Nonstationary Search*

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Abstract

This paper explores optimal matching policies in constant returns to scale search economies with heterogeneous agents. We look for a policy that maximizes the present value of output in the economy, taking the search frictions as given. Our main result is that if agents' characteristics are complements in production, an optimal policy may be nonstationary, with a nontrivial asymptotic limit cycle or possibly chaotic behavior, even though the model has no extrinsic uncertainty. This finding holds for a generic set of parameter values. It is due to a trading externality (Diamond 1982) that is inherent in heterogeneous agent economies. The same forces also generate a continuum of perfect foresight equilibria.

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1 Introduction

Consider the following canonical economic situation: a pair of agents meet and must decide whether to match. Matching gives an immediate payoff, but this is weighed against an opportunity cost, as matching precludes further search for more suitable partners. Examples include the decision of a worker and firm to enter into an employment relationship, the decision of a man and a woman to marry, and the decision of a home buyer and home seller to sign a sales contract.

We model this using the textbook search and matching framework (Pissarides 2000), extended to allow for *ex ante* heterogeneous agents. Agents may be unmatched and searching or matched in pairs and producing. Unmatched agents meet each other at a Poisson rate and must decide whether to match. Once they are matched, they may break up at will, returning to the searching population.

It is well-known that self-fulfilling expectations can generate multiple equilibria in this type of environment (Sattinger 1995, Burdett and Coles 1997). The reason is that matching decisions depend on expectations of the distribution of searching agents, which they in turn determine. This paper sidesteps this multiplicity by asking a related question: how *should* agents match in order to maximize the present value of output, or equivalently, what matching patterns are socially optimal? This matching pattern would be preferred, for example, by a risk-neutral agent who was dropped into the economy under a veil of ignorance. We model this as the solution to the problem of a hypothetical social planner who cannot avoid the search frictions but who can decide whether any two agents who meet should match and whether any two matched agents should break up.

Our main result is that the optimum is nonstationary in an open set of models, in all cases with agents' characteristics complements in the production function. The hypothetical social planner always has the opportunity to pursue a steady state, and yet she will often find that the present discounted value of output is higher if matching decisions are forever changing over time. The time-varying matching decisions induce time-varying matching opportunities. The social planner can profit from this because of an inherent nonconvexity in the economy.

Although it is *not* due to increasing returns to scale in the meeting function (Diamond 1982), the nonconvexity can most easily be understood by considering a related model with increasing returns. Assume all agents are *ex ante* identical, although a particular partnership has an idiosyncratic, stochastic productivity. Unmatched agents search and the meeting rate for each agent is increasing in the number of unmatched agents. Then optimal matching patterns may dictate that the threshold for accepting a match today is an increasing function of anticipated future unmatched rates. If unmatched rates are expected to be higher, it will be easier to form future matches, making low productivity matches today less desirable. Indeed, the economy can cycle through periods with a declining unmatched rate and a declining productivity threshold, ended by the sudden destruction of less productive matches, a surge in the unmatched rate, and thus a sharp increase in the productivity threshold. Moreover, increasing returns to scale implies that for a given mean unmatched rate, greater time-variation in the unmatched rate implies more matches on average, hence more opportunities to form high productivity matches, potentially justifying the time-varying matching patterns.

The same logic works in our economy with a constant returns to scale meeting function and heterogeneous agents. Now, however, it is not time-varying unmatched rates that induce optimal cycles, but rather time-variation in the composition of the searching population. Assume there are two types of agents, high and low productivity. Agents' characteristics are complements in production, so it is desirable to avoid 'mixed' matches between one high productivity and one low productivity agent. If it were possible, the social planner would create separate markets for each type of agent, for example by having them search in geographically distinct locations, thereby avoiding mixed meetings completely. But this is assumed to be impossible. Instead the planner takes advantage of the fact that nonstationary matching patterns generate partially segregated markets using time rather than space. The random meeting assumption implies that the fraction of meetings that are between one high productivity agent and one low productivity agent is $2\gamma(1-\gamma)$, where γ is the share of high productivity agents in the searching population. Since this is a concave function of γ, time-variation in γ can reduce the prevalence of these undesirable mixed matches.

Now suppose there are unequal numbers of high and low productivity agents in the economy. To be concrete, assume high productivity agents are scarcer. This means that the economy can follow a simple cycle: initially the share of high productivity agents γ is large — close to $\frac{1}{2}$ — and so high productivity agents wait until they meet another high productivity agent before matching. The share γ gradually declines as they find matches, and so eventually it is optimal to allow mixed matches. The economy then gradually builds up a stock of high productivity agents in mixed matches. At some point, these are destroyed, unleashing equal numbers of high and low productivity agents, and sending the share γ back up close to $\frac{1}{2}$. The economy thereby generates time variation in γ, which reduces the number of inefficient mixed meetings relative to the number that would occur in an economy with the same average unmatched rate but a constant value of γ.

This logic suggests that if mixed matches are desirable, as would be the case if characteristics are substitutes in production, geographically segregated markets, and hence time variation in γ, would be counterproductive. Indeed, Proposition 2 shows that in this case there is a steady state social optimum. In particular, any steady state solution to the first order conditions of the social planner's problem also satisfies the Arrow-Mangasarian second order sufficient conditions.

On the other hand, we show that the second order sufficient conditions fail if characteristics are complements. Our strategy is instead to develop a set of second order necessary conditions for optimal control problems, and to show that any steady state solution to the first order conditions fails the second order necessary conditions. Section 4 analyzes the simplest version of this model with only two types of agents, high and low, and with low-low matches so unproductive that it is optimal not to form them even if high productivity agents do not match with low productivity agents. We use the first and second order necessary conditions to prove that for an open set of parameter values, optimal matching patterns are nonstationary (Proposition 3). We also use this example to clarify the role of congestion in generating nonstationary matching patterns.

Unfortunately, this example requires a lot of heterogeneity. Low productivity agents must produce much less while matched together in a pair than while unmatched, while high productivity agents enjoy substantial gains from matching together. Were it feasible, output would increase if the low productivity stopped searching. Therefore, Section 5 develops an example of an economy with a more reasonable amount of heterogeneity and with substantial cycles generated by the same mechanism. More precisely, we prove numerically that there are nonstationary paths that dominate any steady state solution. We also argue that the social optimum may exhibit chaotic behavior, although a proof of that possibility eludes us. Although the logic is the same in our two examples, the first proof is analytical while the second proof is numerical.

Section 6 reconsiders the earlier observation that this model can exhibit multiple steady state equilibria due to self-fulfilling expectations. The same forces that generate nonstationary social optima imply the existence of a continuum of perfect foresight equilibria with deterministic cycles or chaos. Allowing for sunspot variables further expands the dimension of the set of equilibria.

Almost all existing work on search and matching models has focused on steady state behavior, vastly reducing the number of equilibria in heterogeneous agent economies. Two earlier papers have considered nonstationary equilibria. Diamond and Fudenberg (1989) find that cycles may occur in search economies with increasing returns to scale. Burdett and Coles (1998) characterize a nonstationary equilibrium in a constant returns to scale, heterogeneous agent economy similar to ours,[1] but do not stress the possibility that a continuum of similar equilibria exist in their model. Previous attempts to characterize socially optimal matching behavior have assumed that the social optimum is stationary, an unjustified assumption.

Section 2 lays out a standard model of search with heterogeneous agents. Section 3 is divided into four parts. First, we develop and provide a concise set of first order conditions for a social optimum. Then we characterize the possible steady state optima when there are two types of agents. If the production function is supermodular, so that two mixed matches produce less than one match between two high productivity agents and another between two low productivity agents, we show that the key question is whether 'mixed' matches are formed. In the opposite case of a submodular production function, the question is whether 'good' and 'bad' matches are formed. Next we look at Arrow-Mangasarian second order sufficient conditions for a steady state optimum. Here we prove that if the production function is submodular, the equilibrium is stationary. But with a supermodular production function, the more common assumption, the sufficient conditions do not have any bite. The last part of this section develops a set of second order necessary conditions, based on the analysis in Colonius (1988). We apply these in the analytically tractable case in Section 4. They also provide some useful intuition for the numerical analysis in Section 5. Section 6 examines the set of decentralized equilibria, showing the close relationship between the centralized and decentralized problems. The important difference is that a social planner can avoid any problems associated with self-fulfilling expectations, while these generate a coordination game

[1] A notable difference is that they assume utility is nontransferable. Our conclusions nevertheless carry over to that environment.

in the decentralized economy, the source of the large multiplicity of equilibria.

2 Model

We develop a continuous time model in which a measure 1 continuum of agents search for partners. All agents are risk-neutral, infinitely-lived, and discount the future at rate $r > 0$. Agents are identified by their type $i \in \{1, \ldots, N\}$,[2] with a fraction $\ell_i > 0$ of agents being of type i. At any point in time t, each agent may be either matched or unmatched. Let $m_{ij}(t)$ denote the measure of type i agents matched with type j agents at time t. This is also the measure of (i, j) matches for $i \neq j$, but $m_{ii}(t)$ is *twice* the measure of (i, i) matches. Then $u_i(t) = \ell_i - \sum_{j=1}^{N} m_{ij}(t)$ denotes the measure of unmatched type i agents.

Every unmatched agent searches, running into another unmatched agent at an exogenous flow rate ρ. The meeting function is anonymous, as the potential partner is drawn randomly from the unmatched population. Let $\gamma_j(t) \equiv u_j(t)/\sum_{k=1}^{N} u_k(t)$ denote the share of type j agents in this population. Then the flow rate at which any unmatched agent i contacts an unmatched type j agent is $\rho\gamma_j(t)$. Note that the aggregate meeting rate depends linearly on the aggregate unmatched rates, with meetings at a flow rate $\rho \sum_{k=1}^{N} u_k(t)$. Thus this is a linear search technology.

