the ability to (potentially) bid state-by-state is brought by the divisible good setting and is present in all three auction formats, the lack of cross-state externality is design-specific and does not appear in the DPA. In Section 4.4, we compare just how much bidders have to know about the market in order to bid optimally for all prices—a concern that arises in practical auction design when attracting new bidders.

[12] With two bidders ($\gamma = 0$), the solution to conditions (7) does not exist.

3.3 Discriminatory Price Auction

In a VA and a UPA, bidders solve a continuum of independent optimization problems, one for each realization of the residual supply. In a DPA, constructing the bid and, hence, the derivation of an equilibrium is more challenging because bidding for one realization of the residual supply *does* affect payments for other realizations. Consequently, in contrast to a UPA and a VA, the probability of different realizations matters for the trade-offs being optimized and the best response in a DPA depends on the distribution of the intercept of the residual supply, $G_i(\cdot)$.

3.3.1 The Basic Trade-off

We report the Euler Equation that describes the optimal behavior in a DPA,

$$-\frac{\partial x(\cdot)}{\partial q}[v-\rho q-p]=\left[\mu_i-\left(\frac{\partial q_i(\cdot)}{\partial p}\right)^{-1}\right]\frac{G_i(x)}{g_i(x)}. \tag{11}$$

As in the UPA and the VA, condition (11) equalizes the marginal benefit and the marginal cost of a small deviation for each realization of the supply. Heuristically, appealing to the mapping between quantities and states, the marginal benefit in a given state is equal to the marginal utility $v-\rho q$ from the greater quantity obtained in that state. The marginal cost has two components: The greater payment in the state, which is equal to p, and a negative externality on the payment that more aggressive bidding inflicts in all higher states (captured in the bracket on the right-hand side of the equation). The latter cost component, which is not present in the UPA or the VA, depends on the probabilistic importance of all higher states relative to the considered state and is, therefore, weighted by the ratio $G_i(x)/g_i(x)$. Thus, the Euler equation in the DPA equalizes the net marginal benefit within the considered state with the negative externality inflicted on the payments in all higher (but not lower) states. In the UPA and the VA, the net marginal change in the utility triggered by a deviation in a given state is zero.

In the symmetric equilibrium, $q=\bar{Q}$ for all i, its distribution coincides with the primitive distribution of $\bar{Q}$. This allows for re-casting the Euler equation in terms of the distribution of the equilibrium quantity q rather than x. Since, in equilibrium, $x(q)$ is an affine decreasing transformation of q, the c.d.f. of q can be found as $\bar{G}_i(q)=1-G_i(x(q))$. It follows that the density of q is $\bar{g}_i(q)=-g_i(x)\times\partial x(\cdot)/\partial q$. The inverse hazard ratio of the equilibrium quantity is defined as

$$\lambda(q)\equiv(1-\bar{G}_i(q))/\bar{g}_i(q). \tag{12}$$

Written to reveal the structure of order shading, the first-order condition (11) becomes,

$$OS \equiv v - \rho q - p = \lambda(q)[\mu_i - (\partial q_i(\cdot)/\partial p)^{-1}]. \tag{13}$$

Condition (13) identifies the role of market power in bidding in a DPA. Unlike in a UPA, the price impact effect for any given quantity (within any given state) is only a second-order effect. Since the marginal payment for every unit is determined by the submitted schedule, a changing bid in one state has a price impact effect only locally, in that state. Nevertheless, price impact does have a first-order effect on the optimal bid. For any quantity, order shading coincides with the negative externality on the payments in higher states (higher quantities), inflicted by more aggressive bidding, and the latter is a function of μ_i. Thus, the magnitude of order shading in a given state can be quantified in terms of two factors: The bidder's price impact (μ_i) and the slope of the individual bid function ($\partial q_i(\cdot)/\partial p$), weighted by the probabilistic importance of higher states relative to that state ($\lambda(q)$).

In the derivation of (13), we assumed that the considered q is in the support of the equilibrium quantity. When density $\bar{g}_i(q)$ is equal to zero, (12) is not well-defined and (13) does not apply. Note that if q is smaller than the quantities in the support, a bidder has an incentive to submit the smallest possible bid. Aggressive bidding for such a q brings no benefit of greater quantity, while it does increase the payments in higher states. Given that submitted bids are required to be non-increasing, the optimal bid has flat parts. Note that the flat-bid parts do not occur for the quantities to the right of the support. For such quantities, the submitted bids have no effect on the equilibrium quantities nor on the payment in any of the possible states, and bidders are indifferent to what they submit. Without loss of generality, we assume that, for quantities that exceed the upper bound of the support, bidders submit continuations of their linear bids given by (13).

3.3.2 Structure of Order Shading

Before exploring the impact of bidder uncertainty and market power on the optimal bid in a DPA, we first investigate the functional form of the bid. In the UPA as well as the VA, bidders' state-by-state optimization assures that the linearity of the optimal bid function then follows from the linearity of the marginal utility and the order shading. In the DPA, the dependence of the shape of the optimal bid on the distribution of the residual supply (x) raises the question of which distributions of x admit linear best responses. Observe first that, since both μ_i and $\partial q_i(\cdot)/\partial p$ are constants, the Euler equation (13) defines a linear schedule only if the inverse hazard ratio is a linear function of the quantity, that is, $\lambda(q) = \lambda_0 + \lambda_1 q$ for some $\lambda_0, \lambda_1 \in \mathbb{R}$. We refer to this condition as the *linear inverse hazard ratio* (*LIHR*). The question then becomes which distributions of equilibrium quantity $\bar{q}$ satisfy LIHR. Lemma 1 characterizes the class of all such distributions.

Lemma 1 (LIHR) *Suppose $\bar{G}_i(q)$ has a convex support. $\bar{G}_i(q)$ exhibits LIHR if, and only if, it belongs to the class of Generalized Pareto distributions, the c.d.f. of which is given by*

$$\bar{G}_i(q) = 1 - (1 + \xi \frac{q-\alpha}{\sigma})^{-\frac{1}{\xi}}, \tag{14}$$

where $\xi, \alpha \in \mathbb{R}$ and $\sigma \in \mathbb{R}_{++}$.

Parameter α determines the location, σ defines the scale, and ξ describes the shape of the distribution.[13] We follow the convention that $\xi = 0$ defines the limit exponential distribution. Lemma 1 identifies the set of all distributions that are compatible with linear bidding strategies in a DPA. As the equilibrium quantity q is an affine transformation of the negative of x, the distribution of $-x$ exhibits LIHR if, and only if, q exhibits it as well.

Apart from providing us with a precise understanding of the kinds of environments we can analyze for the DPA, Lemma 1 uncovers the relationship between bidder uncertainty and the structure of order shading in a DPA. Let us express the inverse hazard ratio in terms of the parameters of the Generalized Pareto distribution,

$$\lambda(q) = \underbrace{\sigma - \xi\alpha}_{\equiv \lambda_0} + \underbrace{\xi}_{\equiv \lambda_1} q. \tag{15}$$

The class characterized by Lemma 1 encompasses distributions with decreasing ($\xi < 0$), constant ($\xi = 0$), and increasing ($\xi > 0$) inverse hazard ratios. Support of any Generalized Pareto distribution has a lower bound, given by α. Whenever $\xi < 0$, the support also has an upper bound, equal to $-(\sigma - \xi\alpha)/\xi$. Among distributions with compact support, $\xi < -1$ concentrates mass on higher quantities, $\xi = -1$ corresponds to a uniform distribution, whereas the distributions with $-1 < \xi < 0$ put relatively greater mass on lower quantities. For $\xi = 0$ (exponential distribution) and $\xi > 0$ (the class of Pareto distributions), the support is unbounded. Figure 1.A illustrates how the shape parameter ξ affects the support and the concentration of mass.

Order shading in a DPA (13) can now be understood through the properties of the supply distributions. Order shading and inverse hazard ratio are decreasing, constant, and increasing

[13] Conveniently, the class of distributions with the LIHR property can be parameterized either by $\lambda_0, \lambda_1, q^*$, where the last parameter is the median of the distribution or, alternatively, as the space of the parameters μ, σ, ξ; and that the linearity of an inverse hazard ratio is preserved under additive (i.e., changing the location) or positive multiplicative (i.e., changing the scale) transformations of a random variable.

in q for negative, zero, or positive ξ, respectively (see Figure 1.B). For the exponential distribution, the inverse hazard ratio is constant, $\lambda(q) = \lambda_0$, and the probabilistic importance of higher states is the same for all q. As a result, order shading is also independent from q and the bid is parallel to the marginal utility. The best response to an exponentially distributed residual supply was first derived by Hortaçsu (2002a) in a DPA for two bidders with independent private values. Lemma 1 reveals that, except when $\xi = 0$, the Generalized Pareto class induces bids that can be flatter ($\xi < 0$) or steeper ($\xi > 0$) than the marginal utility, which reflects the decreasing and increasing with q, respectively, relative importance of higher states. In addition, for $\xi < 0$, the bidding involves no distortion for the highest (rightmost) realization of the residual supply at the upper end of the support. At this point, there is no negative externality on the payments in higher states (as there are no such states), and the bid coincides with the marginal valuation.

In summary, order shading differs qualitatively between the DPA and the UPA in the following respects. Unlike in the UPA, the bid in the DPA is shaded at zero quantity, and for all distributions with compact support, the bid coincides with the marginal utility at the upper end of the support. In the UPA, there is no shading at zero quantity, and bids are strictly below the marginal utility at the upper-end quantity. Moreover, order shading in the DPA need not be increasing in quantity.

3.3.3 Equilibrium in a Discriminatory Price Auction

Having determined how bidder uncertainty is reflected in bidding, we now turn attention to how market primitives, including supply uncertainty, affect order shading through endogenous price impact. The following counterpart of (7) characterizes bidder price impact μ_i in a DPA

$$\mu_i = \left(\sum_{j \neq i} \left(\frac{\xi \mu_j}{1-\xi} + \frac{\rho}{1-\xi} \right)^{-1} \right)^{-1}. \tag{16}$$

Condition (16) sheds light on the subtle nature of strategic interdependence in the DPA and indicates that the nature may differ from that in the UPA, depending on the sign of the shape parameter ξ in the distribution of the market supply. Notably, for markets with $\xi < 0$, the price impact of bidder i depends negatively on the price impact of other bidders; with higher market power, other bidders are induced to shade more, which flattens their (inverse) bids, thereby, reducing the price impact of bidder i. In turn, the mechanism of mutual offsetting (rather than reinforcement) of price impact lowers the overall level of equilibrium market power. Incidentally, the distinct nature of strategic interdependence also explains why, unlike in the UPA, equilibrium in the DPA exists even with two bidders. For markets with $\xi > 0$, order shading steepens the bids of i' trading partners and market power reinforces, as in the UPA. When ξ exceeds $1/(2-\gamma)$, the reinforcement occurs without bounds, and equilibrium fails to exist.

The equilibrium price impact of a bidder in the DPA is given by

$$\mu_i = \frac{(1-\gamma)\rho}{1-(2-\gamma)\xi}. \tag{17}$$

It is apparent from (17) that, in large discriminatory price auctions, bidders become (stop-out) price takers, that is, they do not affect the price by changing their bids. Observe further that, in markets with $\xi < 0$, equilibrium converges more slowly to the price-taking limit in the DPA than in the UPA. Underlying the difference in the adjustment of price impact with market size is precisely the positive (negative) interdependence of market power in the UPA (DPA). The reduction of price impact in aggregation brought about by additional participants in the DPA is partially offset by the steepening of individual bids encouraged by more competitive trading.

