Problem Set 8: Solutions

ECON 301: Intermediate Microeconomics
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Problem 1 (Cobb-Douglas)

(a) To determine the returns to scale, we compare $f(\lambda K, \lambda L)$ to $\lambda f(K, L)$ with $\lambda > 1$.

- For $f(K, L) = K^2 L^2$:

$$f(\lambda K, \lambda L) = (\lambda K)^2 (\lambda L)^2 = \lambda^4 K^2 L^2 = \lambda^4 f(K, L) > \lambda f(K, L) \implies IRS$$

- For $f(K, L) = K^{\frac{1}{3}} L^{\frac{2}{3}}$:

$$f(\lambda K, \lambda L) = (\lambda K)^{\frac{1}{3}} (\lambda L)^{\frac{2}{3}} = \lambda^{\frac{1}{3}+\frac{2}{3}} K^{\frac{1}{3}} L^{\frac{2}{3}} = \lambda f(K, L) \implies CRS$$

- For $f(K, L) = K^{\frac{1}{4}} L^{\frac{1}{4}}$:

$$f(\lambda K, \lambda L) = (\lambda K)^{\frac{1}{4}} (\lambda L)^{\frac{1}{4}} = \lambda^{\frac{1}{4}+\frac{1}{4}} K^{\frac{1}{4}} L^{\frac{1}{4}} = \lambda^{\frac{1}{2}} f(K, L) < \lambda f(K, L) \implies DRS$$

(b) For $w_L = w_K = 1$, we find the cost functions associated with the three Cobb-Douglas production functions using the first secret of happiness, cost minimization, which requires that

$$TRS = -\frac{w_K}{w_L} \tag{1}$$

.

- For $f(K, L) = K^2 L^2$, we have $TRS = -\frac{MP_K}{MP_L} = -\frac{L}{K}$. So from equation (1) we have

$$-\frac{L}{K} = -1 \implies K = L$$

  which gives us the cost minimizing proportion of K and L. We then plug $K = L$ into the production (output) function to get both K and L in terms of output y:

$$y = f(K, L) = K^2 L^2 = K^2 K^2 = K^4 \implies K = y^{\frac{1}{4}} \text{ and } L = y^{\frac{1}{4}} \text{ (since } K = L\text{)}.$$

  Now we plug $K = y^{\frac{1}{4}}$ and $L = y^{\frac{1}{4}}$ into the cost function, getting

$$c(y) = w_K K + w_L L = 1 \times K + 1 \times L = y^{\frac{1}{4}} + y^{\frac{1}{4}} = 2y^{\frac{1}{4}}.$$

- For $f(K, L) = K^{\frac{1}{3}} L^{\frac{2}{3}}$, we have $TRS = -\frac{MP_K}{MP_L} = -\frac{\frac{1}{3}L}{\frac{2}{3}K} = -\frac{L}{2K}$. So from equation (1) we have

$$-\frac{L}{2K} = -1 \implies L = 2K$$

which gives us the cost minimizing proportion of K and L. We then plug $L = 2K$ into the production (output) function to get both K and L in terms of output y:

$$y = f(K, L) = K^{\frac{1}{3}} L^{\frac{2}{3}} = K^{\frac{1}{3}} (2K)^{\frac{2}{3}} = 2^{\frac{2}{3}} K \quad \Longrightarrow \quad K = 2^{-\frac{2}{3}} y \text{ and } L = 2^{\frac{1}{3}} y\,.$$

Now we plug $K = 2^{-\frac{2}{3}} y$ and $L = 2^{\frac{1}{3}} y$ into the cost function, getting

$$c(y) = w_K K + w_L L = 1 \times K + 1 \times L = 2^{-\frac{2}{3}} y + 2^{\frac{1}{3}} y = (2^{-\frac{2}{3}} + 2^{\frac{1}{3}}) y \approx 1.9y\,.$$