When two agents i and j meet, the hypothetical social planner must decide whether they should match. If they do so, they produce a flow output $f_{ij} \equiv f_{ji}$, the only extrinsic source of heterogeneity in this model. This may be viewed without loss of generality as the difference between the amount that i and j can produce while matched and the amount they can produce while unmatched. To make the problem nontrivial, we assume that $f_{ij} > 0$ for some (i, j), but do not impose any other nonnegativity constraints on f. The social planner's objective is to maximize the present value of output:

$$(1) \qquad V(t) = \max \int_t^{\infty} e^{-r(s-t)} \sum_{i=1}^{N} \sum_{j=1}^{N} \tfrac{1}{2} f_{ij} m_{ij}(s) ds.$$

The flow of output at time s is equal to the measure of type i agents matched with type j agents, $m_{ij}(s)$, multiplied by half the output produced in the match, and

[2]Throughout much of the paper, we deal with the case of $N = 2$, since this provides a sufficiently general environment for us to make our main point.

summed across i and j. The $\frac{1}{2}$ avoids double-counting output.

Matched agents forego the opportunity to search, and so never meet any unmatched agents. All matches are exogenously terminated according to a Poisson process with arrival rate $\delta > 0$ and, additionally, the social planner may at any time decide to terminate a match. After either an endogenous or exogenous termination, the pair starts off again unmatched. This implies that the social planner faces a constraint on the evolution of the state variables:

$$dm_{ij}(t) \leq \left(\rho \frac{u_i(t)u_j(t)}{\sum_{k=1}^{N} u_k(t)} - \delta m_{ij}(t) \right) dt \quad \text{and} \quad m_{ij}(t) \geq 0, \tag{2}$$

in addition to an initial condition $m_{ij}(0) = \bar{m}_{ij}$. If the social planner permits all feasible (i, j) matches during any length dt interval of time, inequality (2) will bind. The first term in parenthesis gives the creation rate of new matches, while the second term gives the exogenous destruction rate. But this is only a cap on the growth of $m_{ij}(t)$, since the planner may destroy any existing matches at will. Inequality (2) provides a direct link between current matching decisions and the evolution of the state of the economy, hence the evolution of future matching opportunities.

3 Social Optimum

3.1 First Order Conditions

In this section, we find first order necessary conditions for a social optimum, where

Definition 1. *A social optimum is a sequence of matching rates $\{m_{ij}(t)\}$ that maximizes (1) subject to (2)*

The possibility of discontinuities in the state variable complicates the analysis, but the problem can still be solved using optimal control theory. Here we present the necessary first order conditions for a social optimum and provide some intuition for the results. Appendix A offers a precise derivation of these results using a trick developed by Vind (1967) and Arrow and Kurz (1970), and described in Kamien and Schwartz (1991), pages 240–247.

We begin by writing down a current-valued Hamiltonian with a current-valued costate variable $S_{ij}(t) \equiv S_{ji}(t)$ on the binding version of constraint (2):

$$H(m(t), S(t)) = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\tfrac{1}{2}f_{ij}m_{ij}(t) + S_{ij}(t)\left(\rho\frac{u_i(t)u_j(t)}{\sum_{k=1}^{N}u_k(t)} - \delta m_{ij}(t)\right)\right)$$

There are five types of first order necessary conditions for this problem. First, since $S_{ij}(t)$ represents the shadow value of an (i, j) match, and matches can costlessly be destroyed, the shadow value must be nonnegative. Second, the state variable increases at the fastest possible rate when its shadow value is positive, but otherwise may fall discontinuously:

$$\begin{aligned} &S_{ij}(t) > 0 \Rightarrow \dot{m}_{ij}(t) = \rho\frac{u_i(t)u_j(t)}{\sum_{k=1}^{N}u_k(t)} - \delta m_{ij}(t) \\ &S_{ij}(t) = 0 \Rightarrow 0 \leq m_{ij}(t) \leq m_{ij}(t-) \end{aligned} \tag{3}$$

These conditions are peculiar to environments with discontinuous state variables, and so we derive them carefully in Appendix A.

Condition (3) tells us that the costate variable S is the key to understanding the dynamics of the matching rates. To characterize the dynamics of the costate variable, we turn to the costate equation, which equates the partial derivative of the Hamiltonian with respect to the state variable $m_{ij}(t)$ with the flow value of the costate variable $rS_{ij}(t) - \dot{S}_{ij}(t)$, as in a standard control problem. Noting that the state variable appears in the unmatched rates $u_i(t)$ and $u_j(t)$, we obtain:

$$(r+\delta)S_{ij}(t) - \dot{S}_{ij}(t) = \frac{f_{ij} - v_i(t) - v_j(t)}{2} \tag{4}$$

where $v_i(t)$ is the flow value of an unmatched agent,

$$v_i(t) = \rho\sum_{k=1}^{N}\gamma_k(t)\left(S_{ik}(t) + \sum_{l=1}^{N}\gamma_l(t)(S_{ik}(t) - S_{lk}(t))\right). \tag{5}$$

The flow value is the sum of two terms. First, by searching, an unmatched agent i may meet a type k agent, drawn randomly from the population, and create a new match with value S_{ik}. Second, when she does this, she forces the type k agent to meet her, yielding value S_{ik}, rather than the expected value $\sum_l \gamma_l(t)S_{lk}(t)$ that k would generate from meeting l instead of i. The social planner internalizes this externality, a theme that Shimer and Smith (2001) explore.

When $S_{ij}(t)$ is positive, we can integrate $S_{ij}(t)$ in equation (4) forward to some future date $T > t$ with $S_{ij}(s) > 0$ for all $s \in [t, T)$:

$$S_{ij}(t) = \int_t^T e^{-(r+\delta)(s-t)} \frac{f_{ij} - v_i(s) - v_j(s)}{2} ds + e^{-(r+\delta)(T-t)} S_{ij}(T)$$

Since $S_{ij}(T)$ is nonnegative, the second term on the right hand side is minimized, hence the first term is maximized, by choosing a $T \geq t$ such that $S_{ij}(T) = 0$, if one exists. Otherwise, the transversality condition, the fourth necessary condition,

$$\lim_{T \to \infty} e^{-rT} m_{ij}(T) S_{ij}(T) = 0$$

ensures that the first term is maximized by taking the limit as $T \to \infty$. Either way, we may express $S_{ij}(t)$ more concisely as

$$S_{ij}(t) = \max_{T \in [t, \infty]} \int_t^T e^{-(r+\delta)(s-t)} \frac{f_{ij} - v_i(s) - v_j(s)}{2} ds \tag{6}$$

This, together with the definition of v in (5), provides a forward-looking equation for the match surplus $S_{ij}(t)$. The current surplus in an (i, j) match depends on the future composition of the unmatched population, summarized by the future shares $\gamma_k(s)$, $s \geq t$. Together with the law of motion (2), this is the second link between current matching decisions and future matching opportunities.

The final optimality condition is continuity of the Hamiltonian over time. While following an optimal policy, the Hamiltonian is equal to the average present value of output, $rV(t)$, which clearly cannot have discontinuities. This necessary condition for an optimal control is frequently ignored. In fact, in many control problems, it is implied by the optimality of the control variables, and therefore is redundant. For suppose the Hamiltonian jumps at some time t, for concreteness a jump up between $t-$ and $t+$. If the state variables are continuous over this time interval and the objective function and state equations are continuous in the state variables, this would imply that using the $t+$ controls at $t-$ would raise the value of the Hamiltonian. But that contradicts the optimality of the $t-$ controls. In appendix A, we prove that the Hamiltonian

$$\mathcal{H}(t) = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\tfrac{1}{2}f_{ij}m_{ij}(t) + S_{ij}(t)\left(\rho\frac{u_i(t)u_j(t)}{\sum_{k=1}^{N}u_k(t)} - \delta m_{ij}(t)\right)\right) \tag{7}$$

is continuous in our model as well, even when the state variables are discontinuous, and in Appendix B we establish the equivalence between the Hamiltonian and the average present value of output in a generic control problem.

We summarize these results via a definition and and proposition:

Definition 2. *Assume the triple (v, S, m) satisfies the following four conditions: m solves the state equations (3) given initial conditions and S; v solves (5) given S and m; S solves (6) given v; and the Hamiltonian $H(t)$ in (7) is continuous at all t given S and m. Then the triple is a* candidate optimum. *If each element of the triple is time-invariant, it is a* candidate steady state optimum.

Proposition 1. *If the matching rates m are a social optimum, then there is an associated triple (v, S, m) that is a candidate social optimum.*

One thought is that a candidate steady state optimum might not exist, which would immediately imply that the social optimum is nonstationary. This is not the case. The proof in Shimer and Smith (2000) can be adapted to prove existence of a candidate steady state social optimum for any parameter values. We must therefore turn to second order conditions.

3.2 Candidate Steady State Optimum: $N = 2$

Before doing so, it is worth noting that there are some significant restrictions on the set of candidate steady state social optima. Throughout this section and the remainder of the paper, we focus attention on the case of $N = 2$. In this section, we also assume that the production function is nonnegative, $f_{ij} \geq 0$ for all (i, j). This reduces the number of cases that we must deal analyze at little loss of economic interest.

Our first result is that if the production function is supermodular, $f_{11} + f_{22} > 2f_{12}$, then 'good' $(1, 1)$ and 'bad' $(2, 2)$ matches are always formed in a steady state optimum. Suppose, to find a contradiction, that there is a steady state optimum with $S_{11} = 0$, where we suppress the time arguments on variables that are constant in steady state. With this restriction, (5) implies $v_1 = \rho\gamma_2^2(2S_{12} - S_{22})$. In addition, (6) implies $2(r+\delta)S_{12} = f_{12} - v_1 - v_2$ and $(r+\delta)S_{22} = \frac{1}{2}f_{22} - v_2$. Using the last two equations to eliminate $2S_{12} - S_{22}$ from the previous equation gives

$$v_1 = \frac{\rho\gamma_2^2}{r+\delta+\rho\gamma_2^2}\left(f_{12} - \tfrac{1}{2}f_{22}\right)$$

For S_{11} to be equal to zero in steady state, (6) implies $f_{11} \leq 2v_1$, or equivalently

$$(r+\delta)f_{11} \leq \rho\gamma_2^2(2f_{12} - f_{11} - f_{22}) < 0,$$

which contradicts the assumption that $f_{11} \geq 0$. The proof that $S_{22} > 0$ is symmetric. This means that with a nonnegative, supermodular production function, the only interesting question is whether 'mixed' $(1,2)$ matches are viable.