We are ready to derive equilibrium in a DPA. Recall that, since the bidders are identical, in equilibrium, each bidder receives a *per capita* supply $\bar{Q}$. It follows that the distribution of the equilibrium quantity q can be matched with the primitive distribution, $\bar{G}_i(q) = F(q|v)$. By Lemma 1 and the Euler equation (13), the model admits a linear equilibrium in a DPA only if $F(\bar{Q}|v)$ is given by (14), which we assume. One difficulty in the derivation of a linear equilibrium in a DPA stems from the flat bid parts. As explained in Section 3.3.1, when support of the equilibrium quantity is bounded away from 0, then it is optimal for bidders to submit bids with a kink at the lower end of the support. Such bidding is inconsistent with the linear equilibrium. The subclass of the Generalized Pareto distributions that admits a linear equilibrium consists of the distributions with the lower bound of support equal to 0. Therefore, we hereafter set $\alpha = 0$.

Assumption 1: The conditional distribution of the residual supply $F\left(\bar{Q}|v\right)$ is distributed according to the Generalized Pareto with location parameter $\alpha = 0$.

Proposition 2 characterizes bids in a linear equilibrium of a DPA.[14]

[14] To the best of our knowledge, the characterization of the set of (possibly non-linear) Nash equilibria by Wang and Zender (2002) provided for a class of supply distributions that is a strict subset of ours is the only existing analytical characterization of equilibria in a DPA with I strategic bidders. This paper focuses on linear equilibria and Proposition 2 characterizes equilibrium bids in all environments that admit linear equilibria. In the Supplementary Material, we provide a precise translation between the parameterizations used in Wang and Zender (2002) and this paper.

Proposition 2 *(*Equilibrium in the DPA*) In the unique symmetric linear equilibrium, the strategy of bidder i is given by*

$$q_i(p) = \frac{1-(2-\gamma)\xi}{\rho}(v-p) - (2-\gamma)\sigma. \tag{18}$$

Equilibrium exists only if $\xi < 1/(2-\gamma)$.

One other novel feature of the bid in the DPA, compared to the bid in the UPA, is that bidders shade their valuation irrespective of market size and, in particular, in the limit ($\gamma \to 1$). Although in all auction formats considered bidders become (stop-out) price takers in large markets, in the DPA they still affect the payments for the winning units, and only one of the two determinants of order shading—namely price impact—vanishes in the limit (cf. (13)). Averaging the optimal bids (18) across all bidders and using market clearing solves for the stop-out price in the DPA

$$\bar{p} = v - \frac{\rho}{1-(2-\gamma)\xi}\bar{Q} - \frac{(2-\gamma)\rho\sigma}{1-(2-\gamma)\xi}. \tag{19}$$

4 Rankings

This section presents a comparative analysis of the UPA, DPA, and VA with respect to revenues (Section 4.1), as well as the volatility of the stop-out price (Section 4.2), encouraging bidder participation (Section 4.3), and simplicity (Section 4.4). We examine small and large markets and consider risk averse as well as risk neutral preferences of the bidders and that of the designer. Throughout, two assumptions are maintained. Motivated in Section 3.3.3, Assumption 1 ensures that the DPA admits a linear equilibrium—in particular, that there are no flat bid parts. The possibility of relaxing Assumption 1 is discussed in Section 5. In addition, the following assumption is maintained.

Assumption 2: The support of $\bar{Q}$ does not exceed the threshold at which bidders' preferences are satiated ($v - \rho\bar{Q} \geq 0$).

Assumption 2 simply captures that *per capita* supply is not large enough to cross the inelastic part of the demand; without it, a negative stop-out price would be observed for some realizations of $\bar{Q}$. The non-satiation assumption implies that the support of the distribution of $\bar{Q}$ is bounded, and hence $\xi < 0$. Section 5 relaxes this assumption.

4.1 Revenue

Consider a market with a finite, possibly small, number of strategic bidders ($\gamma < 1$) with decreasing marginal valuations ($\rho > 0$). Theorem 1 establishes the expected-revenue rankings of the UPA, DPA, and VA, and compares the revenues with the total surplus potentially to be extracted.

Theorem 1 *(*Revenue Ranking*) Let $\rho > 0$ and $\gamma < 1$. In the unique symmetric linear equilibrium, for any v,*

$$E(TS|v) > E(R^D|v) > E(R^V|v) > E(R^U|v), \tag{20}$$

whenever equilibria exist.

The result is striking: even though the optimal bidding in a DPA depends on the underlying distribution of the residual supply, the comparison of the mechanisms—including the commonly used UPA and DPA—does not. In particular, the dominance of a DPA over a UPA holds for all Generalized Pareto distributions.[15] One of the arguments often invoked in support of the UPA is that (conjectured) smaller demand reduction may lead to greater revenue. In all of the environments considered in this paper, the UPA design indeed fosters more aggressive bidding, measured by (the inverse of) the expected order shading. (See the proof of Theorem 1 in the Appendix.) Nevertheless, Theorem 1 shows that weaker expected order shading is not sufficient for the UPA to outperform the DPA.

To understand the expected-revenue dominance of the DPA over the UPA in non-competitive auctions, it is useful to pin down the effects from which the revenue differences derive in the two formats. These are (1) coarseness of the uniform price mechanism, which leaves some surplus to a bidder even if he bids his marginal utility; (2) distinct order shading; and (3) greater equilibrium price impact in the UPA. Lemma 2 identifies a link in bidder behavior between the DPA and the UPA that illuminates the role of the three effects in the revenue ranking in our main result. Consider a bidder whose marginal utility coincides with the equilibrium inverse bid in the UPA, $v^*(q) = v - \frac{\rho}{\gamma}q$, and who faces a residual supply with the slope (price impact) as in the UPA equilibrium. The lemma asserts that the expected revenue that a seller could then extract in the DPA corresponds exactly to the expected *per-capita* revenue in the UPA with marginal utility $v(q) = v - \rho q$. Thus, order shading in the DPA by a bidder with preferences $v^*(q)$ is just enough to retain the surplus that is left in the UPA to a bidder with preferences $v(q)$. Let $\bar{p}^U$ denote an equilibrium price in the UPA.

[15] In a study of Turkish Treasury auctions, Hortaçsu (2002b) found that the DPA leads to higher revenues than the revenue obtainable from the Vickrey auction, which Hortaçsu attributed to the allocational inefficiency in a DPA with heterogenous bidders. Our result shows that the discriminatory price mechanism brings higher revenue than does the Vickrey mechanism even when allocation is efficient in both auctions.

Lemma 2 (Strategic Equivalence) *Consider a bidder with marginal utility $v^*(q) = v - \frac{\rho}{\gamma}q$ who faces a residual supply with the same slope as a bidder with marginal utility $v - \rho q$ in the symmetric equilibrium in the UPA. The best-response bid for the q^{th} unit in the DPA is equal to $E(\bar{p}^U|\bar{p}^U \leq v^*(q))$.*

Crucially, it follows from Lemma 2 that order shading in the DPA by a bidder with preferences $v^*(\cdot)$ and price impact as in equilibrium from the UPA reduces the seller's expected revenue by exactly how much surplus is left to a bidder in the UPA with true preferences $v(\cdot)$. Since the actual preferences are $v(\cdot) > v_i^*(\cdot)$ and the expected revenue in the DPA is monotone in ρ (cf. bids (18)), the link in Lemma 2 then implies that the expected revenue in the DPA (with the price impact from equilibrium in the UPA) is strictly higher than in the UPA (effects (1) and (2)). The expected-revenue advantage of the DPA is further boosted by the lower equilibrium price impact in the DPA (effect (3)). Later we show that Lemma 2 can be viewed as a strategic counterpart of the result obtained by Swinkels (2001) for competitive settings.

The revenue dominance from Theorem 1 does not generically hold *ex post*. As the next result demonstrates, the dominance of the Vickrey format over the UPA does hold in the strong state-by-state sense.

Proposition 3 strengthens Theorem 1 beyond the risk neutral preferences of the seller.

Proposition 3 *(*Revenue Ranking; Arbitrary Risk Preference of the Seller*) Let $\rho > 0$ and $\gamma < 1$. For any strictly increasing utility function of the seller $\bar{u}(\cdot)$, in the unique symmetric linear equilibrium,*

$$E(\bar{u}(TS)|v) > E\left(\bar{u}\left(R^V\right)|v\right) > E(\bar{u}\left(R^U\right)|v). \tag{21}$$

This result gives further support to the Vickrey auction, relative to the UPA, when bidders have market power. The strong *ex post* dominance of the Vickrey auction over the uniform price auction can be attributed to two effects, both of which favor the VA and can be explained through the structure of price impact. Suppose first that bidder i in the UPA best responds to truthful bidding of other bidders, so that the residual supplies in the two auction formats coincide, and bidder i's price impact μ_i is the same. Even if the two designs induced identical equilibrium price impact, the VA would then extract more surplus than the UPA from a bidder: In contrast to the UPA, the payment in a Vickrey is not uniform across units; rather, a bidder obtains each unit at a discount which varies in quantity, relative to the stop-out price. The discount is maximal for the first unit, in which case it is equal to $\mu_i\bar{Q}$. In the UPA, bidder i shades his bid in response to his price impact μ_i, and lowers the price—and, hence, the payment for *all* units—exactly by $\mu_i \bar{Q}$ (cf. (5)). Therefore, the bidder payment for any unit obtained in the Vickrey auction is bounded from below by the price in the UPA. The revenue in the UPA is additionally hurt by the higher equilibrium price impact than in the VA, due to the mutual reinforcement of price impact (cf. explanation of (7)) present in the UPA. This reduces the equilibrium price in the UPA strictly below the payment for any unit in the VA.

The key to explaining the state-by-state dominance of Vickrey over a uniform price auction is that the residual supply determines the discount in a Vickrey auction as well as the price impact in the UPA. We should stress that Proposition 3 holds beyond the class of Generalized Pareto distributions (see Section 5). In contrast to the Vickrey and uniform price formats characterized in Proposition 3, the seller's preference for the DPA does depend on his risk attitude. (We provide intuition below by means of Proposition 5.)

We turn next to evaluating the performance of the three mechanisms in large markets. As the market size increases, the expected revenue to be extracted as a fraction of the expected total surplus $E(TS|v)$ increases monotonically in γ. By the continuity of the expected revenues in γ, the ranking from Theorem 1 must hold at least weakly in the limit in large auctions as $\gamma \to 1$. Proposition 4 demonstrates that in fact the DPA, the UPA, and the VA yield the same expected revenue in the competitive limit. Nevertheless, the common limit revenue does not attain the total surplus. Thus, even in the competitive limit, the seller cannot extract the total surplus. When increasing the market size, we keep the distribution of *per capita* supply fixed.

Proposition 4 *(*Revenue Ranking in Large Auctions*) Let $\rho > 0$, and fix the distribution of $\bar{Q}$. In the unique symmetric linear equilibrium,*

$$E(TS|v) > \lim_{\gamma \to 1} E(R^D|v) = \lim_{\gamma \to 1} E(R^V|v) = \lim_{\gamma \to 1} E(R^U|v). \tag{22}$$

In the competitive limit markets, the residual supply faced by each bidder is perfectly elastic, and hence, for each realization of market supply $\bar{Q}$, the payment for each unit coincides in the UPA and the VA, and the revenue in the VA and the UPA is the same. That these expected revenues coincide with that in the DPA becomes noteworthy if one looks more closely at the individual bidding. In the large competitive UPA, bidding becomes truthful, as in the Vickrey auction. With discriminatory pricing, by contract, bidders still shade their marginal utility (see Section 3.3.1). It follows that the reason why the seller is unable to extract the total surplus in the limit differs across the mechanisms: In the DPA, this results from order shading, whereas in the UPA and the VA, it is due to the payment structure itself, which leaves consumer surplus to the bidder.

