Problem Set 3: Solutions

ECON 301: Intermediate Microeconomics
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Problem 1 (Cobb-Douglas Utility Functions)

1.1: Optimal fraction of income spent on (berries) x_2: $\frac{b}{a+b}$. Optimal fraction of income spent on (nuts) x_1: $\frac{a}{a+b}$. (The problem only asks for berries.) *Notice how neither fraction depends on income m or the prices of the two goods, p_1 and p_2. This is always true for Cobb-Douglas utility but not true for all types of utility functions.*

1.2: From the answer above we know that optimal total dollars spent on berries, p_2x_2, will be equal to $\frac{b}{a+b}$ of income m, so this gives us the equation $p_2x_2 = \frac{b}{a+b}m$.

Using the same reasoning (again, the problem does not ask for this), the total amount spent on nuts, x_1, will be $p_1x_1 = \frac{a}{a+b}m$.

1.3: The quantity of nuts consumed will be $x_1 = \frac{a}{a+b} \cdot \frac{m}{p_1}$ (and the optimal quantity of berries consumed will be $x_2 = \frac{b}{a+b} \cdot \frac{m}{p_2}$. We can see this from **(1.2)**: Just divide both sides of those equations by p_1 to find the optimal quantities (i.e., solve for x_1 and x_2).

1.4: For these problems, we first find the optimal bundle (using the equations in **1.3**) and plug these quantities into $MRS(x_1, x_2)$, which gives the slope of the indifference curve at (x_1, x_2). The MRS for Cobb-Douglas utility functions is $MRS(x_1, x_2) = -\frac{ax_2}{bx_1}$ (you can verify this).

- **(a)** Given $a = 4$, $b = 8$, $p_1 = 5$, $p_2 = 10$, and $m = 60$:
  - Optimal quantity of x_1 is $x_1 = \frac{a}{a+b} \cdot \frac{m}{p_1} = \frac{4}{4+8} \cdot \frac{60}{5} = 4$
  - Optimal quantity of x_2 is $x_2 = \frac{b}{a+b} \cdot \frac{m}{p_1} = \frac{8}{4+8} \cdot \frac{60}{10} = 4$
  - Slope of the indifference curve given by $MRS(x_1, x_2) = -\frac{ax_2}{bx_1} = -\frac{4 \cdot 4}{8 \cdot 4} = -\frac{1}{2}$
- **(b)** Given $a = \frac{1}{3}$, $b = \frac{1}{3}$, $p_1 = 4$, $p_2 = 1$, and $m = 12$:
  - Optimal quantity of x_1 is $x_1 = \frac{a}{a+b} \cdot \frac{m}{p_1} = \frac{1/3}{1/3+1/3} \cdot \frac{12}{4} = \frac{3}{2}$
  - Optimal quantity of x_2 is $x_2 = \frac{b}{a+b} \cdot \frac{m}{p_1} = \frac{1/3}{1/3+1/3} \cdot \frac{12}{1} = 6$
  - Slope of the indifference curve given by $MRS(x_1, x_2) = -\frac{ax_2}{bx_1} = -\frac{\frac{1}{3} \cdot 6}{\frac{1}{3} \cdot \frac{3}{2}} = -4$
- **(c)** Given $a = \frac{1}{2}$, $b = \frac{3}{2}$, $p_1 = 5$, $p_2 = 1$, and $m = 20$:
  - Optimal quantity of x_1 is $x_1 = \frac{a}{a+b} \cdot \frac{m}{p_1} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{3}{2}} \cdot \frac{20}{5} = 1$

– Optimal quantity of x_2 is $x_2 = \frac{b}{a+b} \cdot \frac{m}{p_1} = \frac{\frac{3}{2}}{\frac{1}{2}+\frac{3}{2}} \cdot \frac{20}{1} = 15$

– Slope of the indifference curve given by $MRS(x_1, x_2) = -\frac{ax_2}{bx_1} = -\frac{\frac{1}{2}\cdot 15}{\frac{3}{2}\cdot 1} = -5$

Problem 2 (Well-Behaved Preferences)