Conversely, if the production function is submodular, $f_{11} + f_{22} < 2f_{12}$, mixed matches are always formed in a steady state optimum. The logic is similar. In this case, if $S_{12} = 0$, (5) implies $v_1 + v_2 = 2\rho\gamma_1\gamma_2(S_{11} + S_{22})$. On the other hand, (6) yields $(r+\delta)(S_{11} + S_{22}) = \frac{1}{2}(f_{11} + f_{22}) - (v_1 + v_2)$, or

$$v_1 + v_2 = \frac{\rho\gamma_1\gamma_2}{r+\delta+2\rho\gamma_1\gamma_2}(f_{11} + f_{22})$$

Then the condition $f_{12} \leq v_1 + v_2$ implies

$$(r+\delta)f_{12} \leq \rho\gamma_1\gamma_2(f_{11} + f_{22} - 2f_{12}) < 0,$$

which contradicts the assumption that $f_{12} \geq 0$. With a submodular production function, if a steady state optimum exists, 'mixed' matches are consummated. The interesting questions are whether good and bad matches are viable.

3.3 Second Order Sufficient Conditions

Arrow and Mangasarian (see Kamien and Schwartz 1991, pages 221–226) provide the weakest available second order sufficient conditions for optimal control problems: A candidate optimum is an optimum if the Hamiltonian is concave in the state variable when evaluated with an optimal control variable. This section proves that the second order sufficient conditions hold at a candidate steady state optimum if the production function is nonnegative and submodular, $2f_{12} > f_{11} + f_{22}$. In this case, the candidate optimum is a social optimum. On the other hand, if the production function is supermodular, the second order sufficient conditions never hold.

To begin, calculate the second derivative of the Hamiltonian with respect to the vector of state variables (m_{11}, m_{12}, m_{22}):

$$2\rho\frac{S_{11}+S_{22}-2S_{12}}{(u_1+u_2)^3}\left(\begin{array}{ccc} u_2^2 & u_2(u_2-u_1) & -u_1u_2 \\ u_2(u_2-u_1) & (u_1-u_2)^2 & u_1(u_1-u_2) \\ -u_1u_2 & u_1(u_1-u_2) & u_1^2 \end{array}\right),$$

where all variables are evaluated at time t. The eigenvalues of the Hessian are 0 with multiplicity two and

$$4\rho\big(S_{11}(t)+S_{22}(t)-2S_{12}(t)\big)\frac{(u_1(t)-u_2(t))^2+u_1(t)u_2(t)}{(u_1(t)+u_2(t))^3}$$

The fraction is positive since $u_1(t)$ and $u_2(t)$ cannot fall to zero. This implies that the third eigenvalue is negative, and thus the Hamiltonian is concave in the state variables, if and only if $S_{11}(t)+S_{22}(t) < 2S_{12}(t)$, where the $S_{ij}(t)$ satisfy (5) and (6).

Now suppose that the production function is submodular, $2f_{12} > f_{11}+f_{22}$, and that f_{11} and f_{22} are nonnegative. Consider a candidate steady state optimum. If both S_{11} and S_{22} are zero, the assumption that some f_{ij} is strictly positive ensures that S_{12} is positive. On the other hand, if both S_{11} and S_{22} are positive, a steady state version of (6) implies

$$S_{11}+S_{22} = \frac{\frac{1}{2}(f_{11}+f_{22})-v_1-v_2}{r+\delta} < \frac{f_{12}-v_1-v_2}{r+\delta} = 2S_{12},$$

where the inequality uses the submodularity assumption. The proof in the remaining two cases is only slightly more subtle. If $S_{11} = 0 < S_{22}$, equation (5) implies $v_1 = \rho\gamma_2^2(2S_{12}-S_{22})$. Since (6) implies $(r+\delta)(2S_{12}-S_{22}) = 2f_{12}-f_{22}-2v_1$, we obtain

$$2S_{12}-S_{22} = \frac{2f_{12}-f_{22}}{2(r+\delta+\rho\gamma_2^2)}$$

Submodularity of f and nonnegativity of f_{11} implies the right hand side is positive, $2f_{12} > f_{11}+f_{22} \geq f_{22}$. An identical proof works in the case of $S_{11} > 0 = S_{22}$. Thus in all four cases, $S_{11}(t)+S_{22}(t) < 2S_{12}(t)$, and the sufficient conditions for a social optimum are satisfied. Any candidate steady state optimum is a social optimum.

Proposition 2. *Assume that the production function is submodular,* $2f_{12} > f_{11}+f_{22}$*, and that* f_{11} *and* f_{22} *are nonnegative. Then any candidate steady state optimum is a social optimum, and so a steady state social optimum exists.*

On the other hand, if f is weakly supermodular, $f_{11} + f_{22} \geq 2f_{12}$, the second order sufficient conditions have no bite, regardless of whether the candidate solution is a steady state. Let $T_{12} \geq t$ denote the optimal stopping time for $(1, 2)$ matches, so (6) can be written as

$$S_{12}(t) = \int_t^{T_{12}} e^{-(r+\delta)(s-t)} \frac{f_{12} - v_1(s) - v_2(s)}{2} ds$$

Then since T_{12} is a feasible but not necessarily optimal stopping time for $(1, 1)$ and $(2, 2)$ matches, (6) also implies

$$S_{11}(t) \geq \int_t^{T_{12}} e^{-(r+\delta)(s-t)} \frac{f_{11} - 2v_1(s)}{2} ds \text{ and } S_{22}(t) \geq \int_t^{T_{12}} e^{-(r+\delta)(s-t)} \frac{f_{22} - 2v_2(s)}{2} ds$$

Adding these inequalities and subtracting twice the previous equality gives

$$S_{11}(t) + S_{22}(t) - 2S_{12}(t) \geq \int_t^{T_{12}} e^{-(r+\delta)(s-t)} \frac{f_{11} + f_{22} - 2f_{12}}{2} ds$$

Supermodularity of f implies the integrand is nonnegative, and hence that $S_{11}(t) + S_{22}(t) \geq 2S_{12}(t)$, precluding concavity of the Hamiltonian.

3.4 Second Order Necessary Conditions

Since in the supermodular case the Hamiltonian is convex, one might think that the candidate optima are minima rather than maxima. That inference would be incorrect. Second order necessary conditions for an optimal control do not simply weaken the inequalities in second order sufficient conditions. Rather, they also impose that class of admissible deviations be feasible, and that the first order effects of the deviations do not outweigh the second order effects. Colonius (1988) discusses such second order necessary in detail, using variational arguments to prove that for a steady state control to be optimal, local feasible deviations must either lead to a first order reduction in the objective or else must not generate a second order increase in the Hamiltonian.

It is easiest to implement Colonius's second order necessary conditions by considering a social planner who is restricted to continuous state variables. That is, suppose the stock of (i, j) matches must satisfy

$$\dot{m}_{ij}(t) = \rho \frac{u_i(t)u_j(t)}{\sum_k u_k(t)} - \delta m_{ij} - d_{ij} \equiv \psi_{ij}(m(t)) - d_{ij}$$

with initial condition $m_{ij}(0) = \bar{m}_{ij}$. The endogenous destruction rate d_{ij} is restricted to lie in the interval $[0, \bar{d}]$ and the state variable $m_{ij}(t)$ is nonnegative for all t.[3] The planner's objective function (1) is unchanged.

Suppose a steady state (m^*, d^*) is optimal in an economy with initial condition $\bar{m} = m^*$. Consider a deviation $(\tilde{m}(t), \tilde{d}(t))$ that is feasible, i.e. $d^* + \tilde{d}_{ij} \in [0, \bar{d}]$, $\tilde{m}(0) = 0$, and $m^* + \tilde{m}(t) \geq 0$, and that satisfies a linear approximation to the law of motion,

$$\dot{\tilde{m}}(t) = \psi'(m^*) \cdot \tilde{m}(t) - \tilde{d}(t), \tag{8}$$

Then the second order necessary condition is that a small deviation in this direction must either strictly reduce the present value of output,

$$\int_0^\infty e^{-rt} \sum_{i=1}^N \sum_{j=1}^N \tfrac{1}{2} f_{ij} \tilde{m}_{ij}(t) dt < 0, \tag{9}$$

or the Hamiltonian must be weakly concave in the direction of the deviation,

$$\int_0^\infty e^{-rt} \tilde{m}(t) \cdot H^*_{mm} \cdot \tilde{m}(t) dt \leq 0, \tag{10}$$

where H^*_{mm} is the Hessian of the Hamiltonian evaluated at the candidate steady state optimum.

The discussion of concavity of the Hamiltonian when $N = 2$ is useful for interpreting condition (10). If the production function is weakly submodular, the Hessian is negative definite, and the condition holds. On the other hand, if the production function is strictly supermodular, the Hessian is positive definite, and the condition appears to fail. More precisely, it fails unless any feasible deviation $\tilde{m}$ lies in the

[3] An advantage to this framework is that the planners' control variables are additively separable, and so do not enter Hessian of the Hamiltonian.

eigenspace associated with the zero eigenvalues of H^*_{mm}, which can be expressed as $(m^*_{11}, m^*_{12}, m^*_{22}) = (u^*_1, 0, u^*_2) \times (u^*_1 - u^*_2, u^*_2, 0)$.

There is one interesting case in which this feasibility restriction has bite. Suppose there are equal numbers of the two types of agents, $\ell_1 = \ell_2 = \frac{1}{2}$. Recall from Section 3.2 that with a nonnegative and supermodular production function, the only interesting question is whether mixed $(1,2)$ matches are formed, i.e. the value of d^*_{12} and the feasible deviations $\tilde{d}_{12}$. In this case, $u^*_1 = u^*_2$, and so the relevant eigenspace is $(m^*_{11}, m^*_{12}, m^*_{22}) = (1, 0, 1) \times (0, 1, 0)$. No value $\tilde{d}_{12}$ that can get outside this eigenspace, since the destruction of mixed matches has symmetric effects on the future stock of $(1,1)$ and $(2,2)$ matches. But for any other value of ℓ_1 and ℓ_2, any feasible deviation leads to a violation of condition (10).