In a study of large auctions, Swinkels (2001) obtained the expected-revenue equivalence between the UPA and the DPA in a discrete-bid setting with independent private-values.[16] Our model is a continuous-bid, complete information counterpart of the setting analyzed by Swinkels (2001), and our Proposition 4 extends the result by Swinkels to our special small-market setting. Swinkels showed that the bid for unit q in the DPA equals $E(\bar{p}^U|\bar{p}^U \leq v(q))$; i.e., the expected payment per unit conditional on the bid for that unit to be winning coincides with the expected price in the competitive uniform price auction, thereby providing an intuitive connection to the revenue equivalence result in auctions of single objects. Lemma 2 is consistent with this interpretation—in large auctions, equilibrium price impacts become the same (zero) in the two formats, and bidders do not shade their marginal utility in the UPA ($v^*(q) = v(q)$). Lemma 2 also implies that the connection does not carry over to small auctions due to strategic considerations that arise in the presence of market power.

One insight from our simple setting with symmetric information is that the equality of the expected revenue in a large DPA, UPA and VA is not a straightforward generalization of the revenue equivalence theorem for unit demands. In all three auction formats, the allocation is efficient, while the revenue is strictly lower than the revenue brought by the optimal mechanism equal to total surplus. In the competitive limit, $\sigma\rho/(v(1-2\xi)-\sigma\rho)$ percent of the total surplus remains unextracted. Figure 2.A illustrates Theorem 1 and Proposition 4: The DPA dominates the UPA and the VA in expected revenue for all $\gamma < 1$, and the advantage of the DPA disappears in large auctions.

Examining the distribution of payment induced by different formats more carefully reveals that, when bidders have market power, the seller faces a risk-revenue trade-off in design. Consider first the large competitive markets, for which no such trade-off arises. It follows from Proposition 4 that a risk neutral seller who is concerned about the expected revenue will be indifferent among the three mechanisms. Proposition 5 asserts that a risk averse seller strictly prefers the UPA (and the VA) to the DPA in large enough markets. Let $\bar{u}(\cdot)$ denote the utility function of the (revenue-maximizing) seller.

Proposition 5 *(*SOSD*) Let $\rho > 0$. For any strictly concave increasing utility function $\bar{u}(\cdot)$, in the unique symmetric linear equilibrium,*

$$\lim_{\gamma\to 1} E(\bar{u}(TS)|v) > \lim_{\gamma\to 1} E(\bar{u}(R^U)|v) = \lim_{\gamma\to 1} E(\bar{u}(R^V)|v) > \lim_{\gamma\to 1} E(\bar{u}(R^D)|v). \tag{23}$$

[16] For large auctions, Jackson and Kremer (2002) found that ranking is ambiguous when common values are present. In our model, there are no informational considerations arising from correlation of valuations, which makes our model closer to that of Swinkels (2001).

In large auctions, the expected-revenue advantage of the DPA disappears, and the UPA generates a more favorable distribution of revenue (Figure 2.B). UPA should thus be preferred by a risk averse seller.

Theorem 1 and Proposition 5 jointly characterize a trade-off faced by a seller when selecting an auction format in small markets with strategic bidders. For markets with $\gamma < 1$ that are large enough, the DPA is characterized by higher expected revenue, as well as higher variance than the UPA. More generally, for all market sizes $\gamma < 1$, risk averse preferences exist for which either format is strictly preferred by the seller: For any γ, the distribution of revenue in the UPA crosses (once) the one in the DPA from below (as shown in the proof of Proposition 5). Theorem 1 then implies that the second-order stochastic dominance does not extend to small auctions. Our model thus recommends that the UPA is more likely to be superior in markets with many bidders, whereas the DPA might be favored in small markets.

The analysis so far has assumed that the bidders' marginal utility is decreasing. Let $\rho \to 0$. The marginal utility flatten to become constant at v, and so do the bid schedules. In the limit, all three mechanisms lead to bidding in accordance with the marginal utility. In addition, in all auction formats, the competitive price obtains; the limit equilibrium is also the equilibrium for $\rho = 0$.[17] Proposition 6 establishes that, when bidders have constant valuations, the considered designs are revenue-equivalent and permit a full extraction of the expected total surplus by the seller. This holds irrespective of market size.

Proposition 6 *(*Revenue Equivalence with Constant Marginal Utility*) Let $\gamma < 1$. For any increasing utility function of the seller $\bar{u}(\cdot)$, in the unique symmetric linear equilibrium,*

$$E(\bar{u}\,(TS)\,|v) = \lim_{\rho\to 0} E\left(\bar{u}\left(R^V\right)|v\right) = \lim_{\rho\to 0} E(\bar{u}\left(R^D\right)|v) = \lim_{\rho\to 0} E(u\left(R^U\right)|v). \tag{24}$$

Proposition 6 holds state-by-state. When the bidders' marginal utility is constant, the limit revenues in all auction formats amount to the total surplus $v \times \bar{Q}$. Underlying truthful bidding in the limit (Bertrand) equilibrium are two implications of the constant marginal utility. First, both the price impact effect in the UPA and the cross-state externality in the DPA, identified as the determinants of order shading in the respective design (cf. (5) and (13)), disappear. Further, when bidders' marginal utility is independent of quantity, the three auction formats are strategically equivalent to the first-price auction with unit demands: the highest bid wins; winning a positive amount of the good auctioned becomes random; and,

[17] In the limit, the flat bid functions do not determine the allocation at the equilibrium price. In order to close the model for $\rho = 0$, one needs to specify the allocation rule. Since bidders are left without any surplus in any case, flat bidding is actually an equilibrium for any allocation rule.

in the absence of cross-state externalities, bids for all (any) units are perfectly correlated. The second reason behind the optimality of no bid shading is then that the valuations and, therefore, the stop-out price are deterministic at the *interim* stage.

It is interesting to contrast the implications of the convergence reported in Propositions 5 and 6. Unlike the convergence in market size, with the convergence of marginal utility, regardless of market size, none of the designs gives rise to order shading in the limit, bidders have no price impact, and the stop-out price is deterministic at the *interim* stage and equal to v.

For large markets with asymmetric information, Jackson and Kremer (2006) demonstrated that when Q is fixed, as $I \to \infty$ (and, hence, $\bar{Q} \to 0$) the seller extracts the entire surplus in the UPA and the DPA. This result obtains in our simpler setting; the thrust of the mechanism is essentially the same as in Proposition 6.

4.2 Price Volatility

Apart from the fiscal consequences of design, as traditionally emphasized by auction theory, another criterion often used to evaluate the performance of auction formats is the volatility of the stop-out price. For example, the purchase or sale of U.S. government securities is one of the key open market operations of the Federal Reserve. By adjusting the level of reserves in the banking system through trade of securities, the Fed can offset or support cyclical or seasonal shifts in funds. The trade of Treasury securities thereby affects short-term interest rates. Thus, the design of the market for Treasury auctions plays a crucial role in the effectiveness of stabilization policy; the Central Banks tend to prefer auction formats that minimize the variance of the stop-out price. Proposition 7 establishes the rankings of unconditional (as well as conditional on v) volatility in the three auction formats considered.

Proposition 7 *(*Volatility of the Stop-Out Price*) Let $\rho > 0$ and $Cov(v, \bar{Q}) = 0$. In the unique symmetric linear equilibrium,*

$$Var(\bar{p}^U) > Var(\bar{p}^V) > Var(\bar{p}^D). \tag{25}$$

The DPA gives rise to the lowest price volatility among the three formats for any distribution in the Generalized Pareto class with $\xi < 0$. Note that price volatility is inversely related to market size in the uniform price, but not in discriminatory price auctions. In the DPA, the stop-out price becomes more volatile, and the bids steepen as the market grows and the equilibrium price impact weakens. This reflects that, in contrast to the UPA and VA, smaller price impact steepens individual bids in the DPA. Thus, the ranking of price volatility does not follow from the differences in bidders' (endogenous) price impact. Additionally, when $\rho \to 0$, so that the marginal utility becomes flatter, the price volatility monotonically decreases to zero in all three auction formats.

The assumption that v and $\bar{Q}$ are uncorrelated is admittedly strong in many markets. For instance, when the Central Bank is informed about shocks to bidder endowments (which affect valuations), then the Bank is likely to adjust supply $\bar{Q}$ as part of its stabilization policy, which induces a positive correlation between $\bar{Q}$ and v. More generally, a non-zero covariance will be observed when the seller announces the supply before the bids are submitted and after he observes the signals about bidders' valuations. Nowhere has the analysis in previous sections relied on v and $\bar{Q}$ being independent; the revenue rankings derived in Section 4.1 hold for markets with an arbitrary covariance $Cov(v, \bar{Q})$. In contrast, the ranking of the volatility of equilibrium prices depends on the covariance $Cov(v, \bar{Q})$. Specifically, when the positive $Cov(v, \bar{Q})$ exceeds a certain threshold, the conclusions of Proposition 7 reverse in that the UPA generates the lowest price variance of all formats. In studying market design for the stabilization policy of the Central Bank, it is thus important to allow $Cov(v, \bar{Q}) \neq 0$. (The Appendix provides rankings of volatility of the stop-out prices for an arbitrary $Cov(v, \bar{Q})$.)

The advantage of the DPA with respect to price stability is even more pronounced when, instead of price volatility, one compares the volatility of the average payment per unit. In the UPA, per-unit payment coincides with the stop-out price; in the DPA, the volatility per unit amounts only to 25% of the volatility of its stop-out price; in the VA, the volatility of the per-unit payment is at least as high as the volatility of the stop-out price; precisely, it is higher by $25\% \cdot (3-\gamma)^2 > 100\%$.

One of the instruments used in practice to reduce the volatility of the stop-out price is the adjustment of supply to submitted bids. Our model suggests that this policy is more effective in the UPA than in the DPA. This holds because of the mutual reinforcement and offsetting mechanisms that determine equilibrium price impact in the UPA and the DPA with $\xi < 0$, respectively (described in Sections 3.2 and 3.3.3).

4.3 Participation

The analysis so far has been carried out for a given number of bidders I for all auction formats. This describes well many markets with large institutional investors, which often feature natural barriers to the entry of outsiders; similarly, entry is exogenously determined in primary dealer markets that are present in many countries. When the design of a new market is considered, however, one should take into account that the positive surplus left for bidders encourages others to join the auction. We now endogenize the participation rate in the utility of the bidders to examine how bidder incentives to enter the market differ in the three designs and how endogenous entry affects the revenue rankings.

One of the lessons from the auction theory for indivisible goods is that a seller should favor auction formats that encourage more participants (e.g., Bulow and Klemperer (1996)). With an additional bidder, other participants bid more aggressively. We argue that this recommendation is even more relevant in the context of divisible good markets as, apart from the pure *competitive effect* of reduced order shading, each additional participant increases the total surplus in the auction, even if the bidders have identical marginal utilities. We call the latter effect—which is not present in auctions with unit demands—a *surplus effect.* For a quick intuition, consider a seller selling two units of a good to two potential bidders, each with the utility function $2q - 0.5q^2$. Allocating the two units to one bidder brings utility and a total surplus of 2. If, however, the seller attracted an additional bidder, each would receive one unit, which would increase the total utility and the surplus to 3. If the good were indivisible instead, with identical bidder valuations, the total surplus would be independent of the number of bidders.