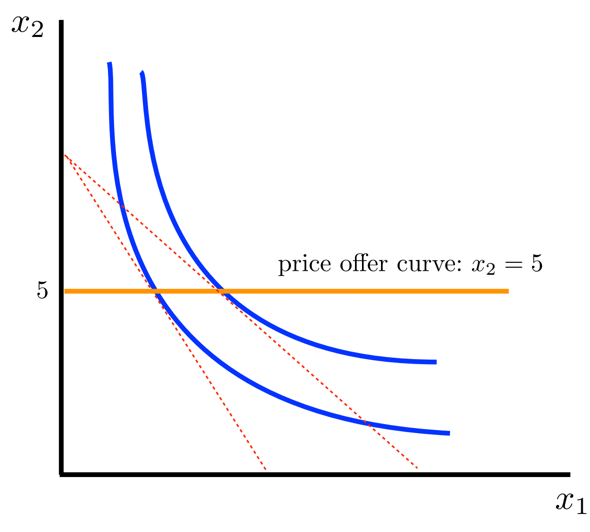
(a) The two secrets of happiness for well-behaved preferences such as these, which we'll use to derive demand, are:

- **(1)** $p_1x_1 + p_2x_2 = m$ (Benjamin spends all of his income, choosing a bundle on the budget line.)
- **(2)** $MRS(x_1, x_2) = -\frac{p_1}{p_2}$ (He chooses the bundle where the budget line is tangent to the indifference curve furthest from the origin.)

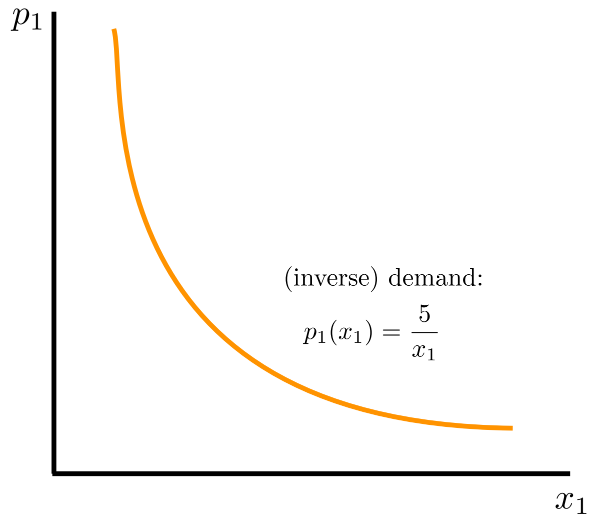
For this utility function, we get that $MRS(x_1, x_2) = -\frac{MU_1}{MU_2} = -\frac{x_2}{x_1}$, for condition **(2)** we have $-\frac{x_2}{x_1} = -\frac{p_1}{p_2}$. Solving these two equations for x_1 and x_2 as we've done before, we get

$$x_1 = \frac{m}{2p_1} \quad \text{and} \quad x_2 = \frac{m}{2p_2}.$$

(b) To find the price-offer curve (which lies in our x_1-x_2 commodity space) with $p_2 = 1$ and $m = 10$, we plug these numbers into demand $x_2 = \frac{m}{2p_2} \implies x_2 = \frac{10}{2\cdot 1} = 5$. So, regardless of the price of x_1, with $p_2 = 1$ and $m = 10$, 5 units of x_2 (MP3s) are always purchased. Plotting this out in the commodity space:

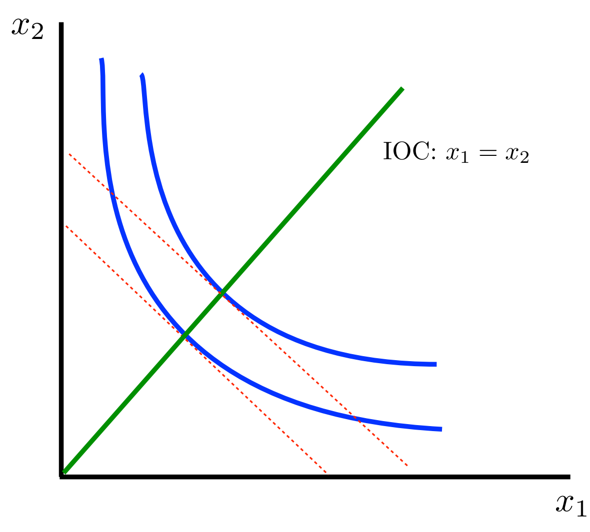


We found the demand function $x_1 = \frac{m}{2p_1}$ above, and with $m = 10$ we have $x_1 = \frac{10}{2p_1}$ (or, equivalently $x_1(p_1) = \frac{m}{2p_1}$). To plot the demand in the p_1-x_1 space, we use the *inverse* of this demand function: $p_1(x_1) = \frac{10}{2\cdot x_1} = \frac{5}{x_1}$.



