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NOTES AND COMMENTS

NECESSARY AND SUFFICIENT CONDITIONS FOR THE PERFECT FINITE HORIZON FOLK THEOREM

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1. INTRODUCTION

FOR A GIVEN n-PLAYER normal form game G, let $G(\delta, T)$, be the T-fold repeated game where the objective function is the average discounted sum of stage payoffs. For this finitely-repeated game, the *perfect folk theorem* is said to hold if the set of subgame perfect equilibrium (SPE) payoffs includes any feasible and strictly individually rational payoff vector of G for large enough $T < \infty$ and $\delta < 1$. Benoit and Krishna (1985) (hereafter BK) produced a perfect folk theorem for $G(\delta, T)$ that obtains when (i) G satisfies the sufficient conditions for the infinite-horizon folk theorem—namely, the full-dimensionality condition of Fudenberg and Maskin (1986), and (ii) each player has distinct Nash payoffs in G.

Recently, Abreu, Dutta, and Smith (1994) discovered an (essentially) iff condition for (i)—*nonequivalent utilities* (NEU)—that neatly supplants full-dimensionality. Wen (1994) subsequently examined games possibly violating NEU, and proved a revised general "folk theorem." Here I provide a simple, easily verifiable, and *necessary and sufficient* condition for the finite-horizon folk theorem that replaced (ii): The intuitive condition, *recursively distinct Nash payoffs*, only insists that players' behavior be *iteratively* leveraged near the end of the repeated game.

The primary contribution of this note is purely conceptual: I wish to finish the work of BK,[2] and thus complete the perfect information folk theorem program. Indeed, just as NEU only differed from full-dimensionality by a nongeneric class of games, so too, within the class of stage games with *recursively* distinct Nash payoffs, the measure without distinct Nash payoffs for all players is zero. But conceptual clarity is not without its own reward, as I later provide a necessary and sufficient condition for the finite-horizon *Nash* folk theorem. I also recast BK in Wen's more encompassing framework.[3] In so doing, I happen upon a simple new proof of the necessity of NEU in Abreu et al. (1994), and of what "necessity" means in Wen's tiered "folk" theorems more generally.

The logic of the BK folk theorem, as best exemplified in Krishna (1989) or Smith (1990/92) is rather simple: Late in the repeated game, because all players have distinct Nash payoffs, the behavior of any one of them can be *leveraged* by threatening to finish off with a fixed number (say S) of plays of that player's worst Nash payoff, rather than cycle through all his best Nash payoffs. Away from the end of the game, the infinite-horizon punishments work perfectly well. And because S is fixed independent of the horizon length, the effect on the average payoff can be made arbitrarily small. This essentially is their proof.

[1] The motivation for this note, namely seeking the *necessary* conditions for a folk theorem, stemmed from collaboration with Dilip Abreu and Prajit Dutta, and in particular from their earlier work (Abreu and Dutta (1991)). The current version reflects a wealth of constructive comments and corrections by two referees and a co-editor.

[2] It should be noted that BK conjecture under weak conditions that only one player need have distinct payoffs. This paper might in part be seen as proving a precise and rigorous statement of this conjecture.

[3] I am grateful to a referee for pressing me on this point.

2, 2, 3	2, 2, 2	2, 2, 2
2, 2, 2	2, 2, 2	2, 2, 2
2, 2, 2	2, 2, 2	−1, −1, 0

2, 1, −1	0, −1, −1	0, −1, −1
−1, 0, −1	−1, −1, −1	−1, −1, −1
−1, 0, −1	−1, −1, −1	−1, −1, −1

FIGURE 1

Rather than formally define my condition, I first motivate it with the three-player example game $\tilde{G}$ in Figure 1. In this game, 1 chooses rows (actions U, M, D), 2 chooses columns (actions l, m, r), and 3 chooses matrices (actions L, R). Player 3 strictly prefers L to R, while action D (resp. r) is weakly dominated for player 1 (resp. player 2). Thus it is easy to see that the only Nash (pure or mixed) payoffs are all convex combinations of (2, 2, 2) and (2, 2, 3). Furthermore, the minimax payoff is 0 for all three players.[4]