To endogenize entry, we analyze our model with an infinite pool of potential entrants. Having learned v, the bidders sequentially choose whether or not to join an auction. Entering involves a fixed cost c. We assume that the auctioneer sells Q units to all bidders, and therefore, the *per capita* supply $\bar{Q} = Q/I$ is endogenous. Define I^U, I^D and I^V as the maximal integers for which the expected payoff is not less than c for any bidder, in the UPA, DPA and VA, respectively. (In the Appendix, these three statistics are shown to be well-defined and unique, as the individual net surpluses are monotonically decreasing to zero in I.) Let $\rho > 0$ and $\gamma < 1$. We first determine the number of participants and the expected revenue in the unique subgame perfect Nash equilibrium, in the second stage of which the linear Nash equilibrium is played: for any v,

$$I^U \geq I^V \geq I^D. \tag{26}$$

There exist values of parameters for which the inequalities are strict (see proof of Proposition 8). The key feature of the UPA in encouraging more entry than other auction formats is that, in equilibrium, it leaves more surplus to the bidders: Given the fixed number and symmetry of the bidders, the allocations in all auction formats are Pareto efficient, and the total surplus is shared between the bidders and the seller. The ranking of market sizes then follows from the revenue rankings in Theorem 1. The dominance of UPA with regard to the number of participants is not significant. Figure 3.A depicts how the number of participants varies with the entry fee c. The market size in the UPA exceeds that for the VA and the DPA. Additionally, for most values of c, the difference between I^U and I^D is one bidder. The weak relative advantage of UPA in encouraging entry is consistent with the evidence from the U.S. Treasury experiment.

The foregoing discussion prompts the following question: Does the small entry advantage suffice to revert the revenue rankings established for markets with a given number of participants? Proposition 8 provides an affirmative answer.

Proposition 8 *(*Revenue Ranking with Endogenous Entry*) There exists a range of values of parameters for which $E(R^U|v) > E(R^D|v)$.*

Indeed, due to the surplus effect as well as the competitive effect, a small difference in the number of bidders translates into a significant revenue change. As a result, the UPA can dominate the DPA and the VA in both expected revenue and participation. (Figure 3.B plots the *per capita* revenue in the three types of auctions with endogenous participation as a function of entry cost c.) The ranking reversal results from a perverse mechanism. With perfectly divisible goods, it is not optimal to implement the mechanism that maximizes revenue, given I. With positive entry cost, such a mechanism leaves zero surplus to the bidders, which may prevent enhancing the potential revenue to-be-extracted from the competitive and the surplus effects.

Our numerical simulations suggest that the revenue reversal is a robust phenomenon due to the high sensitivity of the revenue to the number of bidders. The reversal is more likely when bidders are more risk averse (i.e., have a higher ρ). That is because, even though higher ρ adversely affects the entry advantage of a UPA, it strengthens the surplus effect at the same time, and the latter effect typically dominates.

How do the participation levels in the three auction formats compare with the socially optimal market size? For auctions of single objects, Levin and Smith (1994) demonstrate that markets encourage entry levels that are excessive from a social point of view. Ignoring the integer problem (i.e., treating I as a real number), in the proof of Proposition 8 we show that the endogenized number of participants exceeds the Pareto efficient market size in all three auction formats. This, in turn, provides an argument in favor of the DPA, the market size of which is the closest to being efficient. In all auction formats, excess entry arises because each bidder ignores the negative externality of his participation on the net utility of other bidders. Nevertheless, our simulations indicate that such excess participation is not a quantitatively significant problem.

4.4 Simplicity of an Auction Format

Auction markets are typically anonymous in that bidders and sellers alike know little about the number, the valuations, and the identities of the participants other than themselves. One of the arguments advanced by Friedman (1960) in support of the uniform price design is that bidders need to know less in order to act optimally, as compared to the discriminatory price auction. Just how much do auction participants need to know in either auction format? It follows from our analysis that, in the UPA, the bidding strategy in a linear equilibrium is remarkably simple, namely, to construct an optimal bidding schedule under uniform pricing, the only information a bidder has to have—aside from his own valuation—is his own price impact (see (6)). In particular, to respond optimally to arbitrary profiles of strategies of all other bidders, he does not have to know the (*interim*) distribution of residual supply in the auction, or even the number of participants against whom he is bidding, let alone their identities or preferences. The price impact statistic summarizes all the payoff-relevant information about the strategic environment. While specific to the setting considered in the paper, this insight provides a justification for focusing on the linear equilibrium and might be attractive in models based on mean-variance preferences in financial settings, among others. The low complexity of equilibrium behavior in the UPA stands in marked contrast to the DPA, where bidding optimally for all prices in a linear equilibrium requires that a bidder knows the distribution of the residual supply in addition to knowing his price impact.[18] Clearly, one's own valuation is the only information required for optimal bidding in the VA.

5 Extensions

We briefly discuss which results can and which cannot be extended beyond the environments studied in the paper. In the previous sections, we assumed that preferences are non-satiated for any realization of $\bar{Q}$. In fact, even if the quantities exceed the satiation point for some realizations, all the results except for Proposition 5 hold as long as $\xi < 0$. In this section, we consider departures from the model to: (1) Generalized Pareto distributions of supply with unbounded support; (2) distributions of supply outside of the Generalized Pareto class; (3) independent private values; (4) a non-linear Nash equilibrium; and (5) non-quadratic utilities of bidders.

Supply with Unbounded Support ($\xi \geq 0$). We report how the rankings derived in this

[18] The low complexity of bidding in the UPA versus the DPA could be captured by differential entry costs, in light of which the reversals of the revenue ranking with endogenous participation in favor of the UPA, studied in Section 4.3, might be more significant.

paper are affected when the distribution of the supply is unbounded, $\xi \geq 0$. Since, for non-negative ξ, an (arbitrary) negative stop-out price can be observed with positive probability, we find such values of parameters less plausible than $\xi < 0$. We view this section as a robustness check.[19] To compare the revenues and price volatilities in the three auction formats, we need to restrict attention to distributions of the supply $\bar{Q}$ for which the expected revenues are finite. This requires that the second moment exist, which translates into a restriction on the shape parameter, $\xi < 1/2$.

While the dominance of the DPA over the UPA in expected-revenue terms holds for all $\xi < 1/2$, the revenue ranking between the DPA and the benchmark VA depends on the value of the shape parameter ξ. For example, the two auction formats are revenue equivalent when $\bar{Q}$ is distributed exponentially. More generally, $E(R^D|v) \leq E(R^V|v)$, with the equality of expected revenue attained if, and only if, $\xi = 0$. The equality of revenues established in Proposition 4 and Proposition 6 extends to any $\xi < 1/2$.

The rankings of price volatility, in turn, are affected by the value of ξ. Let $\bar{\xi} \equiv (1-\gamma)/(2-\gamma)$. Then, for all $\xi \in \left(0, \bar{\xi}\right)$,

$$Var\left(\bar{p}^U\right) \geq Var\left(\bar{p}^D\right) \geq Var\left(\bar{p}^V\right), \tag{27}$$

whereas for $\xi \in \left(\bar{\xi}, 1/2\right)$,

$$Var\left(\bar{p}^D\right) \geq Var\left(\bar{p}^U\right) \geq Var\left(\bar{p}^V\right). \tag{28}$$

Beyond Generalized Pareto. By the first-order condition (13) and Lemma 1, a linear Nash equilibrium is inconsistent with the discriminatory price format for distributions of $\bar{Q}$ outside of the Generalized Pareto class. We can still rank the uniform price and Vickrey auctions under quite general conditions. The revenue rankings established in this paper extend to all nondegenerate distributions of $\bar{Q}$. Underlying the strong rankings is the state-by-state dominance of the *ex post* equilibria in the UPA and the VA demonstrated in Section 4.1. The rankings of the volatility of the stop-out price from Section 4.2 extend for the UPA and the VA in a straightforward manner.

Independent Private Values. A comparative analysis that includes the DPA is not feasible with a linear equilibrium.[20] Specifically, with heterogenous valuations, the lower bound

[19] For $\xi \geq 0$, the support of $\bar{Q}$ is unbounded and the supply exceeds the satiation point for some realizations. When the stop-out price is negative, the revenue in a DPA is calculated assuming that bidders pay the area below the bid when the bid schedule is positive and the area above the bid when it is negative. Note that, for any unbounded distribution, the probability of the negative stop-out price can be made arbitrarily small by setting v sufficiently high.

[20] Note that the characterizations of best-response bids from this paper apply regardless of whether the

of support of the equilibrium quantity becomes a function of the realization of the intercept of marginal utility, whereas the supports of $\bar{q}$ that are consistent with a linear equilibrium require a lower bound at zero for all realizations of a bidder's valuation. Otherwise, a best response has flat bid parts when the lower bound of the support is strictly positive, or is discontinuous at zero when zero is in the interior of the support. This problem does not arise in our model with symmetric valuations as, for each bidder, equilibrium quantity $\bar{q}_i$ coincides with $\bar{Q}$, the support of which starts at zero.[21]

Our derivation of the best response in the UPA and the VA entailed no assumptions about support of equilibrium quantity, and we can compare the two formats when private values are introduced. Assume that intercepts v_i are independent of $\bar{Q}$. The Vickrey auction dominates the uniform price auction if the variability of individual intercepts of marginal utility, $Var(v_i)$, is not too high relative to the variability and expectation of the supply.

Proposition 9 *(*Revenue Ranking with IPVs*)* $E\left(R^V\right) > E\left(R^U\right)$ *if, and only if,*

$$Var(v_i) < \frac{(2-\gamma)^2}{2\gamma}\rho^2\left(Var\left(\bar{Q}\right) + E^2\left(\bar{Q}\right)\right). \tag{29}$$

For the intuition, note that in the uniform price auction, the variability of v_i has no impact on the expected revenue except through the average realization $\frac{1}{I}\sum_{i\in I} v_i$, since the revenue is linear in individual intercepts v_i. As is well known, when bidders are asymmetric, the Vickrey auction may yield low revenues. Keeping the same average $\frac{1}{I}\sum_{i\in I} v_i$, realizations for which the valuation of one bidder is significantly higher compared to other bidders' valuations lead to lower revenue than realizations with identical valuations.[22] Hence, greater $Var(v_i)$ reduces the expected revenue of the Vickrey auction, relative to the UPA.

Non-linear Nash Equilibria. Analyzing the linear equilibrium, which is unique in all formats and the selection of which has some empirical support (see Section 2.3), allows us to obtain sharp rankings. How robust are our predictions to relaxing the linearity assumption? As is well known, in games with demand schedules as strategies, Nash equilibrium has weak predictive power—the set of Nash equilibria and the corresponding outcomes is large. In a

uncertainty about residual supply comes from randomness in $\bar{Q}$ or valuations of other bidders. Thus, for a fixed family of residual supplies, best responses with independent private values are the same as derived in this paper in all three formats.

[21] Given the impossibility of analyzing heterogeneous valuations in linear equilibria, Rostek and Weretka (2010) derived revenue rankings for a linear ε-quilibrium for DPA, which allows bidders to bid suboptimally (by linearly extending their bids from the positive quadrant) and such extensions are relevant for an allocation in equilibrium with arbitrary small probability.

[22] An insight of Ausubel and Cramton (2004) from their study of the Vickrey auction is that reserve pricing and restricting quantity allows the seller to overcome the problem of low revenue due to asymmetric bidder valuations.

model of a procurement auction with an exogenous downward-sloping demand, Klemperer and Meyer (1989) demonstrated that when utilities are quadratic and uncertainty (e.g., noise) has unbounded support, Nash equilibrium in the UPA is unique in the set of strategies that are piecewise differentiable functions and the equilibrium is linear. Their result applies directly to our UPA model (with a vertical supply); thus, in our analysis for Generalized Pareto distributions with $\xi > 0$, the UPA equilibrium is unique within a large class. For the empirically more plausible distributions $\xi < 0$, the set of Nash equilibria in the UPA is not determinate. To the best of our knowledge, no results concerning the uniqueness of linear equilibria in the DPA have yet been established, even for unbounded support of uncertainty.

NON-QUADRATIC BIDDER UTILITIES. In the uniform price model by Klemperer and Meyer (1989), bidder preferences need not be quadratic. No results are available in the literature about determinacy of equilibrium for non-quadratic utilities, even for UPA with unbounded support of uncertainty. Recent influential literature with discrete strategy spaces accommodates arbitrary preferences (Swinkels (2001); Jackson and Kremer (2006)), but only for large auctions.