(c) Looking at the demand function for x_1, we can see that as p_1 increases (decreases), the amount of x_1 demanded decreases (increases), so x_1 is an ordinary good. This is consistent with the fact that the inverse demand function we graphed above is downward sloping.

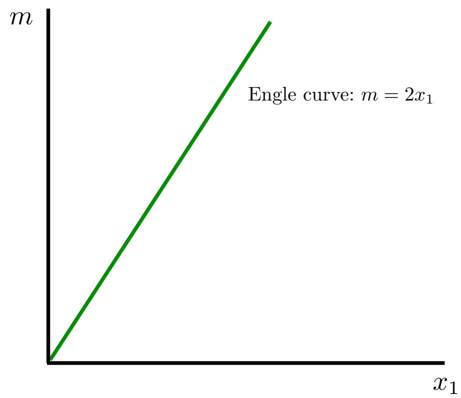
(d) Fixing $p_1 = 1$ and $p_2 = 1$, the income offer curve shows us the optimal bundles in the commodity space for different levels of m. Graphically, we see where the tangencies are (since these are well-behaved preferences) between the indifference curves and budget lines associated with different levels of income.

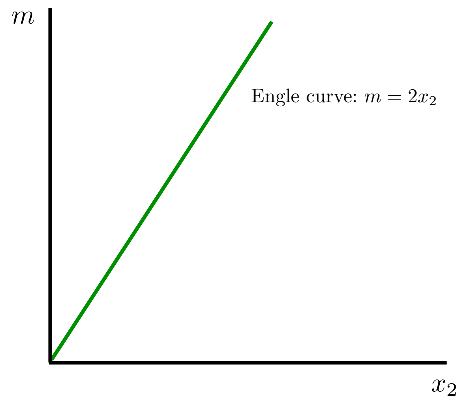


Analytically, for the income offer curve we turn to our demand functions, which with these prices turn into $x_1 = \frac{1}{2}m$ and $x_2 = \frac{1}{2}m$. Solving these two equations (setting $m = 2x_1$ for the first equation and plugging it into the second), we get that our optimal bundles as income changes will lie along the line $x_2 = x_1$.

The Engel curve for each good, movies x_1 and MP3s x_2, plots income m against quantity demanded, holding the price of both goods fixed. Here we can again use the demand curves

$x_1 = \frac{1}{2}m$ and $x_2 = \frac{1}{2}m$. Similar to when we're graphing demand, we solve for what's on the vertical axis (here, m) so we graph $m = 2x_1$ and $m = 2x_2$:





The two commodities are normal rather than inferior since, as we can see in the demand functions analytically and the Engel curve graphically, as income increases, the quantity demanded for both goods increases.

(e) Because the demand function for x_1 does not include p_2, this means a change in the price of MP3s (x_2) has *no* effect on the demand for movies x_1. (Also, the price of movies p_1 does not affect demand for MP3s x_2.) This means the goods are neither gross complements nor gross substitutes.

Problem 3 (Perfect Complements)