Because players 1 and 2 have unique Nash payoffs, $\tilde{G}$ does not satisfy the BK condition (ii). Nonetheless, a folk theorem does obtain! For $\tilde{G}$ satisfies NEU, and thus the standard infinite-horizon folk theorem applies to $\tilde{G}$. Next, player 3 enjoys the distinct (extremal) Nash payoffs of 2 and 3, so that this behavior is leveraged near the end of the game: I need only threaten to switch from (U, l, L) to (M, l, L) for the S-period phase. By choosing S large enough, 3 is willing to play R for as many periods, say S', as I wish just prior to this Nash phase. When player 3 plays R irrespective of what players 1 and 2 do, a new game $\tilde{G}(R)$ is *induced* for players 1 and 2. It has the unique Nash equilibrium payoff vector (2, 1). So player 2's Nash payoff from $\tilde{G}(R)$ is 1, which differs from his unique Nash payoff in $\tilde{G}(R)$ (i.e., when player 3 plays L). I have now leveraged the behavior of player 2 near the end of the game! Iterate this process. By choosing S' (and by implication S) large enough, I can induce player 2 to play l, m, or r for as many periods, say S'', as I wish just prior to this penultimate "Nash" phase. In particular, players 2 and 3 are willing to play (r, R) irrespective of what player 1 does, yielding a new (one-player) game $\tilde{G}(r, R)$ for player 1. It has the unique optimal payoff of 0, which differs from 1's unique optimal payoff of 2 in $\tilde{G}(l, L)$. I have now also leveraged the behavior of player 1 near the end of the game! That $\tilde{G}$ satisfies a folk theorem now follows by the same proofs as before.

I summarize the above procedure by saying that $\tilde{G}$ has *recursively distinct Nash payoffs*. If a game G satisfies NEU, it is *sufficient* for the BK result: So long as such a recursive procedure eventually leverages the behavior of all players, then a perfect folk theorem obtains. And my new condition is *necessary* too for the general perfect finite-horizon "folk theorem." Namely, if for any such chain of recursive reductions, I cannot leverage the behavior of all players, then a "folk theorem" does not obtain.

The intuition behind the necessity is also rather simple. For any given horizon length T, a player's set of SPE payoffs is either point-valued, or it is not. Evidently, if the behavior of a player can be leveraged as above, then his SPE payoffs are multivalued for large enough T, while if his behavior cannot be leveraged, then he entertains a unique equilibrium payoff for all T. Clearly, in the latter case, I cannot possibly hope for anything more. For instance, in $\tilde{G}$ with no discounting, a player 3 has a multivalued SPE payoff set for all T, while players 2 and 1 only receive distinct SPE payoffs for $T \geq 5$ and $T \geq 19$, respectively.

The special payoffs in $\tilde{G}$ speak to the genericity of the distinct Nash payoff requirement when the folk theorem obtains. Indeed, if one player has distinct Nash payoffs, then the game has at least two pure and generically a third *mixed* Nash equilibrium. Thus, for generic games satisfying my condition, all players enjoy distinct Nash payoffs.

[4] But observe that no one player can simultaneously minimax the other two, a fact of some importance later on in my remarks on the necessity of NEU.

Section 2 focuses on the stage game, and describes my iterative procedure. I dwell on sufficiency (folk theorem) in Section 3.1, and necessity (in all respects) in Section 3.2.

2. THE STAGE GAME

2.1. *Basic Definitions*

Let $G = \langle A_i, \pi_i;\ i = 1, \dots, n\rangle$ be a finite normal form n-player game, where A_i is player i's finite set of actions, and $A = \times_{i=1}^{n} A_i$. Let player i's utility function be π_i: $A \to \mathbb{R}$, and set $\pi(a) \equiv (\pi_1(a), \dots, \pi_n(a))$. Let M_i be player i's mixed strategy set, with $M = \times_{i=1}^{n} M_i$. Simply write $\pi_i(\mu)$ for i's expected payoff under the mixed strategy $\mu = (\mu_1, \dots, \mu_n) \in M$.