6 Conclusions

This paper examines the relative performance of two auction formats—uniform and discriminatory price auctions—both commonly used in divisible good markets, as well as the theoretical benchmark of Vickrey mechanism in environments with symmetric valuations. For risk neutral sellers, our results support the DPA relative to the UPA in non-competitive markets with a fixed number of bidders when expected return and price stability are the criteria. Even though the UPA encourages more aggressive bidding, the DPA yields higher expected revenue in all of the considered environments. The expected-revenue advantage of the DPA weakens in larger markets. In competitive markets, the two auction formats bring the same (expected) revenue, but the UPA offers less risky revenue than does the DPA and, hence, is more attractive to risk averse sellers. When bidders have market power, risk averse sellers face a trade-off between the expected return and revenue riskiness. Design preference will then depend on the seller's degree of risk aversion and the size of the market. Table 1 summarizes the performance of all auction formats.

Table 1. Revenue Rankings with Risk Averse Bidders

	Seller	
Auction	Risk neutral	Risk averse
Small	TS≻DPA≻Vickrey≻UPA	TS≻Vickrey≻UPA
Large	TS≻DPA∼Vickrey∼UPA	TS≻Vickrey∼UPA≻DPA

We also show that, when the design can target potential entrants, the UPA encourages a (weakly) greater participation. Even if quantitatively small, the difference in participation may reverse the revenue ranking in small auctions.

The model presented in this paper assumes symmetric bidders. Therefore, we cannot compare the efficiency of the auction formats (since the equilibrium allocations are trivially efficient), or analyze the distribution of awards, the interaction between private information and price impact, or the winner's curse. As argued in Section 5, addressing these important questions requires a departure from the linear Bayesian Nash equilibrium, which we do in Rostek and Weretka (2010) in a model with private and mixed values.

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Appendix

Please note: We provide a more detailed version of Appendix than is intended for publication.

In all three auction formats, to derive a best response to a stochastic residual supply, we first discretize the distribution of x by partitioning its support into a countable number of intervals of equal length Δx (see Figure 4). Each discrete realization of the supply function, which defines a state and is indexed by s, originates at the midpoint of the corresponding interval. The probability associated with the interval in state s, π^s, is equal to the probability of the interval assigned by the distribution function $G_i(\cdot)$. For the discretely distributed residual supply, we find a best response function (Proposition 1, Proposition 2). As the length of the interval goes to zero, $\Delta x \to 0$, the limit bid constitutes a best response to a continuously distributed residual supply.

Proof. Proposition 1 (EQUILIBRIUM IN THE UPA) We derive the first order condition of a bidder in a UPA, who faces a residual supply with slope μ_i. Consider a small deviation from a bid schedule around a price-quantity pair (p, q),a pair that is observed in equilibrium given state s. The deviation increases the equilibrium quantity by dq and the stop-out price by $dp = \mu_i dq$ in state s; in no other state are the stop-out prices or equilibrium quantities altered (cf. Figure 5.B). This translates into the marginal utility gain by $dq \times \pi^s \times (v - \rho q)$. The deviation increases the payment only for state s, in two ways. First, more units are purchased at the stop-out price p, which yields the change in payments of $dq \times p$; second, more aggressive bidding increases the stop-out price for all units q, and hence the payment increases linearly by $q \times \mu_i dq$. Optimality requires that the marginal benefit and cost associated with a small deviation be equal,

$$dq \times \pi^s \times (v - \rho q) = dq \times \pi^s \times (p + \mu_i \times q). \tag{30}$$

The Euler equation (30) holds for any Δx and, by the Maximum Theorem, in the limit as $\Delta x \to 0$. This gives (5). The following steps are in the text. ■

Proof. Lemma 1 (LIHR) (If) The density of q is given by

$$\bar{g}_i(q) = \frac{1}{\sigma}\left(1 + \xi\frac{(x-\alpha)}{\sigma}\right)^{-\left(\frac{1}{\xi}+1\right)}, \tag{31}$$

therefore,

$$\lambda(q) \equiv \frac{1 - \bar{G}_i(q)}{\bar{g}_i(q)} = \sigma - \alpha\xi + \xi q, \tag{32}$$

and hence, the inverse hazard ratio is linear on the whole compact support.

(Only if) Suppose $\bar{G}_i(q)$ satisfies LIHR on the whole support. We now show that it is a Generalized Pareto distribution. LIHR implies that there exist two constants $\lambda_0, \lambda_1 \in \mathbb{R}$ such that the inverse hazard rate can be written as $\lambda(q) = \lambda_0 + \lambda_1 q$. Since $\lambda(q)$ is well-defined, $\bar{g}_i(q) > 0$ on the whole support, and hence, $\bar{G}_i(q)$ is strictly increasing. Therefore, there exists a unique $q^* \in \mathbb{R}$ such that $\bar{G}_i(q^*) = \frac{1}{2}$. For any q in the support, the following condition must be satisfied

$$\bar{g}_i(q) = \frac{1 - \bar{G}_i(q)}{\lambda_0 + \lambda_1 q}. \tag{33}$$

Differential equation (33) is continuous in the interior of the convex support as $\lambda_0 + \lambda_1 q > 0$. Condition (33) defines a unique $\bar{G}_i(q)$ on the interior of the support, up to a constant. It follows that there can exist, at most, one $\bar{G}_i(\cdot)$ such that its inverse hazard ratio is $\lambda_0 + \lambda_1 q$ and its median is given by q^*. If $\lambda_1 = 0$, the only solution to (33) is an exponential distribution; hence, it is within the class. Consider any $\lambda_1 \neq 0$, and define $\xi \equiv \lambda_1$ and

$$\sigma \equiv \frac{\lambda_0 + \lambda_1 q^*}{2^{\lambda_1}}, \tag{34}$$

$$\alpha \equiv \frac{1}{\lambda_1}\left(\frac{\lambda_0 + \lambda_1 q^*}{2^{\lambda_1}} - \lambda_0\right). \tag{35}$$

Notice that since evaluated at the median value, hazard ratio $\lambda_0 + \lambda_1 q^* > 0$ is positive, so is $\sigma > 0$. Therefore, parameters $\xi, \alpha \in \mathbb{R}$ and $\sigma \in \mathbb{R}_{++}$ define a Generalized Pareto distribution. It is straightforward to verify that the hazard rate of this distribution is given by $\lambda_0 + \lambda_1 q$ and its median is q^*. By the previous uniqueness argument, there can be at most one distribution with two such properties and, hence, the obtained Pareto Distribution coincides with $\bar{G}_i(q)$, and belongs to the class of Generalized Pareto functions. ■

Proof. Proposition 2 (Equilibrium in the DPA) We derive the first order condition of a bidder in a DPA, who faces a residual supply with slope μ_i and the c.d.f. of the intercept x of the residual supply $G_i(\cdot)$ and the corresponding density $g_i(\cdot)$. With a discrete x, agent i is bidding against a (at most) countable family of residual supplies, and his best response is a step function. Therefore, in deriving the best response, we can restrict attention to the class of step functions.[23] Consider a local perturbation of a bid to a new step function around (p, q), observed in state s. In state s, the perturbation increases the obtained quantity by dq and the stop-out price by $dp = \mu_i dq$ and does not affect either of the two variables in any other state (cf. Figure 5.C). With the deviation increasing his quantity in state s, bidder i enjoys the marginal benefit of $dq \times \pi^s \times (v - \rho q)$. The cost of the deviation is the following. Similarly to UPA, in state s, the payment increases for two reasons. Additional units purchased at price p augment the payment by $dq \times p$, and more aggressive bidding raises the price by $dp = \mu_i dq$. Since the payment for all units up to $q - \Delta q$ is determined by the upper part of the bid function, the increased price changes the payment for only Δq units. Consequently, the price impact effect in a DPA is smaller than in a UPA, and given by $\Delta q \times \mu_i dq$. The two effects are depicted in Figure 5.C. In addition, more aggressive bidding inflicts a negative externality on the payments in all higher states. In all higher states, the payment increases by $\Delta q \times \mu_i dq$ and $\Delta p \times dq$. The total marginal cost associated with aggressive bidding amounts to

$$dq \times \pi^s \times (p + \mu_i \Delta q) + dq \times (\Delta p + \mu_i \Delta q) \times \sum_{k>s} \pi^k. \tag{36}$$

At the optimum, the marginal benefit balances the marginal cost. Alternatively, the Euler equation equalizes the net marginal benefit in state s with the negative externality inflicted on the payments in all higher (but not lower) states

$$\frac{\pi^s}{\Delta x} \frac{\Delta x}{\Delta q} \times (v - \rho q - p - \mu_i \Delta q) = (\frac{\Delta p}{\Delta q} + \mu_i) \times \sum_{k>s} \pi^k. \tag{37}$$

Condition (37), which defines the optimal bid, characterizes the trade-off faced by each bidder in the discriminatory auction. The Euler equation (37) gives a necessary optimality condition for an arbitrarily fine partition of the intercept into intervals of size Δx. We now let $\Delta x \to 0$. Since, by assumption, the distribution $G_i(\cdot)$ is smooth, we obtain

$$\frac{\pi^s}{\Delta x} = g_i(x) + o(1). \tag{38}$$

By the Maximum Theorem on compact intervals, the best response converges uniformly to the unique linear best reply $q_i(p)$. Thus, the ratio $\frac{\Delta q}{\Delta p}$ converges to the slope of the best

[23] More precisely, we first find a best response within the class of step functions. Since, for an arbitrary downward-sloping function, there exists a step function that dominates it in terms of payoff, the best response found in the restricted class is a best response in the class of weakly decreasing functions.

response,

$$\frac{\Delta q}{\Delta p} = \frac{\partial q_i(\cdot)}{\partial p} + o(1), \tag{39}$$

and the ratio $\frac{\Delta x}{\Delta q}$ converges to the negative of the slope of the affine function $x(q)$ that maps the equilibrium quantities into the intercepts, which is the relation observed in a linear equilibrium,

$$\frac{\Delta x}{\Delta q} = -\frac{\partial x(\cdot)}{\partial q} + o(1). \tag{40}$$

The minus sign reflects the fact that the equilibrium relation $x(q)$ has a negative slope. Note that with an infinitely fine grid, the within-state price impact effect disappears as $\Delta q \to 0$ and the payments for all units up to q are fixed by the upper part of the bid schedule. It follows that

$$\mu_i \Delta q = o(1). \tag{41}$$

Substituting (38), (39), and (40) into (37), ignoring the $o(1)$ elements, and observing that the probability of all higher states coincides with $G_i(x)$, gives the limit Euler equation (11). In equilibrium, $q = \bar{Q}$, and therefore the inverse hazard ratio of the equilibrium quantity, coincides with the inverse hazard ratio of $\bar{Q}$, and it is given by $\lambda(q) = \sigma + \xi q$. The first-order condition becomes

$$v - \rho q - p = (\sigma + \xi q)(\mu_i + \psi_i), \tag{42}$$

where $\psi_i \equiv -(\partial q_i(\cdot)/\partial p)^{-1}$ is the absolute value of the slope of the inverse bid and is constant in the linear equilibrium. Solving for q gives the optimal bid function

$$q_i(p) = \frac{v - p - \sigma(\mu_i + \psi_i)}{\xi(\mu_i + \psi_i) + \rho}, \tag{43}$$

and hence

$$-\left(\frac{\partial q_i(\cdot)}{\partial p}\right)^{-1} \equiv \psi_i = \xi(\mu_i + \psi_i) + \rho, \tag{44}$$

which, after solving for ψ_i, gives

$$\psi_i = \frac{\xi \mu_i}{1 - \xi} + \frac{\rho}{1 - \xi}. \tag{45}$$

Since the residual supply is a horizontal sum of the inverse bids by other bidders, its slope is equal to $1/(I-1) = 1 - \gamma$ of the average slope of the inverse individual bids of other participants, (16). In the symmetric equilibrium, $\mu_i = (1-\gamma)\psi_i$ for all i, which gives price impact

$$\mu_i = \frac{(1-\gamma)\rho}{1 - (2-\gamma)\xi}. \tag{46}$$

Combining (45) and (46) gives

$$\mu_i + \psi_i = \frac{(2-\gamma)\,\rho}{1-(2-\gamma)\,\xi}. \tag{47}$$

The Euler equation (42) can be rewritten in terms of exogenous parameters as

$$q_i(p) = \frac{1-(2-\gamma)\xi}{\rho}(v-p) - (2-\gamma)\sigma. \tag{48}$$

The equilibrium stop-out price is obtained by market clearing $q_i(\bar{p}) = \bar{Q}$ and averaging (48) across all bidders

$$\bar{p} = v - \frac{\rho}{1-(2-\gamma)\xi}\bar{Q} - \frac{(2-\gamma)\rho\sigma}{1-(2-\gamma)\xi}. \tag{49}$$

