Aspirational Bargaining[*]

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Abstract

This paper offers a noncooperative behaviourally-founded solution of the complete information bargaining problem where two impatient individuals wish to divide a unit pie. We formulate the game in continuous time, with unrestricted timing and content of offers. Reprising experimental work from 1960, we introduce and explore *aspirational equilibrium* — a Markovian refinement of subgame perfection where behaviour is governed by aspiration values (expected payoffs). The analysis is tractable, and generates many intuitive aspects of bargaining absent from the standard temporal monopoly paradigm.

We find that discounted aspiration values form a martingale, and thereby compute bounds on the expected bargaining duration. We also deduce some simple implications about consecutive offers, and relate delay times, offers, and acceptance rates. Finally, we draw into question a traditional comparative static: *Ceteris paribus,* more impatient players can expect more of the pie.

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1 Introduction

"The whole creation groans and yearns, desiderating a principle of arbitration..."

— Edgeworth, *Mathematical Psychics* (1881), part II, page 46

The classic bargaining problem presents two individuals with the opportunity to share a dollar if they can first agree on a partition. The rebirth of interest in this subject from a modern noncooperative approach dates to Rubinstein (1982), itself related to Ståhl (1972). He proposes a discrete time alternating offer model with payoff discounting.

We shall focus on three critical "realistic" elements of the above bargaining problem: (*i*) two risk neutral players $1, 2$ must divide a unit pie, by concurring on a feasible split; (*ii*) an agreement means a proposal by one player and an immediate acceptance by the other; (*iii*) the force for timely resolution is the players' impatience. Like Rubinstein (1982), we shall ignore the additional (and important) element of incomplete information.

Harsanyi (1956) observed: "As is well-known, ordinary economic theory is unable to predict the terms on which agreements tend to be reached in cases of ... bilateral monopoly. Only on the basis of additional *assumptions* does the theory of games furnish a determinate solution." Rubinstein's assumptions about the action space converted the intractable bilateral monopoly into a simple sequence of *temporal monopolies:* In his alternating offer bargaining model, players in turn are given the power to ask the other party to accept an offer, or to burn some of the pie by declining. This yielded a unique subgame perfect equilibrium (SPE) with an immediate agreement favouring the proposer.

Temporal monopoly captures situations where the time cost of each negotiation round plays a central role in the players' minds. While there are many situations with this critical feature, this effect intuitively should play no role in procedure-less settings. Players may well care about delay, and yet consider the reply time unimportant for this delay.

This paper explores the bargaining problem without temporal monopoly, and instead aims for a behaviourally motivated noncooperative solution. Since discrete time forces temporal monopoly, we shift to continuous time, where there are infinitely many SPE. View a discrete time model, or more generally temporal monopoly, as a restriction of the feasible action space to a subset of continuous time. We instead build the behavioural foundation into the solution concept — *aspirational equilibrium* (AE). In this Markovian refinement of SPE, players' behaviour is governed by their aspiration values, or expected payoffs in the game. While quite tractable as we will see, the resulting theory is richer than temporal monopoly, and offers many compelling new economic insights into bargaining.

Continuous Time Bargaining and Aspirational Equilibrium. As argued, to avoid the temporal monopoly outcome, we must allow players to react instantaneously, and thereby must start with a model nested from the outset in continuous time. Continuous time models introduce a host of well-known problems, most especially the twin issues that (*i*) outcome profiles need not be well-defined given strategies, and (*ii*) best replies need not exist. We construct an extensive form game that eludes both problems. This extensive form game uses the novel idea of a two-dimensional vector time and a restriction on histories that precludes the players from conditioning their actions on time. We succinctly summarize this feature of our extensive form as *sealed-envelope instructions* — at offer decision nodes, the players simultaneously commit to mixed (offer, time) pairs.

With this new structure, outcome profiles are well-defined, and we introduce four assumptions that refine SPE. The first three essentially ensure time stationary strategies, disallowing time-dependent offers, or bargaining as cheap talk or protocol. The fourth assumption asks that strategies coincide after any histories with the same expected payoffs.