- For $f(K, L) = K^{\frac{1}{4}} L^{\frac{1}{4}}$, we have $TRS = -\frac{MP_K}{MP_L} = -\frac{\frac{1}{4}L}{\frac{1}{4}K} = -\frac{L}{K}$. So from equation (1) we have

$$-\frac{L}{K} = -1 \quad \Longrightarrow \quad L = K$$

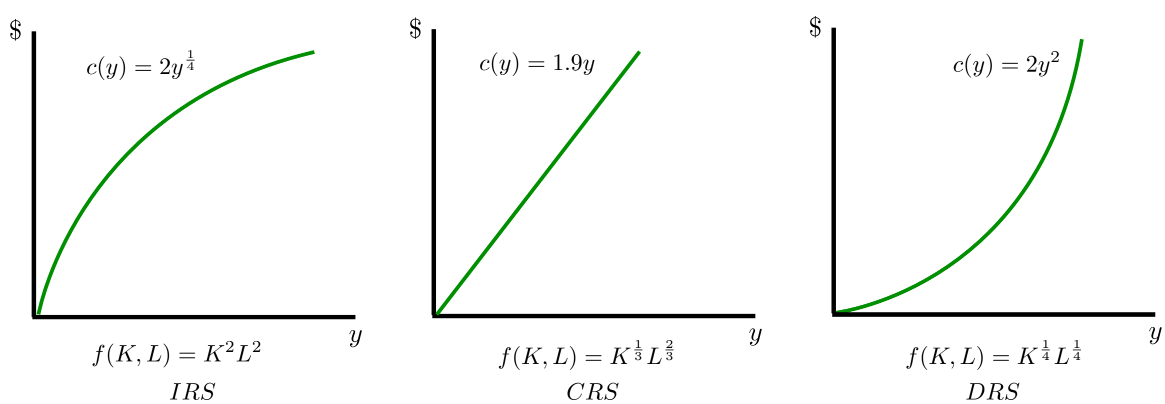
which gives us the cost minimizing proportion of K and L. We then plug $L = K$ into the production (output) function to get both K and L in terms of output y:

$$y = f(K, L) = K^{\frac{1}{4}} L^{\frac{1}{4}} = K^{\frac{1}{4}} (K)^{\frac{1}{4}} = K^{\frac{1}{2}} \quad \Longrightarrow \quad K = y^2 \text{ and } L = y^2\,.$$

Now we plug $K = y^2$ and $L = y^2$ into the cost function, getting

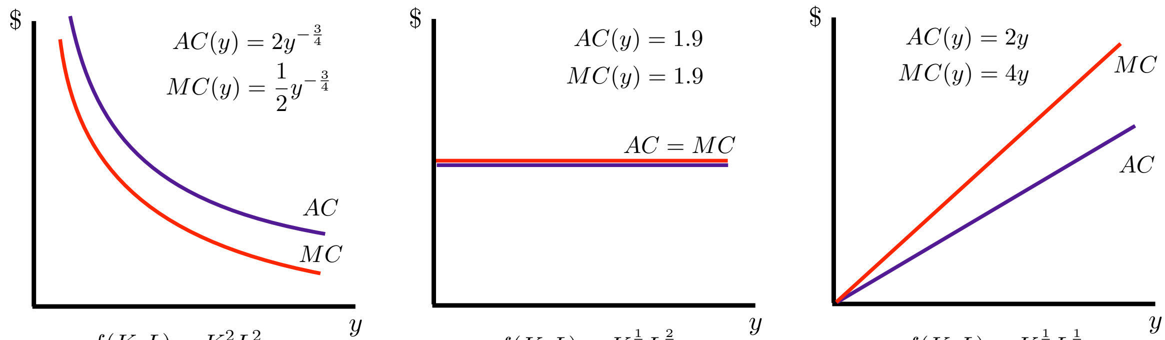
$$c(y) = w_K K + w_L L = 1 \times K + 1 \times L = y^2 + y^2 = 2y^2\,.$$

(c) The cost functions we found in part **(b)** are shown below. Notice how the shape of the cost function is related to the returns to scale for each production function found above in part **(a)**:



(d) The average and marginal cost functions are shown below. Remember that

$$AC(y) = \frac{c(y)}{y} \quad \text{and} \quad MC(y) = c'(y)\,.$$



$f(K,L) = K^2L^2$ *IRS* | $f(K,L) = K^{\frac{1}{3}}L^{\frac{2}{3}}$ *CRS* | $f(K,L) = K^{\frac{1}{4}}L^{\frac{1}{4}}$ *DRS*

Problem 2 (Perfect Complements)

(a) Again, to determine the returns to scale, we compare $f(\lambda K, \lambda L)$ to $\lambda f(K, L)$ with $\lambda > 1$.

- For $f(K,L) = \min\{K, L\}$:

$$f(\lambda K, \lambda L) = \min\{\lambda K, \lambda L\} = \lambda \times \min\{K, L\} = \lambda f(K, L) \implies CRS$$

- For $f(K,L) = (\min\{K, L\})^2$:

$$f(\lambda K, \lambda L) = (\min\{\lambda K, \lambda L\})^2 = \lambda^2(\min\{K, L\})^2 = \lambda^2 f(K, L) > \lambda f(K, L) \implies IRS$$

- For $f(K,L) = \sqrt{\min\{K, L\}}$:

$$f(\lambda K, \lambda L) = (\min\{\lambda K, \lambda L\})^{\frac{1}{2}} = \lambda^{\frac{1}{2}}(\min\{K, L\})^{\frac{1}{2}} = \lambda^{\frac{1}{2}} f(K, L) < \lambda f(K, L) \implies DRS$$

(b) Our first step in finding the cost functions is to determine the cost-minimizing combination of K and L. For the production functions here, K and L are perfect complements and the cost-minimizing combination is such that $K = L$. (These production functions are associated with the L-shaped isoquants, just as when two goods were perfect complements in utility theory, we saw L-shaped indifference curves. We had determined that optimal consumption was along the vertices of the indifference curves; the same thing is going on here, where optimal input combinations are along the vertices of the isoquants.) We can proceed now by substituting $K = L$ into the three production functions to get K and L in terms of y.

- For $f(K,L) = \min\{K, L\}$:

$$y = f(K, L) = \min\{K, L\} = \min\{K, K\} = K \implies K = y \text{ and } L = y\,.$$

Plugging this into the cost function for K and L and $w_K = w_L = 1$ we have

$$c(y) = w_K K + w_L L = 1 \times K + 1 \times L = y + y = 2y\,.$$

- For $f(K,L) = (\min\{K,L\})^2$:

$$y = (K,L) = (\min\{K,L\})^2 = (\min\{K,K\})^2 = K^2 \implies K = y^{\frac{1}{2}} \text{ and } L = y^{\frac{1}{2}}\,.$$

Plugging this into the cost function for K and L and $w_K = w_L = 1$ we have

$$c(y) = w_K K + w_L L = 1 \times K + 1 \times L = y^{\frac{1}{2}} + y^{\frac{1}{2}} = 2y^{\frac{1}{2}}\,.$$

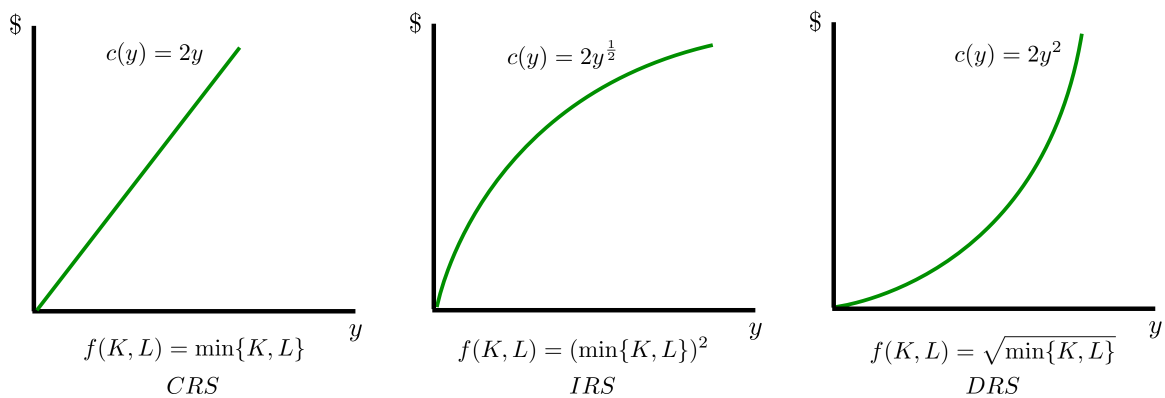
- For $f(K,L) = \sqrt{\min\{K,L\}}$:

$$y = f(K,L) = \sqrt{\min\{K,L\}} = \sqrt{\min\{K,K\}} = K^{\frac{1}{2}} \implies K = y^2 \text{ and } L = y^2\,.$$

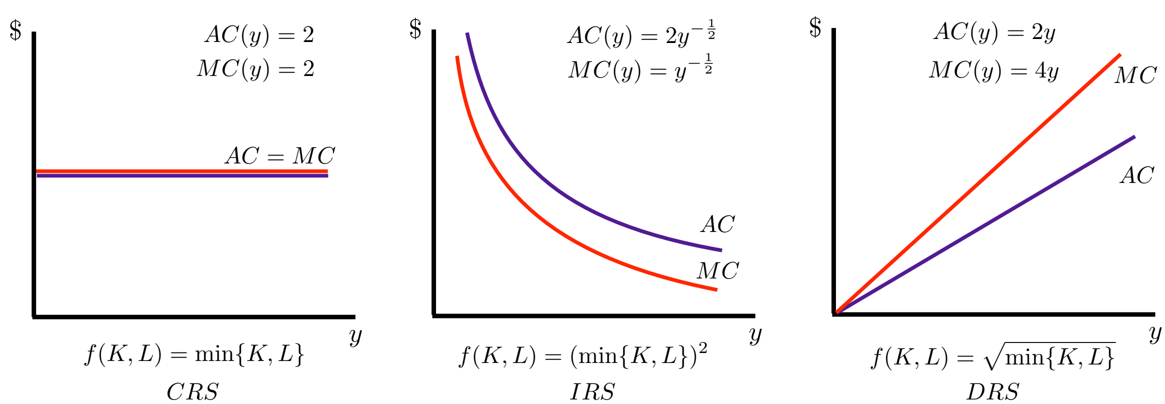
Plugging this into the cost function for K and L and $w_K = w_L = 1$ we have

$$c(y) = w_K K + w_L L = 1 \times K + 1 \times L = y^2 + y^2 = 2y^2\,.$$

(c) The cost functions we found in part **(b)** are shown below. Again, notice how the shape of the cost function is related to the returns to scale for each production function found above in part **(a)**:



(d) The average and marginal cost functions are shown below:



Problem 3 (Perfect Substitutes)

(a) To determine the returns to scale, we compare $f(\lambda K, \lambda L)$ to $\lambda f(K, L)$ with $\lambda > 1$.

- For $f(K, L) = K + 0.5L$:

$$f(\lambda K, \lambda L) = \lambda K + 0.5(\lambda L) = \lambda(K + 0.5L) = \lambda f(K, L) \implies CRS$$

- For $f(K, L) = (K + 0.5L)^2$:

$$f(\lambda K, \lambda L) = (\lambda K + 0.5(\lambda L))^2 = (\lambda(K + 0.5L))^2 = \lambda^2 f(K, L) > \lambda f(K, L) \implies IRS$$

- For $f(K, L) = \sqrt{K + 0.5L}$:

$$f(\lambda K, \lambda L) = (\lambda K + 0.5(\lambda L))^{\frac{1}{2}} = (\lambda(K + 0.5L))^{\frac{1}{2}} = \lambda^{\frac{1}{2}} f(K, L) < \lambda f(K, L) \implies DRS$$

(b) For all of these production functions, $MP_K > MP_L$ (capital is relatively more productive at all levels), and since the costs are equal ($w_K = w_L = 1$), only capital should be used as an input. Hence the cost minimizing choice of inputs requires $L = 0$. (As with **Problem 2** part **(b)**, this is analogous to the argument in utility theory when two goods were perfect substitutes.)