■

Proof. Theorem 1 (Revenue Ranking) For any realization of $\bar{Q}$, the revenues *per capita* in three auction formats are given by

$$R^U = v\bar{Q} - \frac{\rho}{\gamma}\bar{Q}^2, \tag{50}$$

$$R^D = \left(v - \frac{(2-\gamma)\,\rho\sigma}{1-(2-\gamma)\,\xi}\right)\bar{Q} - \frac{1}{2}\frac{\rho}{1-(2-\gamma)\,\xi}\bar{Q}^2, \tag{51}$$

$$R^V = v\bar{Q} - \frac{1}{2}\rho(3-\gamma)\bar{Q}^2, \tag{52}$$

and the total surplus *per capita* is defined in (3). In the linear equilibrium, all the revenue functions are quadratic in $\bar{Q}$. The first two moments of random variable $\bar{Q}$ with Generalized Pareto distribution with $\alpha = 0$, $\sigma > 0$ and $\xi < 1/2$ are given by

$$E(\bar{Q}) = \frac{\sigma}{1-\xi}, \tag{53}$$

$$E(\bar{Q}^2) = \frac{2\sigma^2}{(1-\xi)(1-2\xi)}. \tag{54}$$

We obtain the *interim* expected revenues,

$$E(R^U|v) = \frac{\sigma}{1-\xi}\left[v - \frac{2}{\gamma}\frac{\rho\sigma}{(1-2\xi)}\right], \tag{55}$$

$$E(R^D|v) = \frac{\sigma}{1-\xi}\left[v - \frac{(2-\gamma)\,\rho\sigma}{1-(2-\gamma)\,\xi} - \frac{1}{1-(2-\gamma)\,\xi}\frac{\rho\sigma}{(1-2\xi)}\right], \tag{56}$$

$$E(R^V|v) = \frac{\sigma}{1-\xi}\left[v - (3-\gamma)\,\frac{\rho\sigma}{(1-2\xi)}\right]. \tag{57}$$

For all $\gamma \in (0,1)$, $\rho > 0$ and $\xi < 0$, strict inequalities $E(R^D|v) > E(R^V|v) > E(R^U|v)$ hold, while for $\xi \in (0,1/2)$, one obtains $E(R^V|v) > E(R^D|v) > E(R^U|v)$. Since, for any realization of $\bar{Q}$, the total surplus dominates revenue in any of the three auctions, the surplus is also greater in expectation. We now compare the expected order shading in three auction formats. By truthful bidding, there is no order shading in a VA. The expected order shading in a UPA is

$$E(OS^U) = (1-\gamma)\frac{\rho\sigma}{1-\xi}, \tag{58}$$

whereas in a DPA it is given by

$$E(OS^D) = \left(\frac{(2-\gamma)(1-2\xi)}{1-(2-\gamma)\xi}\right)\frac{\rho\sigma}{1-\xi}. \tag{59}$$

It can be shown that $E(OS^D) > E(OS^U)$ for all

$$\xi < \frac{1}{(1+\gamma)(2-\gamma)}, \tag{60}$$

which holds for all $\xi < \frac{1}{2}$. ■

Proof. Lemma 2 (Strategic Equivalence) From (42) and (45), and given the residual supply with slope $\bar{\mu}^U$, the optimal order shading is given by

$$v - \frac{\rho}{\gamma}q - p = (\sigma + \xi q)\left(\frac{\bar{\mu}^U}{1-\xi} + \frac{\rho}{1-\xi}\right). \tag{61}$$

Using that the equilibrium price impact in the UPA is $\bar{\mu}^U = \frac{1-\gamma}{\gamma}\rho$, the inverse bid is

$$p_i(q) = v - \frac{1}{1-\xi}\frac{\rho}{\gamma}q - \frac{\xi}{1-\xi}\frac{\rho}{\gamma}q. \tag{62}$$

For a fixed q^{th} unit, the conditional expectation of price in the UPA is given by

$$E(\bar{p}^U|\bar{p}^U \leq v^*(q)] = \frac{1}{1-F(q|v)}\int_{q}^{-\frac{\sigma}{\xi}} \bar{p}^U f(\bar{Q}|v)\, d\bar{Q} = \frac{1}{1-F(q|v)}\int_{\bar{q}}^{-\frac{\sigma}{\xi}}\left(v - \frac{\rho}{\gamma}\bar{Q}\right) f(\bar{Q}|v)\, d\bar{Q}. \tag{63}$$

Integrating by parts and substituting for $f\left(\bar{Q}\right)$ and $F\left(\bar{Q}\right)$,

$$\begin{aligned} \int \left(v - \frac{\rho}{\gamma}\bar{Q}\right) f\left(\bar{Q}\right) d\bar{Q} &= \left(v - \frac{\rho}{\gamma}\bar{Q}\right) F\left(\bar{Q}\right) + \frac{\rho}{\gamma}\int F\left(\bar{Q}\right) d\bar{Q} \qquad (64) \\ &= \left(v - \frac{\rho}{\gamma}\bar{Q}\right)\left(1 - \left(1 + \frac{\xi}{\sigma}\bar{Q}\right)^{-\frac{1}{\xi}}\right) + \frac{\rho}{\gamma}\left(\bar{Q} + \frac{\sigma}{1-\xi}\left(1 + \frac{\xi}{\sigma}\bar{Q}\right)^{-\frac{1}{\xi}+1}\right), \end{aligned}$$

and, hence,

$$E(\bar{p}^U | \bar{p}^U \leq v^*\left(q\right)] = v - \frac{1}{1-\xi}\frac{\rho}{\gamma}q - \frac{\rho}{\gamma}\frac{\sigma}{1-\xi} = p_i\left(q\right). \qquad (65)$$

■

Proof. Proposition 3 (Revenue Ranking; Arbitrary Risk Preference of the Seller) The assertion follows from (3), (50), and (52). ■

Proof. Proposition 4 (Revenue Ranking in Large Auctions) Take a limit of (55), (56) and (57), as $\gamma \to 1$. In all three auctions, the expected revenues are equal and given by

$$E(R^U|v) = E(R^D|v) = E(R^V|v) = \frac{\sigma}{1-\xi}\left[v - \frac{2\rho\sigma}{1-2\xi}\right]. \qquad (66)$$

The expected revenue in large auctions is strictly smaller than the expected total surplus. The total expected surplus *per capita* is given by

$$E(R^{Opt}|v) = \frac{\sigma}{1-\xi}\left[v - \frac{\rho\sigma}{(1-2\xi)}\right]. \qquad (67)$$

The fraction of surplus not extracted by the seller is equal to

$$\frac{\rho\sigma}{v(1-2\xi) - \rho\sigma}. \qquad (68)$$

(This value can be seen in Figure 2.A at $\gamma = 1$.) ■

Proof. Proposition 5 (SOSD) For any realization of $\bar{Q}$, the revenue in large uniform price and Vickrey auctions is given by

$$R^U = R^V = v\bar{Q} - \rho\bar{Q}^2, \qquad (69)$$

whereas in a DPA, it is

$$R^D = \left(v - \frac{\rho\sigma}{1-\xi}\right)\bar{Q} - \frac{1}{2}\frac{\rho}{1-\xi}\bar{Q}^2. \qquad (70)$$

(The revenues for different realizations of the supply are depicted in Figure 2.B.) In all formats, the revenues are zero for $\bar{Q} = 0$, and the revenue function in the UPA and the VA is steeper than that of the DPA at zero. Consequently, $R^U(\bar{Q}) > R^D(\bar{Q})$ for all sufficiently small $\bar{Q} > 0$. As the difference $R^U(\bar{Q}) - R^D(\bar{Q})$ is quadratic in $\bar{Q}$ and $R^U(0) = R^D(0) = 0$, there is at most one $\bar{Q}^* > 0$ in the support of both random variables, for which $R^* \equiv R^U(\bar{Q}^*) = R^D(\bar{Q}^*)$. Since, the revenues are equal in expectations, thresholds $\bar{Q}^*$ and $\bar{R}^*$ exist in the interior of the support and the image, respectively. In Figure 2.B, they are given by $R^* = 0.9$ and $\bar{Q}^* = 0.7$. For any scalar $R \in (0, \bar{R})$ where $\bar{R}$ is the maximal realization of revenue in the UPA, define $\bar{Q}^U$ and $\bar{Q}^D$ as the realizations of supply that give revenue R in the UPA and the DPA, respectively (i.e., $R^U(\bar{Q}^U) = R^D(\bar{Q}^D) = R$). By continuity and monotonicity of functions, such scalars exist and are unique. R^* partitions the interval $\left[0, \bar{R}\right]$ into two subsets: for all $R < R^*$, the corresponding scalars satisfy $\bar{Q}^D > \bar{Q}^U$. By monotonicity, the probability of revenue weakly less than R equals $F(\bar{Q}^U|v)$ in the UPA and $F(\bar{Q}^D|v)$ in the DPA. It follows that, in the interval $(0, R^*)$, the c.d.f. of the DPA revenue is strictly higher than that of the UPA. By an analogous argument, for all $R > R^*$, the UPA c.d.f. function is greater than that of the DPA. This establishes single-crossing of the two c.d.f.s at R^*, which is sufficient for the UPA revenue to stochastically dominate the DPA in the second-order sense. ■

Proof. Proposition 6 (Revenue Equivalence with Constant Marginal Utility) Take the limits of (55), (56) and (57) as $\rho \to 0$. In all three auctions, expected revenues are equal and coincide with the expected total surplus

$$E(R^U|v) = E(R^D|v) = E(R^V|v) = E(R^{Opt}|v) = \frac{\sigma v}{1 - \xi}. \tag{71}$$

■

Proof. Proposition 7 (Volatility of the Stop-Out Price) The result follows from the unconditional variances of the equilibrium prices in the UPA, DPA and VA,

$$Var\left(\bar{p}^U\right) = Var(v) + \left(\frac{\rho}{\gamma}\right)^2 Var\left(\bar{Q}\right), \tag{72}$$

$$Var\left(\bar{p}^D\right) = Var(v) + \left(\frac{\rho}{1 - (2-\gamma)\xi}\right)^2 Var\left(\bar{Q}\right), \tag{73}$$

$$Var\left(\bar{p}^V\right) = Var(v) + \rho^2 Var\left(\bar{Q}\right). \tag{74}$$