In contrast, condition (9) holds quite generally. The *first* order necessary conditions impose that there be no feasible local deviation that strictly increase the value of the objective function, a weak version of the inequality in (9). The second order necessary condition may therefore fail only if there is no first order effect of a change in the destruction rate. Intuitively, this is the case when the social planner is just indifferent about creating a particular type of match, $f_{ij} = v_i + v_j$, in the candidate steady state optimum. Indeed, condition (9) is equivalent to[4]

$$\int_0^\infty e^{-rt} \sum_{i=1}^N \sum_{j=1}^N \tilde{d}_{ij}(t)(f_{ij} - v_i - v_j)dt > 0.$$

The first order conditions dictate that (i,j) matches are destroyed as fast as possible when $f_{ij} < v_i + v_j$ and are never endogenously destroyed when the inequality is reversed. This means that any feasible deviation must reduce the destruction rate, $\tilde{d}_{ij}(t) < 0$, in the former case and must increase it in the latter case. Putting this together, $\tilde{d}_{ij}(t)(f_{ij} - v_i - v_j) > 0$ whenever both terms are nonzero. Condition (9)

[4]Define $\tilde{S}_{ij} \equiv \frac{f_{ij} - v_i - v_j}{2(r+\delta)}$, the steady state value of an (i,j) match if it can never be endogenously destroyed. Consider the following chain of equalities:

$$-\int_0^\infty e^{-rt} \sum_i \sum_j \tilde{d}_{ij}(t)\tilde{S}_{ij}dt = -\int_0^\infty e^{-rt} \sum_i \sum_j \left(\sum_k \sum_l \frac{\partial \psi_{ij}(m^*)}{\partial m_{kl}} \cdot \tilde{m}_{kl}(t) - \dot{\tilde{m}}_{ij}(t) \right) \tilde{S}_{ij} dt =$$
$$-\int_0^\infty e^{-rt} \left(\sum_k \sum_l (r\tilde{S}_{kl} - \tfrac{1}{2} f_{kl}) \tilde{m}_{kl}(t) - \sum_i \sum_j r\tilde{m}_{ij}(t)\tilde{S}_{ij} \right) dt = \int_0^\infty e^{-rt} \sum_i \sum_j \tfrac{1}{2} f_{ij} \tilde{m}_{ij}(t)dt$$

The first equality replaces $\tilde{d}_{ij}(t)$ using (8). The second uses the costate equation $r\tilde{S}_{kl} = \frac{1}{2} f_{kl} - \sum_i \sum_j \tilde{S}_{ij} \frac{\partial \psi_{ij}(m^*)}{\partial m_{kl}}$ on the first term, and restates the second term using integration by parts. The third equality cancels terms.

can only fail at a candidate steady state optimum if $f_{ij} = v_i + v_j$ for some (i, j).

To summarize, a candidate steady state optimum fails the second order necessary conditions if three conditions hold: (i) the social planner is just indifferent about some match, $f_{ij} = v_i + v_j$; (ii) the Hessian of the Hamiltonian is positive definite; and (iii) it is feasible for the state of the economy to leave the eigenspace spanned by the Hessian's zero eigenvectors. To illustrate that these conditions may be applied in practice, we now turn to an example in which there is a continuum of candidate steady state social optima, yet none of them satisfy the second order necessary conditions.

4 Analytical Solution of a Special Case

This section analyzes a special case in some detail. There are two types of agents, $N = 2$, and bad $(1, 1)$ matches are sufficiently unproductive that they are never consummated. This reduces the dimensionality of the problem to two state variables m_{12} and m_{22}, which is convenient both graphically and algebraically.

More precisely, we assume that there are equal measures of the two types of agents, $\ell_1 = \ell_2 = \frac{1}{2}$. 'Good' matches between a pair of type 2 agents are productive, while 'mixed matches $(1, 2)$ produce an intermediate level of output, $f_{12} = \frac{\rho\bar{\gamma}_2(1-\bar{\gamma}_2)}{r+\delta+\rho\bar{\gamma}_2(2-\bar{\gamma}_2)} f_{22}$, where $\bar{\gamma}_2 \equiv \frac{\sqrt{\delta}}{\sqrt{\delta}+\sqrt{\delta+\rho}}$. The important part of this assumption is that f_{12} is slightly less than half of f_{22}. This means it is better to create a good match, using two type 2 agents, than a mixed match, which only uses one of this scarce resource. Our precise assumption on the relationship between f_{12} and f_{22} is not generic, and will be relaxed at the end of this section. Our other important assumption is that bad matches actually produce negative net output, sufficiently negative that the social planner never wants to create these matches: $f_{11} < \frac{-\rho\bar{\gamma}_2^2}{r+\delta+\rho\bar{\gamma}_2(2-\bar{\gamma}_2)} f_{22}$.[5] By construction, good matches are always consummated and bad matches are never consummated. The interesting question is whether mixed matches are consummated. In what follows, we first provide a precise analytical answer using the second order necessary conditions, and then offer an intuitive graphical explanation.

[5] It is not enough to assume $f_{11} = 0$. In that case, the social planner might want to create $(1, 1)$ matches, since that improves the process of creating good matches by getting these agents out of the way.

4.1 Analytical Solution

We begin by characterizing the state variables in a candidate steady state optimum. The permanence of good matches implies that the law of motion (2) binds for $(i, j) = (2, 2)$, and that $dm_{22}(t)/dt = 0$ for all t. Solving this equality yields a linear relationship $m_{22} = k(\frac{1}{2} - m_{12})$, where $k \equiv 1 - \sqrt{\delta/(\delta + \rho)}$ is a simple function of model parameters. Crucially, a fraction $\bar{\gamma}_2 \equiv \frac{\sqrt{\delta}}{\sqrt{\delta}+\sqrt{\delta+\rho}} = \frac{1-k}{2-k} = \frac{\frac{1}{2}-m_{12}-m_{22}}{1-2m_{12}-m_{22}}$ of the unmatched population is type 2 in any steady state, regardless of whether mixed matches are consummated. Our measure of matching opportunities $\bar{\gamma}_2$ does not depend on matching decisions in steady state, but rather is a function only of the parameters ρ and δ.

This makes it easy to solve for the social surplus S_{ij} and unmatched values v_i in any steady state. Since the composition of the searching population is unaffected by matching behavior, these too will be unaffected. The unique solution to the stationary versions of (5) and (6) are

$$v_1 = \frac{-\rho\bar{\gamma}_2^2}{2\big(r + \delta + \rho\bar{\gamma}_2(2 - \bar{\gamma}_2)\big)} f_{22}, \quad v_2 = \frac{\rho\bar{\gamma}_2(2 - \bar{\gamma}_2)}{2\big(r + \delta + \rho\bar{\gamma}_2(2 - \bar{\gamma}_2)\big)} f_{22}$$

$$S_{11} = 0, \quad S_{12} = 0, \quad \text{and } S_{22} = \frac{1}{2\big(r + \delta + \rho\bar{\gamma}_2(2 - \bar{\gamma}_2)\big)} f_{22}$$

By construction, $f_{11} < 2v_1$, so bad matches are never consummated, while $f_{12} = v_1 + v_2$, so the social planner is exactly indifferent about consummating mixed matches. This has two implications. Any steady state stock of mixed matches is consistent with the first order necessary conditions for a social optimum; but any steady state stock of mixed matches violates condition (9). A small change in the stock of mixed matches has no first order effect on the present value of output.

Next, observe that $S_{11} + S_{22} > 2S_{12}$, and so the Hessian of the Hamiltonian is convex at any solution to the first order conditions. This means that a small change in the stock of mixed matches induces a second order increase in the Hamiltonian. There is one remaining caveat: it must be possible to leave the eigenspace spanned by the Hessian's zero eigenvalues. To verify that this is possible, pre- and post-multiply the Hessian evaluated at a candidate social optimum by a deviation vector $(0, \tilde{m}_{12}, \tilde{m}_{22})$ to get

$$\frac{2\rho S_{22}}{(2-k)^3(\frac{1}{2} - m_{12})} \big(k\tilde{m}_{12} + \tilde{m}_{22}\big)^2.$$

This is positive unless any feasible deviation has $\tilde{m}_{22} = -k\tilde{m}_{12}$, which is exactly the slope of the locus of feasible steady states, $m_{22} = k(\frac{1}{2} - m_{12})$. Although it is not possible to permanently move m_{12} and m_{22} away from this locus, it is possible to move them away temporarily. More precisely, we can express the local approximation to the dynamics of the state variable, equation (8), as

$$\begin{pmatrix} \dot{\tilde{m}}_{12}(t) \\ \dot{\tilde{m}}_{22}(t) \end{pmatrix} = - \begin{pmatrix} \rho(1 - 2\bar{\gamma}_2 + 2\bar{\gamma}_2^2) + \delta & \rho\bar{\gamma}_2^2 \\ 2\rho\bar{\gamma}_2(1 - \bar{\gamma}_2) & \rho\bar{\gamma}_2(2 - \bar{\gamma}_2) + \delta \end{pmatrix} \cdot \begin{pmatrix} \tilde{m}_{12}(t) \\ \tilde{m}_{22}(t) \end{pmatrix} - \begin{pmatrix} \tilde{d}_{12} \\ 0 \end{pmatrix}$$

Starting from $(\tilde{m}_{12}(0), \tilde{m}_{22}(0)) = (0, 0)$, the local dynamics indicate that, for a negative value of $\tilde{d}_{12}$, initially $\dot{\tilde{m}}_{12}(0) > 0$ and $\dot{\tilde{m}}_{22}(0) = 0$. Thereafter, $\dot{\tilde{m}}_{22}(t)$ becomes negative, although asymptotically the system approaches a new steady state. During the entire transition, $\tilde{m}_{22}(t) > -k\tilde{m}_{12}(t)$, although asymptotically we can see from the equation for $\dot{\tilde{m}}_{22}(t)$ that $\tilde{m}_{22} = -k\tilde{m}_{12}$. The inequalities are reversed for a positive deviation $\tilde{d}_{12}$. In other words, a transitory movement away from the steady state generates a second order increase in the Hamiltonian. Both conditions (9) and (10) are violated at any candidate steady state optimum, proving that the social optimum can never settle down to a steady state.