- For $f(K, L) = K + 0.5L$:

$$y = f(K, L) = K + 0.5L = K + 0.5(0) = K \implies K = y$$

Plugging this into the cost function for K and L and $w_K = w_L = 1$ we have

$$c(y) = w_K K + w_L L = 1 \times K + 1 \times 0 = y\,.$$

- For $f(K, L) = (K + 0.5L)^2$:

$$y = f(K, L) = (K + 0.5L)^2 = (K + 0.5(0))^2 = K^2 \implies K = y^{\frac{1}{2}}$$

Plugging this into the cost function for K and L and $w_K = w_L = 1$ we have

$$c(y) = w_K K + w_L L = 1 \times K + 1 \times 0 = y^{\frac{1}{2}}\,.$$

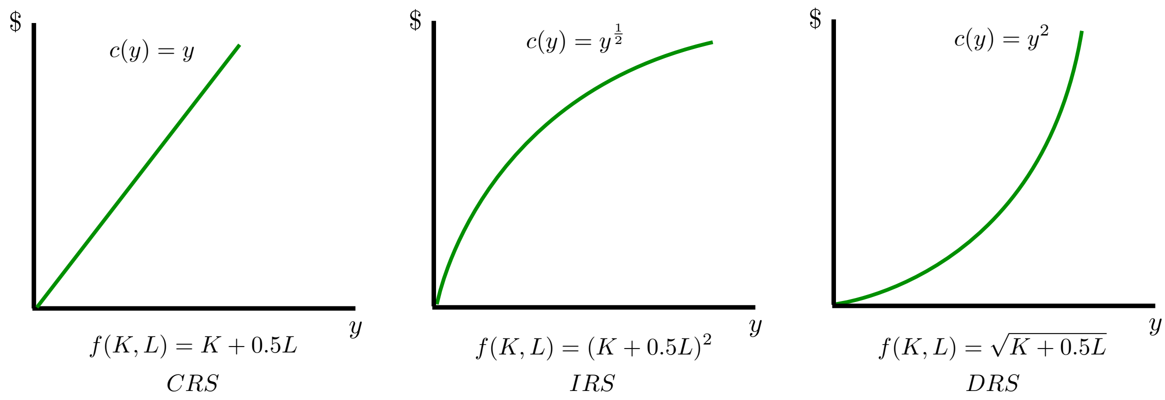
- For $f(K, L) = \sqrt{K + 0.5L}$:

$$y = f(K, L) = (K + 0.5L)^{\frac{1}{2}} = (K + 0.5(0))^{\frac{1}{2}} = K^{\frac{1}{2}} \implies K = y^2$$

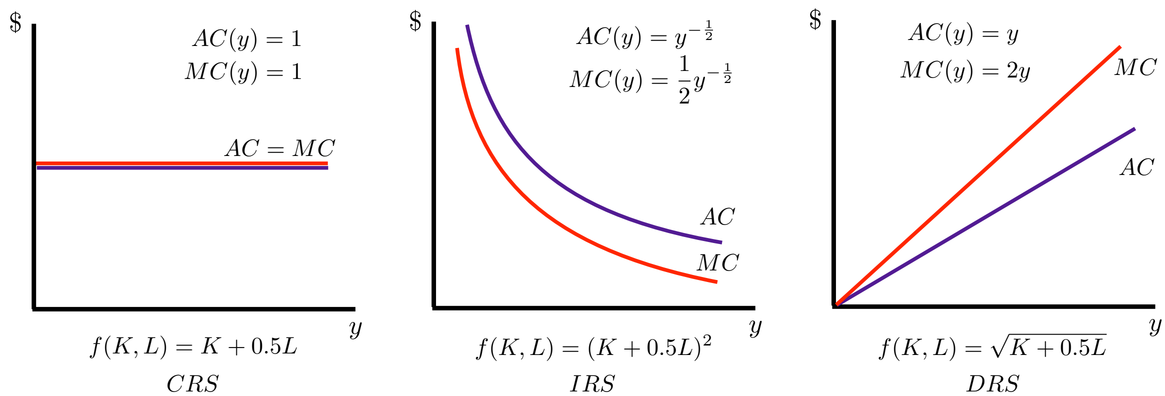
Plugging this into the cost function for K and L and $w_K = w_L = 1$ we have