The game G has *nonequivalent utilities* (NEU) if no two players' von Neumann Morgenstern utility functions are equivalent, i.e., for all i and j, $\pi_i(\cdot)$ is not a positive affine transformation of $\pi_j(\cdot)$. Denote the set of players as $\mathscr{I} = \{1, 2, \dots, n\}$. Following Wen (1994), let $\{\mathscr{I}_b \subseteq \mathscr{I},\ b = 1, \dots, B\}$ be the coarsest partition of $\mathscr{I}$ such that any two players in the same $\mathscr{I}_b$ have equivalent utilities; denote by $\mathscr{I}(i)$ all players with equivalent utilities to i. Normalize payoffs so that $\pi_i(\cdot) \equiv \pi_j(\cdot)$ for all $j \in \mathscr{I}(i)$. The *effective minimax* payoff level $\pi_i(w^i) = \min_a \max_{j \in \mathscr{I}(i)} \max_{a_j} \pi_i(a_j, a_{-j})$ of player i is the best payoff that any one in $\mathscr{I}(i)$ can guarantee himself, i.e. the greatest minimax payoff over all players in $\mathscr{I}(i)$.[5]

Normalize $\pi_i(w^i) = 0$ for all i. Define $F = \text{co}\{\pi(\mu)\colon \mu \in M\}$, and call $F^* = \{w \in F\colon w_i > 0$, for all $i\}$ the *feasible and* (*strictly*) *rational* payoff set. To sidestep trivialities, let $F^* \neq \varnothing$. Note that under NEU, w^i reduces to the standard minimax strategy for player i, and F^* the feasible and (strictly) *individually rational* payoff set.

2.2. *Recursively Distinct Nash Payoffs*

Given a subset of players[6] $\mathscr{J}' = \{j_1, j_2, \dots, j_m\} \subset \mathscr{I}$ and their (possibly mixed) actions

(1) $$a_{\mathscr{J}'} \equiv (a_{j_1}, a_{j_2}, \dots, a_{j_m}) \in M_{j_1} \times M_{j_2} \times \cdots \times M_{j_m} \equiv M_{\mathscr{J}'},$$

let $G(a_{\mathscr{J}'})$ be the induced $(n-m)$-player game for players $\mathscr{I} \setminus \mathscr{J}'$ obtained from G when the actions of players $\mathscr{J}'$ are fixed to $a_{\mathscr{J}'}$.

Define a *Nash decomposition* of G as an increasing sequence of $h \geq 1$ nonempty subsets of players from $\mathscr{I}$, namely

(2) $$\{\varnothing = \mathscr{J}_0 \subset \mathscr{J}_1 \subset \mathscr{J}_2 \subset \cdots \subset \mathscr{J}_h \subseteq \mathscr{I}\},$$

so that for $g = 1, \dots, h$, actions $e_{\mathscr{J}_{g-1}}, f_{\mathscr{J}_{g-1}} \in M_{\mathscr{J}_{g-1}}$ exist with a pair of Nash payoff vectors $y(e_{\mathscr{J}_{g-1}})$ of $G(e_{\mathscr{J}_{g-1}})$ and $y(f_{\mathscr{J}_{g-1}})$ of $G(f_{\mathscr{J}_{g-1}})$ different *exactly* for players in $\mathscr{J}_g \setminus \mathscr{J}_{g-1}$, i.e.

(3) $$y(e_{\mathscr{J}_{g-1}})_i \neq y(f_{\mathscr{J}_{g-1}})_i$$

for all $i \in \mathscr{J}_g \setminus \mathscr{J}_{g-1}$. For instance, players in $\mathscr{J}_1$ have distinct Nash payoffs in G.