■

Proof. Proposition 8 (Revenue Ranking with Endogenous Entry) To determine the number of participants and the expected revenue in the unique subgame perfect Nash equilibrium, we first ignore the integer problem. In any auction format, the endogenous number of bidders is determined by the equality between expected utility from participation and entry cost

$$U = vE(q) - \frac{1}{2}\rho E(q^2) - E(Payment) = c. \tag{75}$$

We assume that the *total* supply Q is fixed and Generalized Pareto with exogenous parameters $\sigma' > 0$, $\xi' \in \mathbb{R}$ and $\alpha' = 0$. For the UPA, using that the revenue is given by (50), condition (75) becomes

$$\frac{2-\gamma}{2\gamma}\rho\frac{1}{I^2}E(Q^2) = c. \tag{76}$$

Substituting from the definition of γ in (2) , we solve (76) for the positive root

$$I^U(c) = 1 + \left(1 + \frac{\rho\sigma'^2}{c(1-\xi')(1-2\xi')}\right)^{0.5}, \tag{77}$$

where we used the second moment formula for Q from (53). Condition (77) determines participation in the UPA as a function of c. Next, we derive an analogous function for DPA. Note that since *per capita* supply $\bar{Q}$ coincides with Q scaled down by coefficient $1/I > 0$, it is in the class of Pareto distribution functions and is defined by parameters $\sigma = \sigma'/I$, $\xi = \xi'$ and $\alpha = 0$. For the DPA, condition (75) becomes

$$\frac{(2-\gamma)\,\rho\sigma'}{1-(2-\gamma)\,\xi'}\frac{1}{I^2}E\left(Q\right) + \left[\frac{(2-\gamma)\,\xi'}{1-(2-\gamma)\,\xi'}\right]\frac{1}{2}\frac{1}{I^2}\rho E\left(Q^2\right) = c. \tag{78}$$

Applying the definition of γ and the expectation (53), gives

$$(1-\xi')cI^2 - cI - \frac{\rho\sigma'^2}{(1-2\xi')} = 0, \tag{79}$$

which has the (positive) solution given by

$$I^D\left(c\right) = \frac{1}{2\left(1-\xi\right)} + \left(\frac{1}{4\left(1-\xi\right)^2} + \frac{\rho\sigma'^2}{c(1-2\xi')\left(\left(1-\xi\right)\right)}\right)^{0.5}. \tag{80}$$

For the VA, condition (75) is

$$\frac{(2-\gamma)}{2I^2}\rho E\left(Q^2\right) = c, \tag{81}$$

the (positive) solution of which is given by

$$I^V(c) = \frac{1}{2} + \left(\frac{1}{4} + \frac{\rho\sigma'^2}{c\,(1-\xi')\,(1-2\xi')}\right)^{0.5}. \tag{82}$$

$I^U(\cdot) > I^D(\cdot)$ and $I^U(\cdot) > I^V(\cdot)$, and the ranking between the DPA and the VA depends on the sign of ξ'. In particular, for $\xi' < 0$, we have $I^V(\cdot) > I^D(\cdot)$. In the three auction formats, the number of participating bidders (given c) is the greatest integer that is less than or equal to derived $I^U(c)$, $I^D(c)$ and $I^V(c)$ and, hence, the asserted participation ranking holds with weak inequalities. Simulations indicate that for some values of parameters, the rankings are strict (see Figure 2.A). Using the derived market sizes in expected revenues (55) and (56), numerical simulations show that there exists a range of values of parameters for which $E(R^U|v) > E(R^D|v)$.

The Pareto efficient entry maximizes the surplus

$$vE\,(Q) - \frac{1}{2I}\rho E\left(Q^2\right) - cI. \tag{83}$$

The optimal number of entrants I^{PE} is equal to

$$I^{PE} = \left(\frac{\rho\sigma'^2}{c\,(1-\xi')\,(1-2\xi')}\right)^{0.5} \tag{84}$$

and is strictly smaller than I^U, I^V and I^D. ■

Proof. Proposition 9 (Revenue Ranking with IPVs) Using the revenue in the UPA (50) with $\bar{v} = \frac{1}{I}\sum_i v_i$, the expected revenue is

$$E\left(R^U\right) = E\,(v_i)\,E\left(\bar{Q}\right) - \frac{\rho}{\gamma}E\left(\bar{Q}^2\right). \tag{85}$$

In the VA, the residual supply of bidder i is found by solving $I\bar{Q} = \frac{1}{\rho}\sum_{j\neq i}(v_j - p) + q_i$ for p

$$p\,(q_i) = \bar{v}_{-i} - \rho\,(2-\gamma)\,\bar{Q} + \rho\,(1-\gamma)\,q_i, \tag{86}$$

and the individual quantity obtained is

$$q_i = \frac{1}{\rho\,(2-\gamma)}\,(v_i - \bar{v}_{-i}) + \bar{Q}. \tag{87}$$

The residual supply and quantity obtained in equilibrium gives *per-capita* revenue

$$\begin{aligned} R^V &= \frac{1}{2}\left[\bar{v}+\bar{v}_{-i}-(3-\gamma)\,\rho\bar{Q}\right]\left[\bar{Q}+\frac{1}{(1-\gamma)\,\rho}\left(\bar{v}-\bar{v}_{-i}\right)\right]= \\ &= \frac{1}{2}\left[\left[\bar{v}+\bar{v}_{-i}\right]\bar{Q}-(3-\gamma)\,\rho\bar{Q}^2+\frac{1}{(1-\gamma)\,\rho}\left[\bar{v}^2-\bar{v}_{-i}^2\right]-\frac{(3-\gamma)}{(1-\gamma)}\left(\bar{v}-\bar{v}_{-i}\right)\bar{Q}\right]. \end{aligned} \tag{88}$$

In expectations,

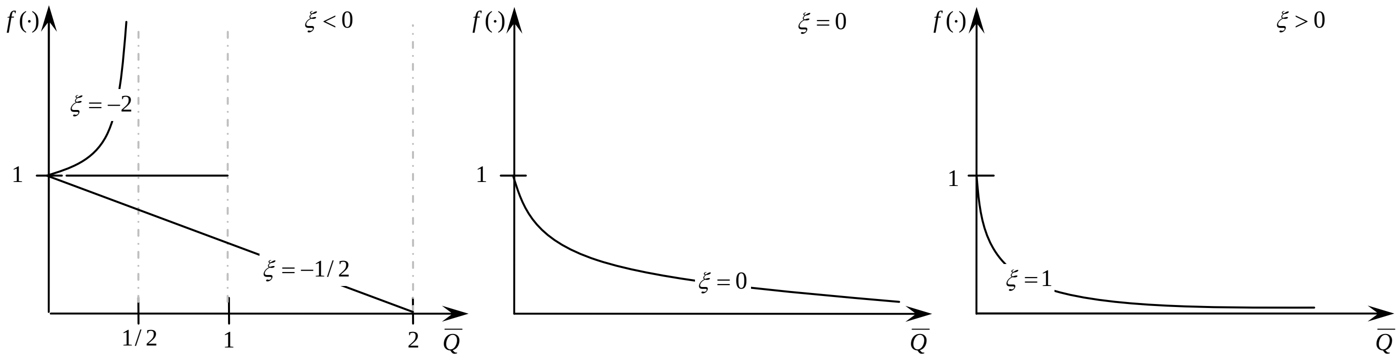
$$E\left(R^V\right) = \left[E\left(v_i\right)\right]\bar{Q}-\frac{(3-\gamma)\,\rho\bar{Q}^2}{2}+\frac{1}{(1-\gamma)\,\rho}E\left[\bar{v}^2-\bar{v}_{-i}^2\right]= \tag{89}$$

$$= \left[E\left(v_i\right)\right]\bar{Q}-\frac{(3-\gamma)\,\rho E\left(\bar{Q}^2\right)}{2}-\frac{(1-\gamma)}{(2-\gamma)\,\rho}V\left(v_i\right). \tag{90}$$

The ranking then follows from (85) and (90). ■

FIGURE 1: DENSITY OF $\overline{Q}$ AND OPTIMAL BIDS IN DPA ($\mu = 0, \sigma = 1$)

A.) DENSITY



B.) OPTIMAL BIDS

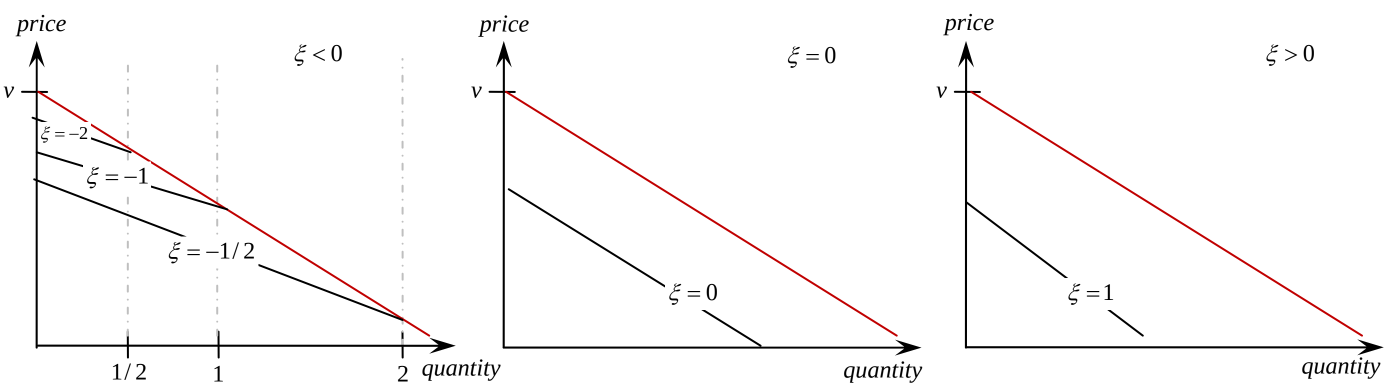


FIGURE 2: REVENUE AND TOTAL SURPLUS ($\mu = 0, \sigma = \rho = 1, \xi = -1, v = 3$)

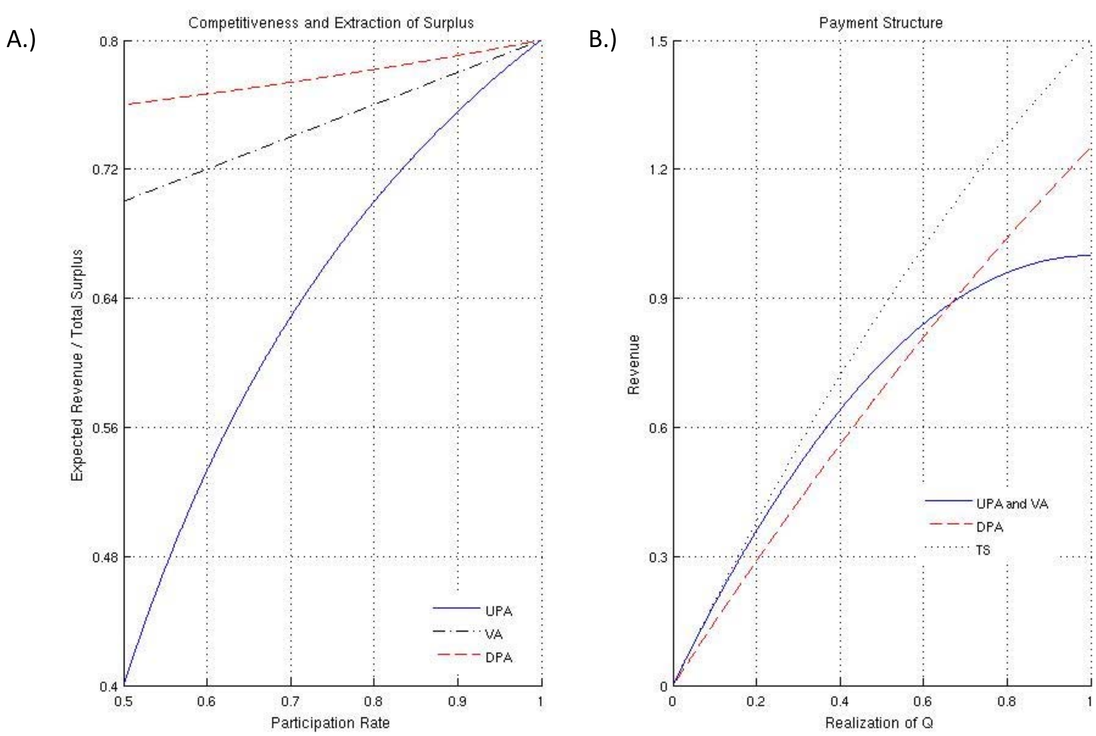


FIGURE 3: ENDOGENOUS PARTICIPATION AND REVENUE RANKING REVERSAL ($\mu = 0, \sigma = 3, \rho = 1, \xi = -1, v = 1$)