$$c(y) = w_K K + w_L L = 1 \times K + 1 \times 0 = y^2\,.$$

(c) The cost functions we found in part **(b)** are shown below. Again, notice how the shape of the cost function is related to the returns to scale for each production function found above in part **(a)**:



(d) The average and marginal cost functions are shown below:



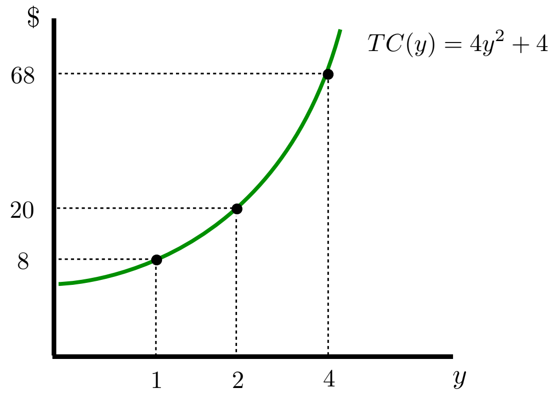
Problem 4 (Cost Curves)

(a) The production function associated with cost curve $c(y) = 4y^2$ must exhibit decreasing returns to scale. With this cost curve, doubling output more than doubles the cost; this is a result of the fact that, with decreasing returns to scale, doubling output would require more than doubling inputs, and since input costs are linear, the cost will more than double.

(b) The total cost function is

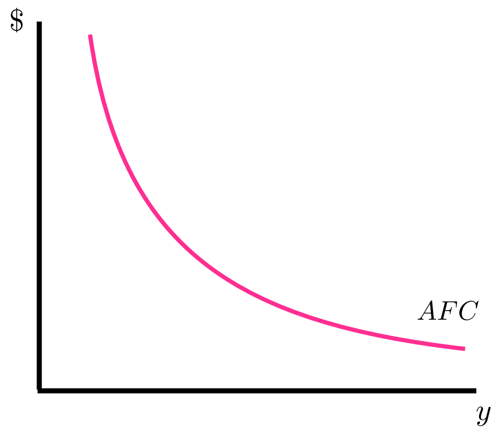
$$TC(y) = 4y^2 + 4$$

and so $TC(1) = 8$, $TC(2) = 20$, and $TC(4) = 68$ as shown below along with the fixed and variable cost curves:

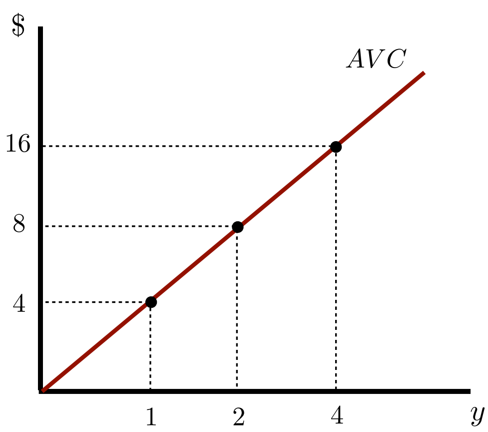


(c) The average fixed cost is $AFC(y) = \frac{FC}{y} = \frac{4}{y}$, so $AFC(1) = 4$, $AFC(2) = 2$, and $AFC(4) = 1$. As y becomes infinitely large, AFC goes to zero (the fixed cost of 4 is being spread over a very large number of units); when y is very close to zero, AFC is infinitely large ($\lim_{y \to 0^+} \frac{4}{y} = \infty$).