The game G has *recursively distinct Nash payoffs* if there is a Nash decomposition with $\mathscr{J}_h = \mathscr{I}$. So if, as in BK, all players have distinct Nash payoffs in G, then this condition holds. The converse is not true, as illustrated earlier. I later consider games G that do

[5] Wen (1994) does not describe it as such, but it is true. Indeed, w^i is certainly a minimax profile for some player $j \in \mathscr{I}(i)$. To see that $\pi_i(w^i)$ weakly exceeds any minimax payoff for all $j \in \mathscr{I}(i)$, let $\hat{\pi}_j(a) \equiv \max_{a'_j} \pi_j(a'_j, a_{-j})$ for $j \in I$. Analogous to the fact that the minimax $\geq$ maximin in constant sum games,

$$\pi_i(w^i) = \min_a \max_{j \in I} \hat{\pi}_j(a) \geq \max_{j \in I} \min_a \hat{\pi}_j(a) = \max_{j \in I} \left(\min_{a_{-j}} \max_{a_j} \pi_j(a_j, a_{-j}) \right).$$

[6] Throughout, $A \subset B$ means that A is a *strict* subset of B, i.e. $A \subseteq B$ and $A \neq B$.

not have recursively distinct Nash payoffs, i.e., $\mathscr{I}_h = \mathscr{I}$ is impossible. An ambiguity may then arise, since Nash decompositions need not be unique. It is not inconceivable that the union of *all* Nash decompositions could include every player. In fact, this cannot occur.[7]

LEMMA: *There is a well-defined maximal set of players $\mathscr{I}^* \subseteq \mathscr{I}$ who have recursively distinct Nash payoffs.*

3. FINITELY-REPEATED GAMES

3.1. *Sufficiency of Recursively Distinct Nash Payoffs*

I shall analyze finitely-repeated games with *perfect monitoring*, allowing each player to condition his current actions on the past actions of all players.

Let $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{iT})$ be a behavior strategy for player i, and $\pi_{it}(\alpha)$ his expected payoff in period t with the strategy profile α. Player i's objective function in $G(\delta, T)$ is the expected discounted sum of his payoffs: $((1-\delta)/(1-\delta^T))\Sigma_1^T \delta^{t-1}\pi_{it}(\alpha)$. The set of SPE payoffs is $V(\delta, T)$.

Beyond replacing the distinct Nash payoff requirement, Theorem 1 differs from BK's Theorem 3.7 in three ways. First, it admits payoff discounting, where δ and T can vary independently over the relevant range. Second, it is a more general "folk theorem," as pursued by Wen (1994), which will imply BK's result with NEU. Third, unlike the (more intuitive) use of long deterministic cycles in BK, *public randomization* is used: In every period, players can condition on the outcome of a publicly observed exogenous continuous random variable.[8] Just as in BK, I shall assume that players can observe deviations within the support of strictly mixed strategies. Alternatively, just assume that the folk theorem refers to the pure strategy minimax payoff level.[9]

THEOREM 1 (The "Folk Theorem"): *Suppose that the stage game G has recursively distinct Nash payoffs. Then for the finitely-repeated game $G(\delta, T)$, $\forall u \in F^*$ and $\forall \varepsilon > 0$, $\exists T_0 < \infty$ and $\delta_0 < 1$ so that $T \geq T_0$ and $\delta \in [\delta_0, 1] \Rightarrow \exists v \in V(\delta, T)$ with $\|v - u\| < \varepsilon$.*

PROOF: Fix a Nash decomposition (2) for which inequality (3) obtains, and define

$$c_g = \min_{i \in \mathscr{I}_g \setminus \mathscr{I}_{g-1}} \left\| y(e_{\mathscr{I}_{g-1}})_i - y(f_{\mathscr{I}_{g-1}})_i \right\| > 0$$

for $g = 1, 2, \ldots, h$. Let the $\rho > 0$ be the largest payoff range (i.e. the difference between best and worst payoffs) for any player in G, and $\psi_g(k)$ the least even number above $2k\rho/c_g$. Saving on notation, let y^g denote $y(e_{\mathscr{I}_{g-1}})$ in even periods and $y(f_{\mathscr{I}_{g-1}})$ in odd ones. Further, let $z^{g,i}$ be the *less preferred* Nash payoff vector amongst $y(e_{\mathscr{I}_{g-1}})$ and $y(f_{\mathscr{I}_{g-1}})$ for player $i \in \mathscr{I}_g \setminus \mathscr{I}_{g-1}$. Since for all $i \in \mathscr{I}_g \setminus \mathscr{I}_{g-1}$,

$$-k\rho + \psi_g(k) y_i^g > \psi_g(k) z_i^{g,i},$$

it follows that without payoff discounting, any player i has a strict incentive to conform to k consecutive periods of any action profile followed by $\psi_g(k)$ periods of y^g, if deviations are punished by switching each y^g to $z^{g,i}$.