4.2 Graphical Solution

The intuition for this result can best be understood graphically. Figure 1 indicates the set of feasible steady states with the downward sloping line segment SS. The earlier analysis indicates that for any constant m_{12}, the steady state value of m_{22} is $k(\frac{1}{2} - m_{12})$. A similar logic shows that the measure of mixed matches m_{12} cannot exceed $\frac{1}{2}k$, the value obtained if condition (2) holds an an equality. On the other hand, by never creating these matches, it can fall as low as zero. At any point along the steady state locus, the fraction of type 2 agents in the searching population γ_2 is constant and equal to $\bar{\gamma}_2$. The figure also indicates that at (unsustainable) points above the steady state locus, there are more type 2 agents matched with other type 2 agents, and so γ_2 is less than $\bar{\gamma}_2$. The opposite inequality holds at points below the steady state locus.

By construction, if $\gamma_2(t) = \bar{\gamma}_2$ for all t, the social planner is just indifferent about creating mixed matches, and any steady state matching behavior is a candidate optimum. On the other hand, if it were possible to sustain a permanently higher

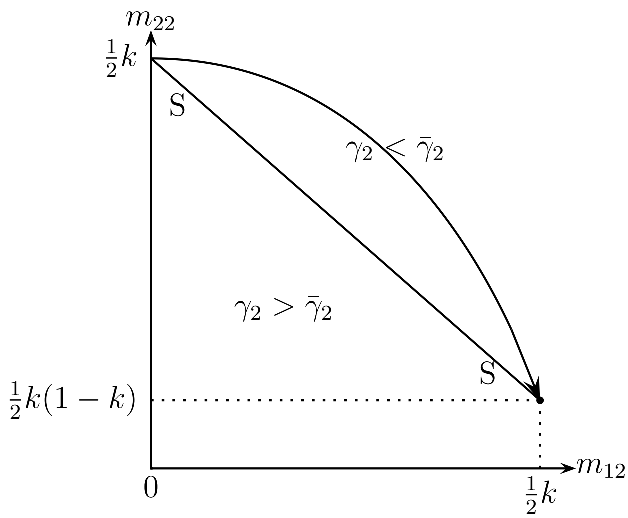

Figure 1: The set of feasible steady states and a nonstationary deviation.

value of $\gamma_2(t)$, the first order conditions tell us that it would be optimal to never create mixed matches. Intuitively, it is easier for high productivity agents to meet, and so they should hold out for such meetings. Conversely, with a permanently lower value of $\gamma_2(t)$, it would be optimal to always create these matches.

This is not just a thought experiment. The social planner is capable of attaining a permanent decrease in $\gamma_2(t)$, assuming that initially $m_{12} < \frac{1}{2}k$. Start from a steady state without the maximum feasible rate of creating mixed matches, for example at the point $(m_{12}, m_{22}) = (0, \frac{1}{2}k)$. Suppose that at time 0, the social planner decides never again to endogenously destroy a mixed match. Since the stock of matches is a state variable, it is unchanged at the initial date, and therefore $\dot{m}_{22}(0) = 0$ as well. On the other hand, the decision not to destroy mixed matches implies $\dot{m}_{12}(0) > 0$. This moves the system directly to the right in Figure 1, into the region of the parameter space with $\gamma_2(t) < \bar{\gamma}_2$. Thereafter, it becomes harder to create good matches, and so $m_{22}(t)$ starts to fall. Asymptotically, the economy converges towards the steady state at the opposite corner of the feasible set, with $m_{12} = \frac{1}{2}k$ and $m_{22} = \frac{1}{2}k(1-k)$. But throughout the transitional dynamics, the share of unmatched type 2 workers is low, justifying the decision to create mixed matches. Alternatively, by destroying a stock of mixed matches, the economy moves into the region of the parameter space in which unmatched type 2 agents are abundant, potentially justifying that decision.

It is worth emphasizing why the first order necessary conditions are not sufficient in this problem. They take the stock of matches as given, while the social planner has the ability to control the stock of matches. The second order conditions tell us that if the social planner is just indifferent about matching when she maintains a constant stock of matches, she can do strictly better by considering a small nonstationary deviation from that constant stock.

4.3 The Optimal Limit Cycle

The Poincaré-Bendixson theorem ensures that if the social optimum does not have a steady state, it must converge to a limit cycle. More complicated dynamics, such as chaos, are impossible in this two-dimensional continuous time system. The discussion of second order conditions suggests the nature of the optimal limit cycle. Start from a point on the steady state locus SS in Figure 1, with $m_{12} < \frac{1}{2}k$, and begin creating mixed matches as fast as possible. Initially the congestion created by type 1 agents worsens, or equivalently γ_2 falls, but later the system converges back towards the steady state locus and γ_2 starts to increase. At some point, destroy the large stock of mixed matches. By unleashing equal numbers of each type of agent into the unmatched population, γ_2 increases sharply towards $\frac{1}{2}$. Until that share is worked back down, the social planner does not allow mixed matches. Eventually, however, the cycle starts again, with the planner allowing mixed matches in anticipation of future low values of γ_2.

To find the optimal limit cycle, we use the following algorithm:

1. Conjecture a limit cycle with only good matches created for T_0 periods, and then mixed matches permitted for a subsequent T_1 periods, followed by the destruction of all mixed matches.

2. Solve for the asymptotic path of the state variable given T_0 and T_1.

3. Check whether the social planner is exactly indifferent about creating mixed matches at the beginning of the period in which mixed matches are created, time T_0.

    (a) If $f_{12} > v_1(T_0) + v_2(T_0)$, reduce T_0 and return to step 2.

    (b) If $f_{12} < v_1(T_0) + v_2(T_0)$, increase T_0 and return to step 2.

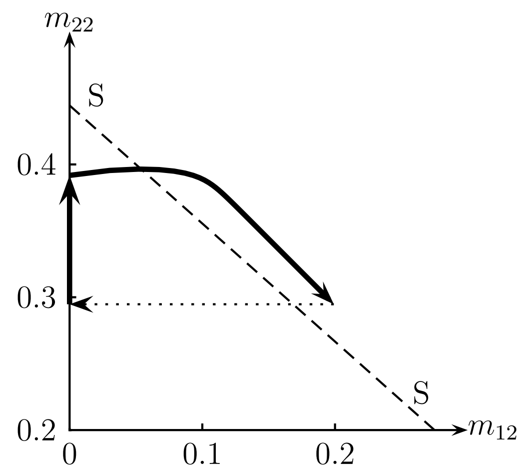

Figure 2: The optimal limit cycle.

4. Check the continuity of the Hamiltonian at the time of destruction of mixed matches, $T_0 + T_1$.

   (a) If $\mathcal{H}((T_0 + T_1)-) < \mathcal{H}((T_0 + T_1)+)$, reduce T_1 and return to step 2.

   (b) If $\mathcal{H}((T_0 + T_1)-) > \mathcal{H}((T_0 + T_1)+)$, raise T_1 and return to step 2.

By changing the initial conditions, it is possible to search for alternative nonstationary solutions to the social planner's problem. In practice, this is not an issue.

A concrete example clarifies the nature of the optimal limit cycle. Set $\rho = 4$, $\delta = r = 1/20$, and $\ell_1 = \ell_2 = 1/2$. Normalize $f_{22} = 1$ and set $f_{12} = 18/43 \approx 0.42$, which leaves the planner indifferent about creating mixed matches in a candidate steady state optimum. Any constant mixed matching pattern is a candidate steady state optimum, but that none is an actual optimum. Instead, the optimal policy is cyclical. For $T_0 = 0.83$ time units, do not allow mixed matches; then for $T_1 = 6.46$ time units, allow these matches; then destroy them and repeat. The state variables cycle clockwise, as indicated in Figure 2, spending much of the time beyond the set of points achievable in steady state, which is indicated by the dashed line segment.

4.4 Robustness of the Example

In order to use the second order necessary conditions, it is necessary to find parameter values such that the planner is indifferent about creating mixed matches in any steady state. Unfortunately, such parameters are nongeneric, possibly casting doubt on the relevance of nonstationary policies. This section argues that while that the proof requires nongeneric parameter values, the logic of nonstationary optimal policies does not.

With the nongeneric parameter values, we proved that there is no first order cost to pursuing a nonstationary policy and there is a second order gain. In other words, the steady states are local minima. For nearby parameter values, there will be a first order cost to a nonstationary policy, which implies that a small deviation is not optimal. But the second order gain outweighs the first order cost when a larger deviation is contemplated. Thus although the steady states may be local maxima with nearby parameter values — condition (9) is satisfied — they are not global maxima. In particular:

Proposition 3. *There is an open set of parameter values such that the social optimum is nonstationary.*

5 A More Plausible Example

A reasonable criticism of the model in Section 4 is that if the social planner could keep type 1 agents from searching, output would increase further. This is a consequence of the large degree of heterogeneity in the model, with good matches producing more than twice as much as mixed matches, and bad matches yielding negative output. Congestion effects seem more plausible in environments with only a modest degree of heterogeneity, and therefore in environments in which the social planner would prefer that all agents search. This section develops an example of nonstationary search in such an environment.

We consider the simplest example with two types of agents, $N = 2$; a supermodular production function $f_{11} + f_{22} > 2f_{12}$, so that the second order sufficient conditions for a steady state optimum fail; and nonnegative net output $f_{ij} \geq 0$, so that heterogeneity is moderated. Even in this case, there is always a candidate steady state optimum in which the second order necessary condition (9) holds, implying that the policy is a local maximum. On the other hand, the second order necessary condition (10) is violated, suggesting that the candidate steady state optima may not be global maxima. This is analogous to the rationale for perturbed parameter values in the nongeneric example. Thus section provides an example in which there are two candidate steady state optima that are local maxima, but both policies are dominated by a limit cycle. Our proof is necessarily computational.