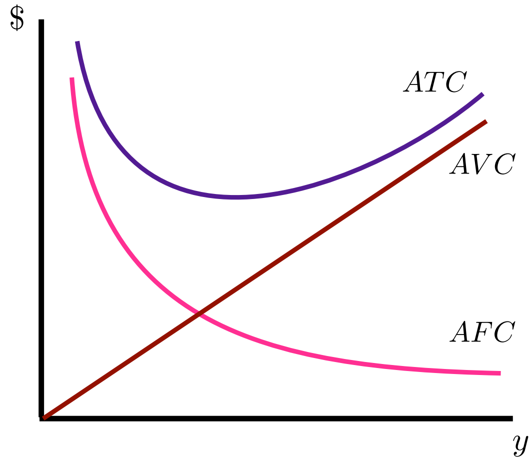
Average fixed cost is shown below:



(d) The average variable cost is $AVC(y) = \frac{VC(y)}{y} = \frac{4y^2}{y} = 4y$, so $AVC(1) = 4$, $AVC(2) = 8$, and $AVC(4) = 16$:



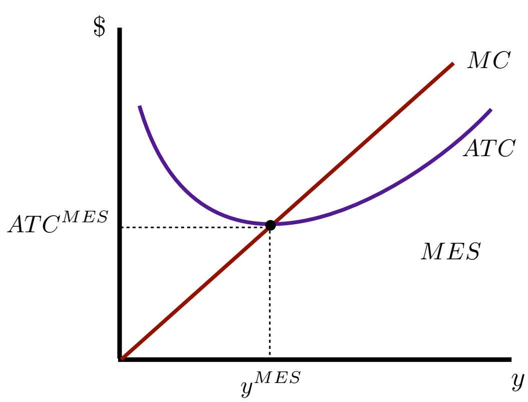
(e) Average total cost is $ATC(y) = AVC(y) + AFC(y) = 4y + \frac{4}{y}$, so $ATC(1) = 8$, $ATC(2) = 10$, and $ATC(4) = 17$, as shown below:



For smaller levels of output y, the average variable cost is negligible and hence ATC is dominated by the high AFC. When production is large, ATC becomes large due to the high AVC for large y.

(f) The minimum efficient scale occurs at the output y for which ATC obtains its minimum (i.e., the y such that $ATC'(y) = 0$). Here $ATC'(y) = 0 \implies -\frac{4}{y^2} + 4 = 0 \implies y = 1$ ($y = -1$ also solves this equation, but we cannot have negative production!). So $y^{MES} = 1$ and $ATC^{MES} = ATC(y^{MES}) = 8$.

(g) We know that $MC(y) = TC'(y)$ so $MC(y) = 8y$. This is shown below along with ATC. Notice that they intersect at $y^{MES} = 1$.



(h) Why, intuitively, the MC curve intersects ATC at its minimum, y^{MES}: For output $y < y^{MES}$, $MC(y)$ is below $ATC(y)$, so producing an additional unit is cheaper than the average cost per unit, which brings that average down. This means $ATC(y)$ is decreasing to the left of y^{MES}. For output $y > y^{MES}$, $MC(y)$ is above $ATC(y)$, so producing an additional unit is more costly than the average cost per unit, which brings that average up. This means $ATC(y)$ is increasing to the right of y^{MES}.

Since $ATC(y)$ is decreasing to the left of y^{MES} and increasing to the right of y^{MES}, the minimum of $ATC(y)$, which is ATC^{MES}, must be at y^{MES}.