[7] The proof of this result is omitted; see my (1994) working paper.

[8] One may also take advantage of the correlating device to produce a nearly *exact*, rather than an *approximate*, folk theorem. That is, one can use the correlation device to do away with the ε-approximation for all payoff vectors not on the boundary of F. I omit such a laborious exercise.

[9] This assumption is not necessary to detect deviations from the minimax phase (step 3, below), as noted in my (1994) working paper. But for the two recursive Nash phases, no such workaround exists. It would thus be interesting to know whether Theorem 2 goes through when $\mathscr{I}^*$ is defined given pure actions (1), in which case this would not be a concern.

For any $m > 0$, recursively define $s_h(m) = \psi_h(m)$ and

$$s_g(m) = \psi_g\big(m + s_{g+1}(m) + \cdots + s_h(m)\big)$$

for $g = h-1, h-2, \ldots, 1$. Define $t_0(m) = 0$ and $t_g(m) = s_1(m) + \cdots + s_g(m)$ for $g = 1, \ldots, h$. The T-period equilibrium outcome sequence is

$$a, \ldots, a; y^h, \ldots, y^h; \ldots; y^1, \ldots, y^1$$

where a is played for $T - t_h(q+r)$ periods,[10] and y^g is played for $s_g(q+r)$ periods. Fix $\varepsilon > 0$. If v is the δ-discounted average payoff vector, then $\|v - u\| < \varepsilon$ for big enough δ and T.

Abreu et al. (1994) have established the existence of feasible payoff vectors $x^1, \ldots, x^n$ such that for all $i \neq j$, $x^i \gg 0$ (strict IR), $x_i^i < x_i^j$ for all $j \notin \mathscr{I}(i)$ (payoff asymmetry), and $x_i^i < u_i$ (target payoff domination).

I now explicitly describe the players' strategies which support this equilibrium.[11] For ease of exposition, *late* deviations are those occurring during the final $q + r + t_h(q+r)$ periods of the repeated game; all others are called *early* deviations.

1. Main Path: Play a until period $T - t_h(q+r)$. (*If any i deviates early, start* 3; *if some player in $\mathscr{J}_{g'}$ deviates late, start* 5.)
2. Good Recursive Nash Phase: For $g = h, \ldots, 1$: Play y^g in periods $T - t_g(q+r) + 1, \ldots, T - t_{g-1}(q+r)$. (*If some $i \in \mathscr{J}_{g'}$ deviates late, where $g' < g$, start* 5.)
3. Minimax Phase: Play w^i for q periods. (*If $j \notin \mathscr{I}(i)$ deviates, start* 4.) Set $j \leftarrow i$.
4. Reward Phase: Play x^j for r periods. (*If any i deviates early, restart* 3; *if some $i \in \mathscr{J}_{g'}$ deviates late, start* 5.) Then return to step 1 or 2.
5. Bad Recursive Nash Phase: Play $z^{g',i}$ until period $T - t_{g'-1}(q+r)$. (*If $j \in \mathscr{J}_{g''}$ deviates, where $g'' < g'$, set $g' \leftarrow g''$ and $i \leftarrow j$ and restart* 5.) Then go to step 2.

It is straightforward (or see my (1994) working paper) to verify that these strategies constitute an SPE for some q, r and big enough δ_0 and T_0.[12]

3.2. *Necessity of Recursively Distinct Nash Payoffs*

If a game G does not have recursively distinct Nash payoffs, the consequences are rather stark. The following result has a flavor of the "zero-one" laws of probability theory. Namely, as $T \to \infty$, either $V(\delta, T)$ tends to the strictly individually rational payoff set F^*, or some players receive a *unique* SPE payoff. There is no middle ground.