To understand the intuition for the existence of nonstationary optima, it helps to think about why there may be multiple candidate steady state optima. The analysis in Section 3.2 establishes that with a nonnegative, supermodular production function, good and bad matches are always consummated in any steady state optimum. The only question is whether mixed matches are consummated and in steady state that hinges on whether

$$(r+\delta)f_{12} \gtreqless \rho\gamma_1\gamma_2(f_{11}+f_{22}-2f_{12}).$$

If the left hand side is larger than the right hand side, the first order necessary conditions indicate that mixed matches are consummated, while if the left hand side is smaller, mixed matches are not consummated.

Note, however, that $\gamma_1\gamma_2$ depends on whether mixed matches are consummated, at least if $\ell_1 \neq \ell_2 \neq \frac{1}{2}$. To be concrete, suppose $\ell_1 > \frac{1}{2} > \ell_2$. Then if all matches are consummated, every meeting results in a match, and so the average duration of an unmatched spell is $1/\rho$ periods, independent of the agent's type. With only exogenous destruction at rate δ, both types of agents are unmatched a fraction $\frac{\delta}{\delta+\rho}$ of the time, and so the share of type i agents in the unmatched population is just $\gamma_i = \ell_i$. On the other hand, if mixed matches are rejected, the average duration of an unmatched spell increases to $1/\gamma_i\rho$ periods for type i agents by reducing the relevant contact rate. This is lower for type 1 agents than for type 2 agents, since $\gamma_1 > \gamma_2$, and so type 1 agents are unmatched less frequently. This raises the share of unmatched type 2 agents and reduces the share of unmatched type 1 agents towards $\frac{1}{2}$. In particular, the product $\gamma_1\gamma_2$ is larger when mixed matches are rejected, potentially justifying rejecting the mixed matches.

As long as $\ell_1 \neq \ell_2$, this logic implies that there is an open set of parameter values with two candidate steady state optima. In one, mixed matches are created, $\gamma_i = \ell_i$, and $\gamma_1\gamma_2$ is small, while in the other, mixed matches are not created and $\gamma_1\gamma_2$ is larger.[6] In the former steady state, $v_1 + v_2$ is strictly smaller than its value in the latter steady state, and so it is impossible for the planner to be indifferent about creating mixed matches in both situations. The second order necessary conditions can therefore be used to prove that one steady state is not optimal, but in this event,

[6] If both these candidate optima exist, a third candidate exists as well. In it, mixed matches are sometimes created, with the probability chosen so that $(r+\delta)f_{12} = \rho\gamma_1\gamma_2(f_{11}+f_{22}-2f_{12})$. This leaves the planner indifferent about creating mixed matches. It is straightforward to establish that such a policy violates the second order necessary conditions.

the other steady state is a local maximum. The second order necessary conditions never imply that both steady states are not optimal.

The only apparent exception occurs if $\ell_1 = \ell_2 = 1/2$. In this case, any mixed matching pattern still implies $\gamma_1 = \gamma_2 = 1/2$. If

$$(r+\delta)f_{12} = \tfrac{1}{4}\rho(f_{11} + f_{22} - 2f_{12}),$$

the planner is indifferent about creating mixed matches, regardless of how many of these matches are created. Condition (9) is violated, so there is no first order cost to pursuing a nonstationary policy. But recall from Section 3.4 that there is also no second order gain in this case. The state vector is stuck in the eigenspace of the Hessian's zero eigenvalues. Even by pursuing a nonstationary policy, the γ_i do not change, eliminating the gains from such a policy. Indeed, in this special case, nonstationary policies generate the same output as stationary policies.

We can still use this logic to look for examples in environments with multiple candidate steady state optima. The multiplicity implies that, depending on the composition of the searching population, either policy for mixed matches may be optimal and, moreover, that the first order loss from pursuing a nonstationary policy is relatively small. Thus we set $r = 1/20$, $\rho = 5$, and $\delta = 1/8$. We assume $\ell_1 = 2/3 = 1 - \ell_2$, so nonstationary policies can affect the composition of the unmatched population. We normalize $f_{11} = 1$ and set $f_{12} = 1$ and $f_{22} = 1.15$. The extent of heterogeneity in this economy is modest, with no productivity differences in excess of 15%.

It is easy to verify that there are multiple candidate steady state optima with these parameter values. We prove by construction that none of these candidates are optimal. We use the numerical algorithm in Section 4.3 to search for the optimal limit cycle, alternating between an interval of length $T_0 = 0.643$ with no mixed matches, an interval of length $T_1 = 3.277$ with mixed matches, followed by the instantaneous destruction of all mixed matches, restarting the cycle. We then calculate the present value of moving to the cycle starting from the steady state. For example, starting from the economy with any mixed matches, we begin by destroying all the mixed matches. We then allow T_0 periods without these matches, T_1 periods with the matches, followed by a new round of destruction. Asymptotically, the economy converges to the limit cycle, and so it is possible to compare the present value of steady state output, 9.919 with the present value of output along this particular nonoptimal path, 10.221. Conversely, suppose we start from an economy without any mixed matches by having a period T_1 with these matches, followed by destruction and the usual cycle. The present value of output would have been 10.009, while it is 10.318 along this particular path. Of course, in neither case is the nonstationary path optimal, and so the optimal policy would produce more still more output. Still, in both cases, cycles raise the present value of output by at least 3%. This is surprisingly large in an economy with a modest degree of heterogeneity.

A number of simulations using a wide range of parameter values support the following conjecture:

Conjecture 1. *If multiple candidate steady state optima exist, the optimum is nonstationary.*

In other words, if the feedback from matching opportunities to matching decisions is sufficiently strong, and the social planner is sufficiently close to indifferent about creating some type of match, the optimal policy has periods in which those matches are created and periods in which they are destroyed.

We have so far restricted attention to limit cycles, but in this three dimensional problem, there is no reason to believe that the optimal policy is cyclical. In principal, an optimal policy maps the current state $\{m_{ij}(t)\}$ into future states in a nonlinear fashion. With $N \geq 2$, the state of the economy is $\frac{1}{2}N(N+1) \geq 3$ dimensional. It is well-known that with generic parameter values, a system of three or more nonlinear differential equations can converge to a 'strange attractor'. If this were the case, the limiting behavior of this economy would be chaotic, not periodic. We do not know how to solve for optimal chaotic controls.

6 Equilibrium

In this section, we return Sattinger's (1995) and Burdett and Coles's (1997) observation that search models with heterogeneous agents may have multiple equilibria. We extend our methodology to show that the multiplicity is much worse than previously thought. Even the simplest example, the one analyzed in Section 4, may exhibit a continuum of perfect foresight equilibria.

In a perfect foresight equilibrium, each agent maximizes the expected present value of her income, taking the actions of the other agents — and hence the time path of the matched rates m — as given. In particular, a pair of agents match whenever it is in their mutual interest. Since the matching process gives rise to a bilateral monopoly, we must specify how output is divided. We follow the literature and impose a symmetric Nash bargaining solution, so that the expected present value of the surplus that one agent enjoys in a match in excess of her unmatched value must be equal to the expected present value of her partner's surplus. This implies that all bilateral gains from trade are exploited, and in particular that a pair breaks up if and only if it is in the interest of both agents. Nevertheless, agents ignore the effect of their actions on third parties, e.g. potential partners, which drives a wedge between equilibrium and optimal behavior.

6.1 Characterization

We characterize an equilibrium using recursive equations. Start with the expected value of an agent i matched with an agent j at time t, $W_{i|j}(t)$, expressed as a function of future unmatched values $W_i(s)$, $s \geq t$:

$$(11) \qquad W_{i|j}(t) = \max_{T \in [t,\infty]} \int_t^T e^{-(r+\delta)(s-t)} \big(\pi_{i|j}(s) + \delta W_i(s)\big) ds + e^{-(r+\delta)(T-t)} W_i(T)$$

An (i, j) match ends at an optimally chosen future date $T \in [t, \infty]$, when it is in the mutual interest of i and j. Until then, i gets an endogenous flow payoff $\pi_{i|j}$, and suffers exogenous match destruction with flow probability δ, which leaves her unmatched. If the match survives until date T, she gets her unmatched value $W_i(T)$, discounted back to date t.

We can express (11) in a more useful form by noting that for any $T \in [t, \infty]$,

$$W_i(t) \equiv \int_t^T e^{-(r+\delta)(s-t)} \big((r + \delta)W_i(s) - \dot{W}_i(s)\big) ds + e^{-(r+\delta)(T-t)} W_i(T),$$

as can be verified using integration by parts on the right hand side of this identity. Subtracting this from (11) implies

$$(12) \qquad R_{ij}(t) \equiv W_{i|j}(t) - W_i(t) = \max_{T \in [t,\infty]} \int_t^T e^{-(r+\delta)(s-t)} \big(\pi_{i|j}(s) - w_i(s)\big) ds$$

where $w_i(s) \equiv rW_i(s) - \dot{W}_i(s)$ is the flow value of an unmatched agent. $R_{ij}(t)$ is the surplus that i gets from being matched with j at time t, analogous to the social surplus S.