(i) If the fixed cost is F, $ATC(y) = VC(y) + FC = 4y + \frac{F}{y}$. Then y^{MES} solves $ATC'(y) = 0$ so $y^{MES} = \frac{1}{2}\sqrt{F}$ and $ATC(y^{MES}) = 4\sqrt{F}$. For higher fixed costs, the minimum efficient scale increases.

Problem 5 (Supply Curve of GMC)

(a) With $TC(y) = 4y^2 + 4$, condition $p = MC(y)$ (the first-order condition for profit maximization) becomes

$$p = 8y \implies y = \frac{1}{8}p\,.$$

- For $p = 4$, $y = \frac{1}{8} \times 4 = \frac{1}{2}$, giving profit

  $$\pi = py - TC(y) = 4 \times \frac{1}{2} - \left(4\left(\frac{1}{2}\right)^2 + 4\right) < 0 \implies \text{should produce } y = 0 \text{ instead,}$$

  resulting in profit $\pi = 0$.
- For $p = 8$, $y = \frac{1}{8} \times 8 = 1$, resulting in profit

  $$\pi = py - TC(y) = 8 \times 1 - \left(4\,(1)^2 + 4\right) = 0\,,$$

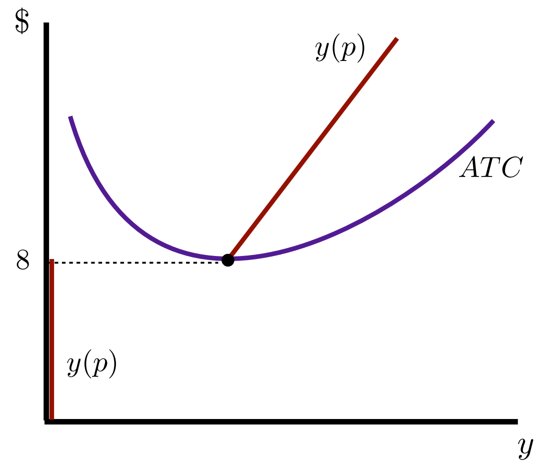
  so $y = 1$ is produced with $\pi = 0$.
- For $p = 16$, $y = \frac{1}{8} \times 16 = 2$, giving profit

  $$\pi = py - TC(y) = 16 \times 2 - \left(4\,(2)^2 + 4\right) = 12\,.$$

(b) The supply function $y(p)$ will correspond with the marginal cost curve for all prices that give positive profits, and will be $y(p) = 0$ for all prices yielding strictly negative profits. Positive profits are associated with all p above $ATC(y^{MES}) = ATC(1) = 8$. Or, alternatively prices such that profits are positive: $py - (4y^2 + 4) \geq 0 \implies p\left(\frac{1}{8}p\right) - (4\left(\frac{1}{8}p\right)^2 + 4) \geq 0$ (using $p = MC$ condition) $\implies p \geq 8$. Negative profits are associated with $p < 8$ and hence the optimal choice is $y = 0$ for those prices. Our supply function is then:

$$y(p) = \begin{cases} 0 & \text{for } p < 8 \\ \frac{1}{8}p & \text{for } p \geq 8 \end{cases}$$

(c) The supply curve and ATC are shown below:



(d) For a new fixed cost $F = 1$, the slope of the supply curve will not change, however the price at which the firm will choose not to produce ($y = 0$) does change.

In part **(b)**, we found that the firm will only produce when $p > ATC^{MES} = 8$. Now, we need to find the MES with the new $F = 1$ (at which point $MC = ATC$):

$$MC(y) = ATC(y) \implies 8y = \frac{1}{y} + 4y \implies y = \frac{1}{2}, \text{ then}$$

$$y^{MES} = \frac{1}{2} \text{ and so } ATC^{MES} = ATC(y^{MES}) = \frac{1}{1/2} + 4\left(\frac{1}{2}\right) = 4\,.$$

This gives us the following piecewise defined supply function:

$$y(p) = \begin{cases} 0 & \text{for } p < 4 \\ \frac{1}{8}p & \text{for } p \geq 4 \end{cases}$$