Theorem 2: *For any $T < \infty$ and any $\delta \in (0,1]$, players in $\mathscr{I} \setminus \mathscr{J}^*$ receive a payoff in any SPE of $G(\delta, T)$, equal to their unique Nash equilibrium payoff of G.*[13]

Proof: Every player in $\mathscr{I} \setminus \mathscr{J}^*$ has a unique Nash and hence SPE payoff in $G(\delta, 1) \equiv G$. Assume that everyone in $\mathscr{I} \setminus \mathscr{J}^*$ has a unique SPE payoff in $G(\delta, T)$ equal to his Nash payoff in G, for $T = 1, \ldots, T_0$. Then in the first period of $G(\delta, T_0 + 1)$,

[10] Note that q and r are implicitly defined in steps 3 and 4 below.

[11] Below, i and j denote arbitrary players, and g, g', and g'' arbitrary indices in $\{1, 2, \ldots, h\}$. For clarity, I shall use the simple notation $j \leftarrow i$ to mean "assign j the value i." Also, steps always follow sequentially, unless otherwise indicated. Bracketed remarks refer to off-path play, i.e. following deviations.

[12] Absent public randomization, the simple direct proof presented above would require two modifications: First, the target outcome a would have to be replaced by an approximating finite outcome profile. Second, ditto for each x^i vector. Finally, to my knowledge, public randomization is essential if one wishes to work with the mixed minimax strategy, which I noted earlier was possible. See, for instance, Abreu et al. (1994).

[13] Note that this theorem says nothing about the payoff set of players in $\mathscr{J}^*$.

players in $\mathscr{I}\setminus\mathscr{I}^*$ must play a Nash equilibrium of $G(e_{\mathscr{I}^*})$, for any $e_{\mathscr{I}^*}\in M_{\mathscr{I}^*}$, because they have a unique SPE continuation payoff by assumption. By induction, the result obtains for all T. *Q.E.D.*

REMARKS: 1. This stark necessity result contrasts markedly with the partial failure of NEU, where we need only replace the minimax payoff level with the effective minimax payoff level.

2. Let $\tilde{F}^* = \{v \in F^* | v_i > \min_a \max_{j\in\tilde{\mathscr{I}}(i)} \max_{a_j}\pi_i(a_j, a_{-j})\ \forall i\}$, where the partition $\{\tilde{\mathscr{I}}_b\}$ strictly refines $\{\mathscr{I}_b\}$.[14] If ($\star$) *for no such partition* $\{\tilde{\mathscr{I}}_b\}$ *does there exist any strategy profile separately holding two or more players in any* $\mathscr{I}_b$ *to their worst feasible payoff in* $\tilde{F}^*$, then necessarily $\pi_i(w^i) > \min_a \max_{j\in\tilde{\mathscr{I}}(i)} \max_{aj}\pi_i(a_j, a_{-j})$, for $\tilde{\mathscr{I}}(i)\subseteq\mathscr{I}(i)$. But Wen (1994) proves that all SPE payoffs Pareto dominate the effective minimax outcome.[15] This yields "necessity" in another respect: The folk theorem fails when F^* is replaced by $\tilde{F}^*$. By corollary, *the necessity result of Abreu et al.* (1994) *follows*! For under NSM (*no simultaneous minimizing*, a particularization of ($\star$)), if NEU fails, then $\{\mathscr{I}_b\}$ admits a strict refinement, and the (standard) finite- or infinite-horizon folk theorem cannot obtain!

3. The recursively distinct Nash payoffs condition is also necessary and sufficient for the finite horizon Nash folk theorem, due to Benoit and Krishna (1987).[16]

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[14] That is, $\varnothing \neq \tilde{\mathscr{I}}(i) \subseteq \mathscr{I}(i)$ for all i, and $\tilde{\mathscr{I}}(i) \subset \mathscr{I}(i)$ for some i.

[15] Unlike his folk theorem, this result of his actually doesn't rely on observable mixed strategies. And it holds for finitely-repeated games too.

[16] I am grateful to a referee for suggesting this. For a proof of sufficiency, consider the following strategies: Let the equilibrium path follow steps 1 and 2. Off-path, retain the minimax phase, step 3, but allow it to last until the end of the game, and omit any reference to play after deviations, including step 4 (as it is not needed in a Nash equilibrium); finally, retain step 5, but omit the references to off-path play. To see necessity, just observe that Theorem 2 still obtains if SPE is replaced by Nash equilibrium.