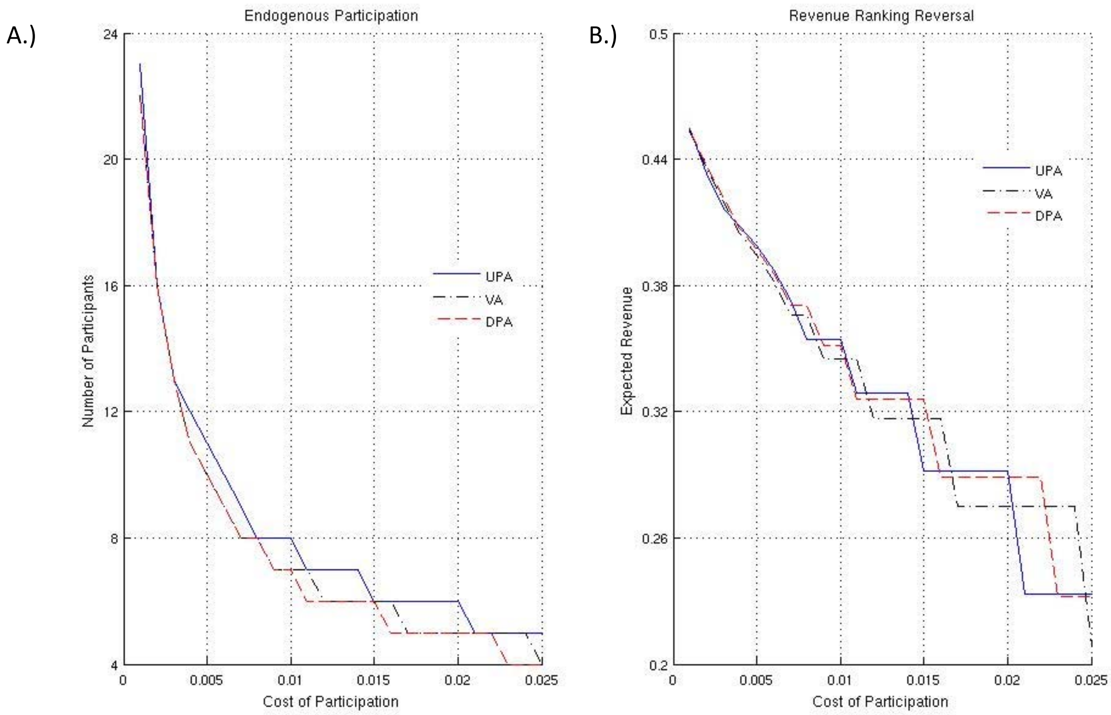


FIGURE 4: DISCRETE RESIDUAL SUPPLY

A.) RESIDUAL SUPPLY

B.) PROBABILITIES

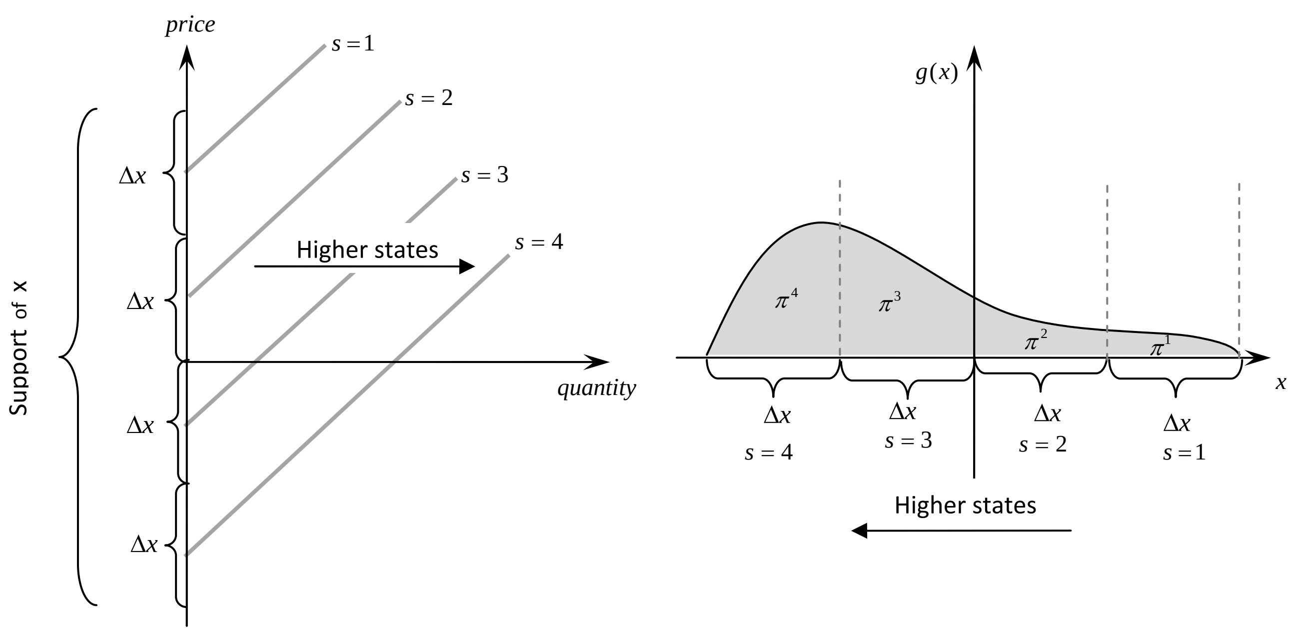
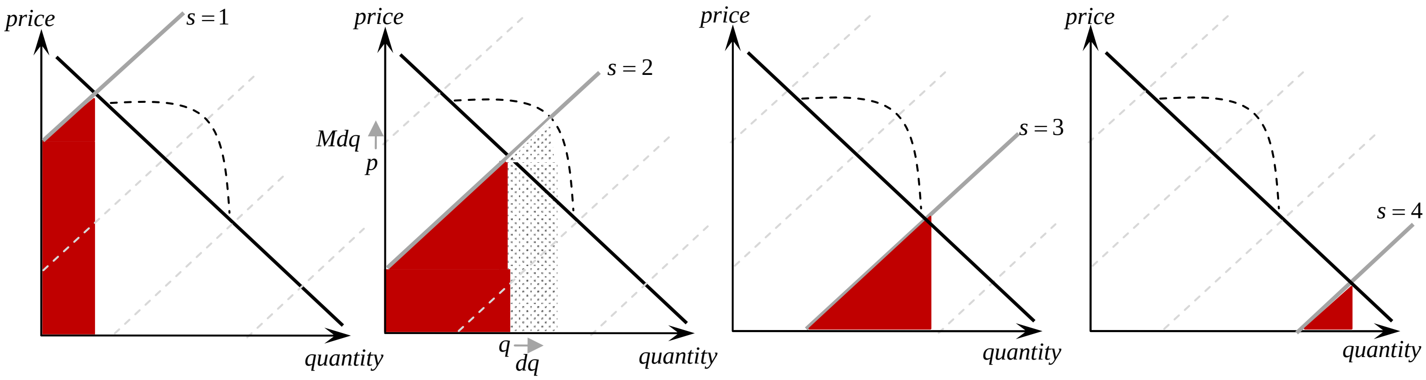
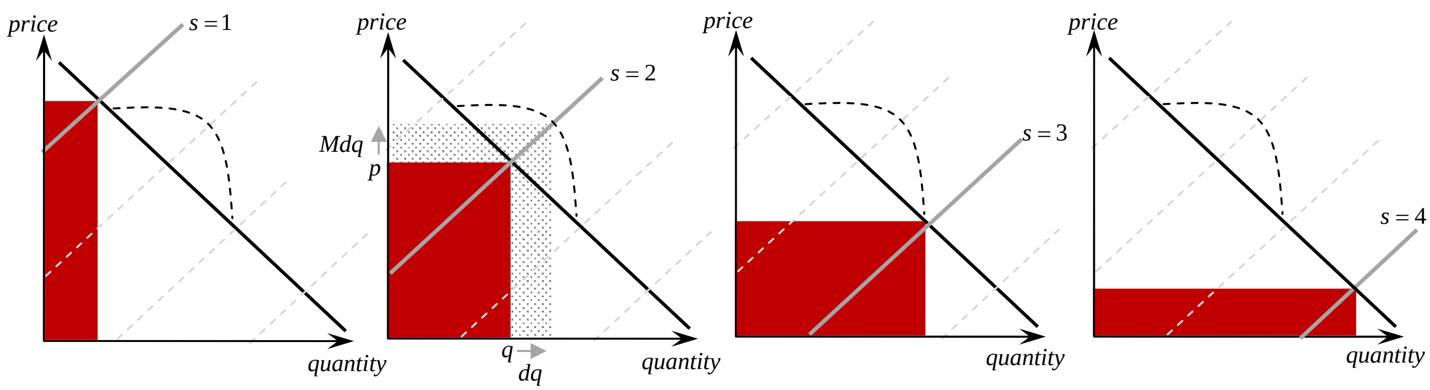


Figure 5: Effects of a Deviation on Payment

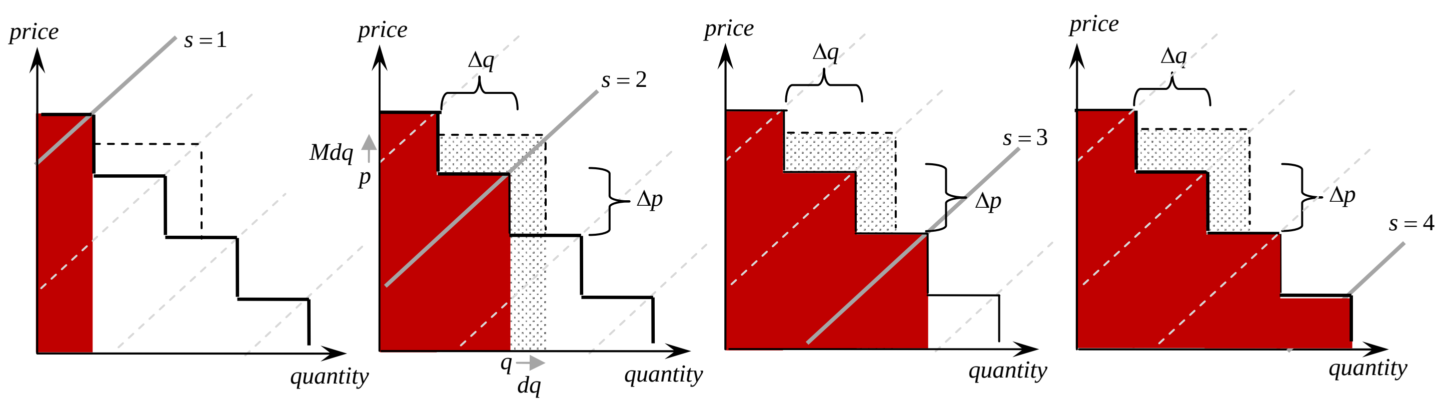
A.) Payment in a Vickrey Auction



B.) Payment in a Uniform Price Auction



C.) Payment in a Discriminatory Price Auction



Legend:

- Optimal Bid
- Deviation
- Realized Residual Supply
- Not Realized Residual Supply
- Payment Without Deviation
- Increase in Payment