Since a pair always has the option to destroy the match immediately, $R_{ij}(t)$ is nonnegative. If it is positive, the pair matches and divides up the gains from trade according to a dynamic Nash bargaining solution.[7] The threat of each agent is to destroy the match, making $W_i(t)$ and $W_j(t)$ the relevant threat points, while the payoff to be divided up is $W_{i|j}(t) + W_{j|i}(t)$. Thus Nash bargaining imposes $R_{ij}(t) = R_{ji}(t)$. Equivalently, the integrand of (12) must be equal for each party to the match,

$$\pi_{i|j}(s) - w_i(s) = \pi_{j|i}(s) - w_j(s) = f_{ij} - \pi_{i|j}(s) - w_j(s)$$

where the second equality uses the fact that output f_{ij} is divided between i and j, and hence equal to $\pi_{i|j}(s) + \pi_{j|i}(s)$. This pins down individual flow payoffs π. Substituting into (11) implies

$$R_{ij}(t) = \max_{T\in[t,\infty]} \int_t^T e^{-(r+\delta)(s-t)} \frac{f_{ij} - w_i(s) - w_j(s)}{2} ds \tag{13}$$

Each agent's share of surplus in an (i, j) match is half the present discounted value of future flow surplus, which is output in excess of the sum of the flow unmatched values. Finally, we close the system by writing the recursive equation for the flow value of an unmatched agent:

$$rW_i(t) - \dot{W}_i(t) \equiv w_i(t) = \rho \sum_{k=1}^{N} \gamma_k(t) R_{ik}(t) \tag{14}$$

An unmatched agent's flow payoff is zero, while the probability of meeting a new potential partner is ρ, and the expected capital gain is the appropriately weighted average of $R_{ij}(t)$.

The match surplus $R_{ij}(t)$ depends on the future composition of the searching population $\gamma_k(s)$, $s > t$, through the composition's effect on future unmatched values $w_i(s)$. This is because the opportunity cost of matching is forgoing the possibility of searching in the future. Note in particular that the agents do not

[7] Note that agents bargain over the expected present value of payoffs, not current payoffs. Coles and Wright (1998) is the first application of this bargaining rule in a search model.

care how hard it was for them to meet at time t. Thus optimal matching (and match termination behavior) yields one dynamic linkage between matching decisions (whether $R_{ij}(t) > 0$) and matching opportunities (the $\gamma_k(s)$).

Another dynamic linkage comes from how matching decisions affect the matched rates. When $R_{ij}(t)$ is positive, the measure of (i, j) matches increases at the maximum possible rate, so (2) is binding. Otherwise, the measure of (i, j) matches may fall:

$$\begin{aligned} R_{ij}(t) > 0 &\Rightarrow \dot{m}_{ij}(t) = \rho \frac{u_i(t)u_j(t)}{\sum_{k=1}^{N} u_k(t)} - \delta m_{ij}(t) \\ R_{ij}(t) = 0 &\Rightarrow 0 \leq m_{ij}(t) \leq m_{ij}(t-) \end{aligned} \tag{15}$$

where $m_{ij}(t-)$ denotes the left hand limit of $m_{ij}(s)$ as $s \to t$. A precise definition of a perfect foresight equilibrium follows from these expression:

Definition 3. *A perfect foresight equilibrium is any tuple* (w, R, m) *such that:* R *solves (13) given* w*;* w *solves (14) given* R *and* m*; and* m *solves the state equations (15) given some initial conditions and* R*.*

Shimer and Smith (2000) prove that a steady state equilibrium exists in a similar model, and their argument carries through to this environment.

Note that any tuple satisfying these three equations is a perfect foresight equilibrium. This is in contrast to the first order necessary conditions for a social optimum. We called any tuple satisfying those conditions a candidate optimum, but emphasized that a candidate optimum was not necessarily a social optimum. There is one other important difference between the definition of an equilibrium and a candidate optimum,[8] the absence of a condition like the continuity-of-Hamiltonian requirement. This condition reflects the social planner's ability to choose the timing of a mass of match destruction. In a decentralized equilibrium, private agents play a coordination game, and so there is no reason to leave one match until other agents have left their match.

[8]Another difference is the distinction between the social unmatched value v and the private unmatched value w. Shimer and Smith (2001) discusses this in detail, explaining how it is related to search externalities. Although this quantitatively affects the behavior of the equilibrium, it is qualitatively unimportant.

6.2 Analytically Tractable Example

This section analyzes an example based on the one in Section 4. We assume $N = 2$ and $\ell_1 = \ell_2 = \frac{1}{2}$. Also, $f_{12} = \frac{\rho\bar{\gamma}_2}{2(r+\delta+\rho\bar{\gamma}_2)} f_{22}$,[9] where $\bar{\gamma}_2 = \frac{\sqrt{\delta}}{\sqrt{\delta}+\sqrt{\delta+\rho}}$, while $f_{11} < 0$. In any equilibrium, good matches are always created and bad matches are never created. As was the case with candidate optima, any steady state mixed matched rate is consistent with equilibrium, since by construction the share of type 2 agents is $\gamma_2 = \bar{\gamma}_2$ in any steady state; and with this share of type 2 agents, $f_{12} = w_1 + w_2$.

On the other hand, nonstationary paths may be equilibria as well. Again, the key observation is that if there is a transitory increase in the rate of creating mixed matches, the share of type 2 agents in the unmatched population, γ_2, falls. Since it is harder for type 2 agents to find each other, they are willing to accept mixed matches, justifying the nonstationary path of the economy. The opposite logic prevails with a transitory decrease in the mixed matching rate.

More formally, suppose $m_{12}(0) > 0$ and $m_{22}(0) = k(\frac{1}{2} - m_{12}(0))$, the usual steady state relationship. We claim that, in addition to the steady state equilibrium, there is an equilibrium in which all mixed matches are destroyed and never recreated, $m_{12}(t) = 0$ for all $t > 0$. In such an equilibrium, $m_{22}(t)$ gradually increases to $\frac{1}{2}k$, while $\gamma_2(t) = \frac{\frac{1}{2} - m_{22}(t)}{1 - m_{22}(t)}$ monotonically declines towards $\bar{\gamma}_2$.

To prove that this is an equilibrium, we must show that $R_{11}(t) = R_{12}(t) = w_1(t) = 0$ for all t, and hence that there is negative surplus in a mixed match. For this to be the case, it must be that $w_2(t) = \rho\gamma_2(t)R_{22}(t) > f_{12}$, where the equality uses condition (14), while the inequality imbeds the requirement that $R_{12}(t) = 0$. First differentiate $w_2(t) = \rho\gamma_2(t)R_{22}(t)$ with respect to time to get

$$\begin{aligned} \dot{w}_2(t) &= \frac{\dot{\gamma}_2(t)}{\gamma_2(t)} w_2(t) + \rho\gamma_2(t)\dot{R}_{22}(t) \\ &= \frac{\dot{\gamma}_2(t)}{\gamma_2(t)} w_2(t) + (r + \delta + \rho\gamma_2(t))w_2(t) - \rho\gamma_2(t)\frac{f_{22}}{2}, \end{aligned}$$

where the second equation follows by time-differentiating $R_{22}(t)$ in (13). Now suppose $w_2(t) \leq f_{12} \equiv \frac{\rho\bar{\gamma}_2}{2(r+\delta+\rho\bar{\gamma}_2)} f_{22}$, its steady state value. Substituting this into the

[9] This is the same condition as before if $r = 0$, but is otherwise somewhat different. The distinction again reflects the search externalities, which is not the focus on this paper.

previous equation gives

$$\dot{w}_2(t) \leq \frac{\dot{\gamma}_2(t)}{\gamma_2(t)} w_2(t) + \left(\frac{\rho(r+\delta)(\bar{\gamma}_2 - \gamma_2(t))}{2(r+\delta+\rho\bar{\gamma}_2)} \right) f_{22}$$

Since $\gamma_2(t)$ is greater than $\bar{\gamma}_2$ and decreasing, both terms are negative, and so $\dot{w}_2(t) < 0$. This contradicts the requirement that asymptotically $w_2(t)$ converges to f_{12}, and hence proves that $w_2(t) > f_{12}$ for all t, as desired.

Alternatively, suppose $m_{12}(0) = 0$, although we can handle any initial condition with $m_{12}(0) < k/2$ with equal ease. Now consider the following deviation: Create all mixed matches during the interval $[0, T)$. Then destroy enough of these matches so as to revert to the steady state locus, i.e. so that $m_{12}(T) = \frac{1}{2} - m_{22}(T)/k$. We will show that for $T > 0$ sufficiently small, this is an equilibrium, meaning $R_{12}(t) > 0$ for all $t \in [0, T)$.[10] The key is to approximate the differential equations for R, which can be done very well when T is small. We do this via quadratic functions:[11]

$$\tilde{R}_{12}(t) = \frac{gf_{12}}{4}(t^2 - T^2) \text{ and } \tilde{R}_{22}(t) = \bar{R}_{22} + \frac{gf_{12}}{2}(t^2 - T^2),$$

where $g \equiv -\rho\frac{(1-k)(k-2m_{12}(0))}{(2-k)^3(1-2m_{12}(0))} < 0$ is the initial growth rate of γ_2 following the shock, $\dot{\gamma}_2(0)/\gamma_2(0)$. The level of the quadratic approximations are correct at one end point, $R_{i2}(T) = \tilde{R}_{i2}(T)$, $i = 1, 2$; the slope of the quadratic approximations are correct at the same end point, $\dot{R}_{i2}(T) = \dot{\tilde{R}}_{i2}(T)$; and the curvature of the quadratic approximations are correct at the opposite end point, $\ddot{R}_{i2}(0) = \ddot{\tilde{R}}_{i2}(0)$. Since the approximate surplus function $\tilde{R}_{12}(t) > 0$ for $t \in [0, T)$, this is prices are consistent with the proposed nonstationary strategies.

Of course, there are many other nonstationary perfect foresight equilibria in this model. Adding a sunspot variable further increases the dimensionality of the set of equilibria. Moreover, since the equilibria are strict, they will survive perturbations in all the parameter values. In summary:

Proposition 4. *There is an open set of parameter values such that a continuum of nonstationary perfect foresight equilibria exist.*

[10] Simulations show that this policy is not an equilibrium for large T.

[11] A linear approximation yields locally constant values of the surplus functions, and hence does not tell us whether the surplus in a mixed match is positive or negative at $t \in [0, T)$.

7 Conclusion

We close by summarizing our results. This paper has explored the simplest heterogeneous agent extension to a standard random model, focusing on optimal matching patterns in order to sidestep the problems introduced by self-fulfilling expectations. Our main finding is that time-varying matching behavior can improve the efficiency of the matching process if agents' characteristics are complements in production. This is true even though the meeting function exhibits constant returns to scale. Nonstationary policies allow for periods in which particular types of matches are easier to create. The most important assumption for generating nonstationary optima is that some matching decisions are nontrivial, in the sense that whether it is desirable to consummate a particular match depends on the composition of the searching population. This is a realistic assumption in practical applications.