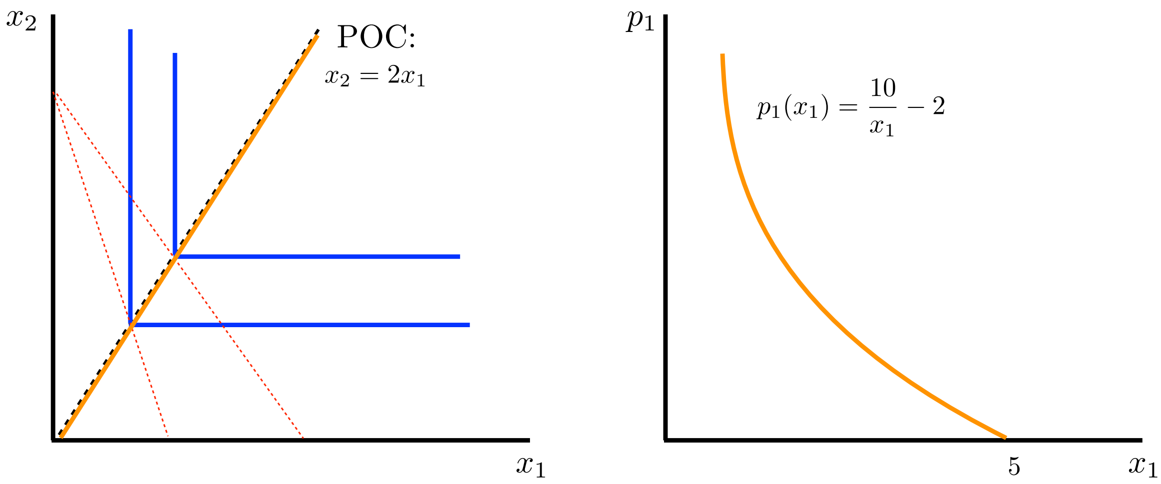
In Problem Set 2 (in **Problem 3(e)**), we found that Trevor's preferences can be represented by utility function $U(x_1, x_2) = \min\{2x_1, x_2\}$
(a) To find Trevor's demand for milk (x_1) and strawberries (x_2), we use the two secrets of happiness *for complementary goods*:

- **(1)** $p_1x_1 + p_2x_2 = m$ (Trevor spends all of his income.)
- **(2)** $2x_1 = x_2$ (He chooses a bundle along the optimal proportion line.)

Solving these equations for x_1 and x_2 we get

$$x_1 = \frac{m}{p_1 + 2p_2} \quad \text{and} \quad x_2 = \frac{2m}{p_1 + 2p_2}.$$

(b) For $p_2 = 1$ and $m = 10$, we have the price-offer curve $x_2 = 2x_1$. (Shown below on the left.) Demand for x_1 is $x_1 = \frac{10}{p_1+2}$. We solve this for p_1 to get the inverse demand curve to graph: $p_1 = \frac{10}{x_1} - 2$. (Shown below on the right.)



(c) Milk, x_1 is an ordinary good for Trevor. Looking at the demand function, we see that as p_1 increases (decreases), quantity demanded x_1 decreases (increases).

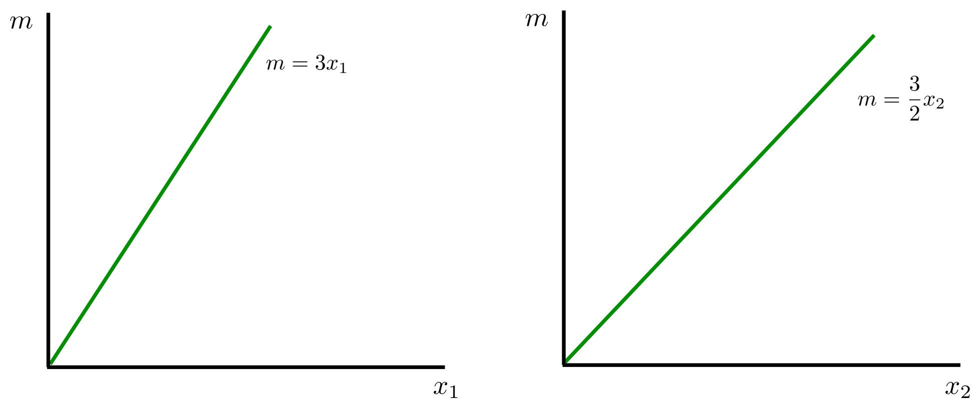
(d) Given $p_1 = p_2 = 1$, the optimal choices given m (which define the Engel curve) are given by

$$x_1 = \frac{m}{p_1 + 2p_2} = \frac{m}{3}$$

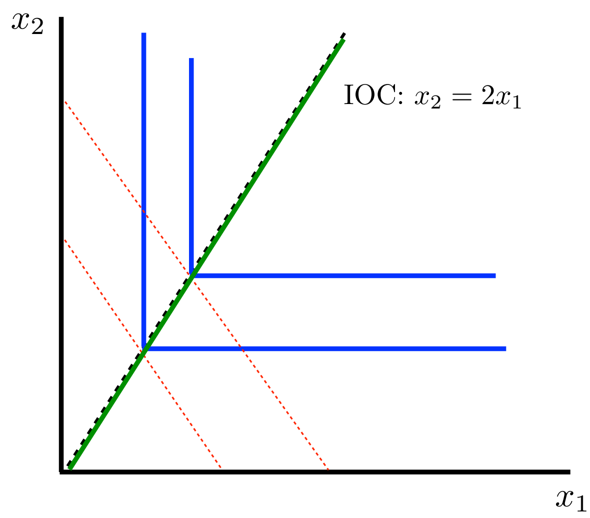
and

$$x_2 = \frac{2m}{p_1 + 2p_2} = \frac{2m}{3}$$

We graph the Engel curves for each good in x_1-m and x_2-m space:



The income-offer curve coincides with the optimal proportion line (solve each of the demand functions to m, set them equal to each other and solve for x_2).



(e) Since $\frac{\Delta x_1}{\Delta p_2} < 0$ (as p_2 goes up, demand for x_1 goes down) and $\frac{\Delta x_2}{\Delta p_2} < 0$ (as p_1 goes up, demand for x_2 goes down), x_1 and x_2 are gross complements.

Problem 4 (Perfect Substitutes)

(a) Our demand functions for x_1 and x_2 will be depend on what the price ratio is relative to the MRS (the slope of the indifference curves, which is constant for perfect substitutes like these).

Here, $MRS(x_1, x_2) = -\frac{MU_1}{MU_2} = -\frac{2}{1}$. If the budget line is less steep than the slope of the indifference curves, i.e., if $|-\frac{p_1}{p_2}| < |MRS(x_1, x_2)|$, then Kate consumes only x_1 (Red Delicious). Otherwise, if if the budget line is more steep than the indifference curves $|-\frac{p_1}{p_2}| > |MRS(x_1, x_2)|$, Kate consumes only Jonagolds x_2, as shown graphically in Problem Set 2.

So we have that

$$\left|-\frac{p_1}{p_2}\right| < |MRS(x_1, x_2)| \quad \Longrightarrow \quad x_1 = \frac{m}{p_1},\ x_2 = 0$$

and

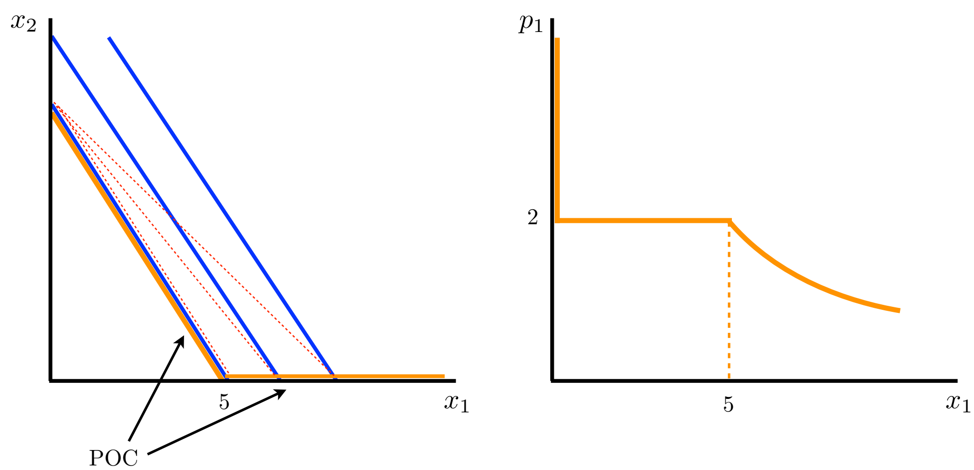
$$\left|-\frac{p_1}{p_2}\right| > |MRS(x_1, x_2)| \quad \Longrightarrow \quad x_1 = 0,\ x_2 = \frac{m}{p_2}\ .$$

(b) With $m = 10$ and $p_2 = 1$, we have $|-\frac{p_1}{p_2}| = |-\frac{p_1}{1}| = p_1$. Since $|MRS(x_1, x_2)| = |-2| = 2$,

- For $p_1 > 2$, demand for x_1=0 and $x_2 = \frac{m}{p_2} = \frac{10}{1} = 10$.