The Intuitive Structure of Aspirational Equilibria. For a flavour of this refinement, consider an AE without immediate agreement. As strategies depend on expected payoffs alone and not on time, delay owes solely to the players' randomization. With strict time preference, there are rents from ending this bargaining hiatus. And since any player is indifferent about proposing, only the opponent strictly benefits from this offer. In other words, any delay implies that the players are locked in a *war of attrition*: Each strictly prefers that his counterpart and not himself stop the clock and propose. To summarize, an endogenous *'proposee' advantage* arises, since receiving an offer makes one better off. This is precisely opposite to the hard-wired proposer advantage with temporal monopoly, and is the ultimate source of the different implications between these bargaining paradigms.

Next consider what transpires when some player, say Mr. 1, finally tenders an offer to Mr. 2. The latter might accept it with probability one. If so, game over. Assume not. Agreement brings us to the Pareto frontier, and is efficient, while rejection incurs further delay, and is inefficient. Since Mr. 2 weakly prefers to reject, Mr. 1 must be disappointed by this outcome, suffering a strict payoff loss if Mr. 2 declines, and payoff jump otherwise. This yields several desirable and realistic bargaining features: (*i*) wars of attrition explain negotiation lags; (*ii*) serious offers are concessions; (*iii*) offers may be turned down; (*iv*) proposers are strictly disappointed from rejection, and strictly pleased by acceptance.

An AE specifies exogenously-given aspiration levels for the players — the initial state of a Markov process. Once Mr. 1 offers x, Mr. 2's aspiration value jumps to this new level x. Afterwards, the concession that is offered may yet be rejected. (See Figure 1.) This leads to a tractable Markov and martingale stochastic process on the space of possible pairs of discounted aspiration levels (Theorem 4). We can then conclude that bargaining almost surely ends in finite time; we also use this process to provide a simple lower bound on the duration of bargaining based upon observed alternating offers (Corollaries 3–4).

Contrast with Temporal Monopoly. Temporal monopoly in no way precludes endogenous timing — as Perry and Reny (1993), Sakovics (1993), and Stahl (1993) have clearly demonstrated. They posit continuous time bargaining games where players have tiny 'waiting' and 'reaction' times after offers. This temporal monopoly setting yields very small monopoly rents, and an accordingly small proposer advantage. Consequently, the normative predictions of our model do not obtain — wars of attrition and strict concessions. Since their results intuitively obtain for random but boundedly positive *mean* waiting times after offers, at first blush one might presume that our AE are merely special cases. But our timing is truly unrestricted — and to the player with an aspiration level approaching its highest level, offers must be tendered arbitrarily quickly (Theorem 3).

We explore some key links between when and what to offer, and what to accept. The second of consecutive offers by a player is more generous (Corollary 5). Also, incentive constraints force a trade-off between the offer content and delay time (Corollary 6), as well as the chance an offer is rejected and the surplus it concedes (Corollary 7). We show how some properties — sweetening an offer raises its acceptance chance (Corollary 8), for one — turn on a decreasing or convex aspiration set, which we also characterize (Lemma 8–9).

The distinction between a proposer and proposee advantage yields an enlightening contrast between the models. In our final result, we explore what happens if player 1 becomes more impatient. In the temporal monopoly framework, this increases player 2's temporal advantage, and forces both players to propose pie splits more favourable to player 2. In our aspirational framework, notwithstanding the multitude of equilibria, we believe that there is insight gained from posing this question. At the very least, it questions the validity of the conclusion in the temporal monopoly world. In a word, we ask what happens to the whole equilibrium set. We argue that there is a natural bijection between AE in the two worlds where player 1 is more and less impatient. Provided player 2 offers more rapidly, all incentive conditions are restored at the current aspiration levels. With this mapping, we can see that the final outcome splits in the AE with a more impatient player 1 place player 2 more often in the strategically disadvantageous offering role. We show in Theorem 5 that this raises the expected ultimate pie share of player 1.