Appendix

A Necessary Conditions for a Social Optimum

Vind (1967) and Arrow and Kurz (1970) show how to solve optimal control problems in which the state variables may be discontinuous. We apply their technique to this environment. Introduce an artificial time index τ, and divide artificial time between periods when natural time runs normally, $x(\tau) = 1$, and periods when natural time stops, $x(\tau) = 0$, so natural time evolves according to $\dot{t}(\tau) = x(\tau)$. Then we can restate the control problem in artificial time:

$$\max \int_{\tau}^{\infty} e^{-r(t(s)-t(\tau))} x(s) \sum_{i=1}^{N} \sum_{j=1}^{N} f_{ij} m_{ij}(s)/2 \quad ds \quad \text{subject to}$$

$$\dot{m}_{ij}(\tau) = x(\tau) \left(\rho \frac{u_i(\tau) u_j(\tau)(\tau)}{\sum_{k=1}^{N} u_k(\tau)} - (\delta + d_{ij}(\tau)) m_{ij}(\tau) \right) - (1 - x(\tau)) y_{ij}(\tau),$$

$$\dot{t}(\tau) = x(\tau),$$

$$d_{ij}(\tau) \geq 0,\ y_{ij}(\tau) \geq 0,\ x(\tau) \in \{0, 1\}, \text{ and } m_{ij}(\tau) \geq 0$$

The objective is to maximize the present value of output, which is only accumulated during, and only discounted in, natural time. Endogenous destruction of matches is achieved by stopping natural time, i.e. setting $x(\tau) = 0$. When natural time is stopped, $y_{ij} \geq 0$ indicates the destruction rate of (i, j) matches. Since there is no point in stopping natural time without destroying matches, we assume without loss of generality that $x(\tau) = 1$ if $y_{ij}(\tau) = 0$ for all i and j. During natural time, the control variable $d_{ij} > 0$ allows the possibility of not consummating a new (i, j) match. Technically, it is not necessary to introduce the control d since match destruction can be achieved by stopping natural time and destroying any undesired matches. But the use of this control simplifies the interpretation of natural time stoppages. In our formulation, natural time stops when a positive measure of existing matches are destroyed and otherwise flows ahead at full speed. Finally, we introduce the nonnegativity constraint on m_{ij} to keep matching rates from becoming negative during stoppages in natural time, while still allowing matching rates to fall to zero in finite time.

Represent the reformulated control problem using a present-valued Hamiltonian with shadow value $e^{-rt(\tau)}S_{ij}(\tau) \equiv e^{-rt(\tau)}S_{ji}(\tau)$ on the matched rate $m_{ij}(\tau) \equiv m_{ji}(\tau)$ and shadow value $e^{-rt(\tau)}z(\tau)$ on natural time $t(\tau)$:

$$H_a(\tau) = e^{-rt(\tau)}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\left(x(\tau)\left(f_{ij}m_{ij}(\tau)/2 + S_{ij}(\tau)\left(\rho\frac{u_i(\tau)u_j(\tau)}{\sum_{k=1}^{N} u_k(\tau)}\right.\right.\right.\right.$$
$$\left.\left.\left.\left. - \big(\delta + d_{ij}(\tau)\big)m_{ij}(\tau)\right)\right) - (1 - x(\tau))S_{ij}(\tau)y_{ij}(\tau)\right) + x(\tau)z(\tau)\right)$$

As in the text, optimality rules out negative values of $S_{ij}(\tau)$ since otherwise the value of the Hamiltonian is unboundedly large. Thus (3) must hold during natural time, i.e. whenever $x(\tau) = 1$, as must the complementary slackness conditions $y_{ij}(\tau)S_{ij}(\tau) = d_{ij}(\tau)S_{ij}(\tau) = 0$. In particular, jumps in the state variable, $y_{ij}(\tau) > 0$, can only occur when $S_{ij}(\tau) = 0$. This is analogous to condition (13) on page 243 of Kamien and Schwartz (1991).

Next turn to the costate equation for matching rates, which states that the partial derivative of the Hamiltonian with respect to the state variable $m_{ij}(\tau)$ plus the time derivative of the shadow value of a new match $e^{-rt(\tau)}S_{ij}(\tau)$ is equal to zero. In natural time, this reduces to equations (4) and (5) in the text. During a stoppage in natural time, $\dot{t}(\tau) = x(\tau) = 0$, while the artificial time Hamiltonian H_a does not depend on m_{ij}. This means that $S_{ij}(\tau)$ is constant when natural stops, and so we can integrate up the costate equation in natural time to obtain (6).

Finally, look at the costate equation for natural time, which imposes that the derivative of the Hamiltonian with respect to the state variable t plus the time derivative of the shadow value $e^{-rt(\tau)}z(\tau)$ is equal to zero. Letting $H_a(\tau)$ denote the value of the artificial Hamiltonian at τ, this implies

$$\dot{z}(\tau) = rx(\tau)(H_a(\tau) + z(\tau))$$

A transversality condition pins down the initial value of $z(\tau)$ to ensure that this differential equation does not explode. But more importantly, note that $z(\tau)$ is constant during stoppages in natural time. Moreover, the Hamiltonian H_a is equal to zero during these stoppages since optimality implies $y_{ij}(\tau)S_{ij}(\tau) = 0$. Thus continuity of the artificial time Hamiltonian H_a implies that it is equal to zero just before and just after a break in natural time. That is, the standard natural time Hamiltonian (7), which differs from H_a in natural time only by the shift term $z(\tau)$, is continuous in natural time, even across jumps in the state variable. This is equivalent to condition (17) on page 243 in Kamien and Schwartz (1991).

B The Hamiltonian = The Average Present Value

Consider the following generic optimal control problem:

$$V(t) \equiv \max \int_t^\infty e^{-r(s-t)}\phi(x(s), y(s))ds$$
$$\text{subject to } \dot{x}(t) = \psi(x(t), y(t)) \text{ and } y(t) \in Y$$

Assume ϕ and ψ are continuously differentiable. The vector of state variables x follows the stated law of motion, while the control variables y may be constrained to lie in a set Y. Note that we assume the state variables are continuous, which can be justified by appealing to the artificial time extension of the model in Appendix A.

Put a multiplier $e^{-r(s-t)}\lambda(s)$ on the constraint in each period and add it to the objective:

$$V(t) \equiv \int_t^\infty e^{-r(s-t)}\big(\phi(x(s),y(s)) + \lambda(s)\big(\psi(x(s),y(s)) - \dot{x}(s)\big)\big)ds$$
$$= \int_t^\infty e^{-r(s-t)}\big(H(x(s),y(s),\lambda(s)) - \lambda(s)\dot{x}(s)\big)ds$$

Since the Hamiltonian is continuous in x, y, and λ, and x and λ are continuous over time, using the value of the control variable y from a moment before or a moment after s will have only a second order effect on the value of the Hamiltonian. For example, with a standard concave problem, the derivatives of the Hamiltonian with respect to the control variables are zero optimally, and so the claim follows trivially from the continuity of the control variable in such a problem. This observation enables us to perform integration by parts on the integrated Hamiltonian, without worrying about how the control variable changes over time, and how those changes affect the Hamiltonian:

$$\int_t^\infty e^{-r(s-t)}H(x(s),y(s),\lambda(s))ds = \frac{H(x(t),y(t),\lambda(t))}{r} + \frac{1}{r}\int_t^\infty e^{-r(s-t)}\big(H_x(x(s),y(s),\lambda(s))\dot{x}(s) + H_\lambda(x(s),y(s),\lambda(s))\dot{\lambda}(s)\big)ds$$

where H_x and H_λ represent the partial derivatives of the Hamiltonian. Use the costate equation $H_x(x(s),y(s),\lambda(s)) = r\lambda(s) - \dot{\lambda}(s)$ to eliminate H_x from this expression, and the trivial relationship $H_\lambda(x(s),y(s),\lambda(s)) = \psi(x(s),y(s)) = \dot{x}(s)$ to get rid of H_λ. Making these substitutions into the equation for $V(t)$ yields $rV(t) = H(x(t),y(t),\lambda(t))$, as desired.

References

Arrow, Kenneth and Mordecai Kurz, *Public Investment, the Rate of Return, and Optimal Fiscal Policy*, Baltimore: Johns Hopkins University Press, 1970.

Burdett, Kenneth and Melvyn Coles, "Marriage and Class," *Quarterly Journal of Economics*, 1997, *112* (1), 141–168.

—— **and** —— , "Separation Cycles," *Journal of Economic Dynamics and Control*, 1998, *22* (7), 1069–1090.

Coles, Melvyn and Randall Wright, "A Dynamic Equilibrium Model of Search, Bargaining, and Money," *Journal of Economic Theory*, 1998, *78* (1), 32–54.

Colonius, Fritz, *Optimal Periodic Control*, Berlin: Springer-Verlag, 1988.

Diamond, Peter, "Aggregate Demand Management in Search Equilibrium," *Journal of Political Economy*, 1982, *90* (5), 881–894.

___ **and Drew Fudenberg**, "Rational Expectations Business Cycles in Search Equilibrium," *Journal of Political Economy*, 1989, *97* (3), 606–619.

Kamien, Morton and Nancy Schwartz, *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, second ed., New York: Elsevier Science Publishing, 1991.

Pissarides, Christopher, *Equilibrium Unemployment Theory*, second ed., Cambridge, MA: MIT Press, 2000.

Sattinger, Michael, "Search and the Efficient Assignment of Workers to Jobs," *International Economic Review*, 1995, *36* (2), 283–302.

Shimer, Robert and Lones Smith, "Assortative Matching and Search," *Econometrica*, 2000, *68* (2), 343–370.

___ **and** ___ , "Matching, Search, and Heterogeneity," *Advances in Macroeconomics*, 2001, *1* (1), Article 5.

Vind, Karl, "Control Systems with Jumps in the State Variables," *Econometrica*, 1967, *35*, 273–277.