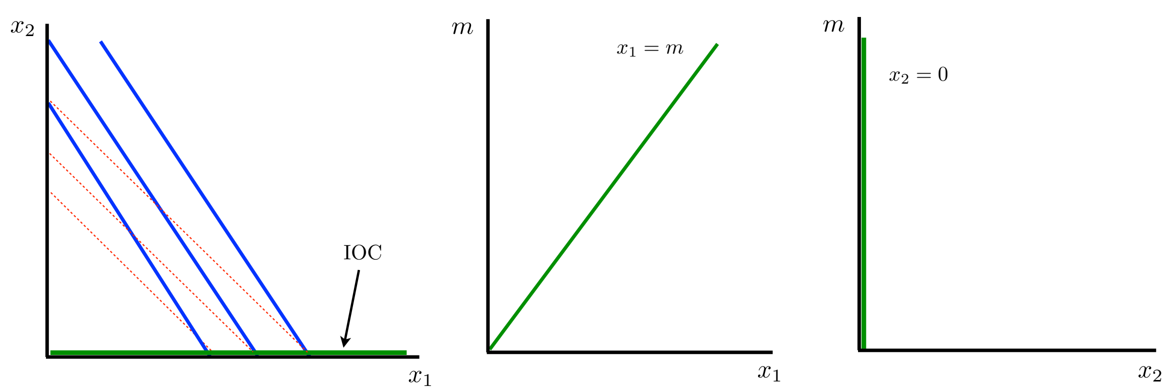
- For $p_1 < 2$, demand for $x_1 = \frac{m}{p_1}$ and $x_2 = 0$.
- For $p_1 = 2$, $MRS = -\frac{p_1}{p_2}$ everywhere, so any bundle along the budget line is optimal.

Tracing out all these optimal bundles in the commodity space as p_1 varies, we get the price-offer curve shown below one the left, and the demand for Red Delicious apples x_1 is on the right:



(c) Since demand for x_1 does not increase as its price p_1 increases, x_1 is an ordinary good.

(d) Fixing $p_1 = 1$ and $p_2 = 1$, for this price ratio we know that Kate will always choose $x_1 = \frac{m}{p_1} = m$ Red Delicious apples and $x_2 = 0$ Jonagold apples. We get the following income-offer curve (left) and Engel curves for x_1 and x_2 (two on the right):



(e) The two goods are gross substitutes since demand for x_1 does increase as p_2 increases. This happens discontinuously: Suppose p_1 and p_2 are such that $|-\frac{p_1}{p_2}| > |MRS(x_1, x_2)|$, then $x_1 = 0,\ x_2 = \frac{m}{p_2}$. As soon as p_2 increases enough for $|-\frac{p_1}{p_2}| < |MRS(x_1, x_2)|$, $x_1 = \frac{m}{p_1},\ x_2 = 0$ (so x_1 went up from $x_1 = 0$ to $\frac{m}{p_1}$).

Problem 5 (Quasilinear Preferences)

For $U(x_1, x_2) = x_1 + 10x_2 - \frac{1}{2}x_2^2$, we have that $MRS(x_1, x_2) = -\frac{1}{10-x2}$

(a) Using the two secrets of happiness and $p_1 = 1$, $p_2 = 2$, and $m = 10$:

- **(1)** $p_1x_1 + p_2x_2 = m \implies x_1 + 2x_2 = 10$
- **(2)** $MRS(x_1, x_2) = -\frac{p_1}{p_2} \implies -\frac{1}{10-x_1} = -\frac{1}{2}$

If we solve for this we get $x_1 = -6$ and $x_2 = 8$. But since consumption must be non-negative, we must have that $x_1 = 0$ and $x_2 = \frac{m}{p_2} = 5$ (a corner solution).

(b) Using the same two conditions with $m = 20$, we get interior solution $x_1 = 4$ and $x_2 = 8$.

(c) In part **(a)** marginal utility per dollar for stamps (x_1) is $\frac{MU_1}{p_1} = \frac{1}{1} = 1$ and for clothes (x_2) is $\frac{MU_2}{p_2} = \frac{10-(5)}{2} = 2.5$. At this corner solution, marginal utility per dollar is not equalized across goods.

In part **(b)**, marginal utility per dollar for stamps (x_1) is $\frac{MU_1}{p_1} = \frac{1}{1} = 1$ and for clothes (x_2) is $\frac{MU_2}{p_2} = \frac{10-(8)}{2} = 1$. At this interior solution, marginal utility per dollar is equalized across goods.