Experimental Support for Aspirational Approach. The salience of aspirations in decision-making has long been established in the psychology literature (Siegel 1957). In fact, its significance in bargaining has long ago been investigated in laboratory tests. Siegel and Fouraker (1960) studied a simple buyer-seller bargaining game. The project's goal was to investigate the role of differential information on the bargaining split, seeing whether the better-informed party excelled. But among the "interesting implications", they conclude that "the basis of both the bargainer's 'expectancy' and, at least partially, of his 'bargaining strength' may very well be his *level of aspiration*" (p. 60). While the authors expected the 50-50 equal split under complete information, their experiments showed how aspirations acted as a dynamic anchor on the bids made, and explored how the exchange of offers affected these aspirations. A key role played by offers in an AE — to ratchet up the opponent's aspiration level — was observed in a specific experiment (pp. 80–81), as was the intertemporal non-monotonicity of a player's offer (pp. 77–90).

Relative to the literature, we take some liberty in our use of the term "aspiration." We take Siegel and Fouraker's phrase 'expectancy' literally, as our aspiration is a rational expectation and not a purely arbitrary benchmark or "reference point" against which gains and losses are compared (Gilboa and Schmeidler (1994)). For we wish to provide a non ad hoc basis for the aspirations discussed by the psychological studies. This gives the desired dynamic anchor for behavior sought by Siegel and Fouraker. A few papers (recently, Karandikar, Mookherjee, and Ray (1998)) have studied models where aspirations evolve over time. With our rational aspirations, the evolution is endogenous to the model.

Outline. Section 2 develops an extensive form with which we may formally speak of subgame perfect equilibria. This is essential, for in section 3, we develop our aspirational refinement of subgame perfection. Some immediate properties are explored in section 4, and sensitivity analysis is performed in section 5. The conclusion in section 6 links some of the details of our approach to the literature. We appendicize two technical proofs.

2 The Continuous Time Bargaining Model

A. Overview. Two players $i \in \{1,2\}$ (often denoted i, j) must split a unit pie. The players are impatient, discounting future payoffs by possibly different positive interest rates $r_1, r_2 > 0$. The players' outside options are each zero. All parameters are common knowledge. Bargaining transpires in real time on $[0,\infty)$. At a time of his choosing, either party may propose a pie split. Proposals must lie on the *Pareto frontier* where pie shares sum to one — and so may be summarized by the share $x \in [0,1]$ offered to the other party. To capture the irreversibility of tendering an offer and the risk of rejecting, offers are assumed final and 'exploding': They must be immediately either accepted — thereby ending the game — or rejected, with no explicit future commitment. (On the other hand, we shall see that in equilibrium, rejected offers will have *implied* future repurcussions.)

We now formally introduce the model and the notion of subgame perfection. In so doing, there are two main problems we must tackle. First, is the possibility of simultaneous offers and immediate counteroffers. For this, we introduce a notion of vector time, thereby nuancing between the simultaneous and the instantaneous. Second, and more delicate, the existence of a well-defined outcome path h^σ from a given strategy profile σ is problematic. For since the continuum is not well-ordered (there is no first time before or after a given moment), there need not be an initial historical cause for any current action profile.[1] To handle this problem, we develop a richer notion of an action space, and maintain a standard extensive form.[2] In our approach every outcome path of the extensive form is countably discrete. This concept will better reflect the fact that a strategy must be a *plan of action*. That is, it must dictate what happens now given the past history.

B. Real Time Bargaining. Let $S = [0,1] \cup \{Y, N\}$ be the set of the available actions when players 'speak', where $x \in [0,1]$ is the share of the unit pie offered *to the other player*, and the response is Y = 'yes' or N = 'no'. The *vector time domain* is $\mathcal{T} = [0,\infty) \times \mathbb{N}$. For any $(t,k) \in \mathcal{T}$, t refers to the *real time*, and k refers to the *artificial time* at any real moment t. That is, the second component k counts the number of events that occur at a moment in time, and increments whenever players reply instantaneously, thereby 'stopping the clock'.[3] Let $\succ$ denote the natural strict lexicographic order on $\mathcal{T}$, namely, $(t,k) \succ (t',k')$ if $t > t'$ or $t = t'$ and $k > k'$. Let $\succeq$ be the corresponding weak order.

A *path* is a countable subset $h \subset \{1,2\} \times S \times \mathcal{T}$ satisfying the following. Each element $(i, s, (t,k)) \in h$ is an *event*, where i is the player acting, s the action taken, and (t,k) the time. If $(i, s, (t,k)), (i', s', (t',k')) \in h$ then $(t,k) \neq (t',k')$. If $(i, s, (t,k)) \in h$ and $k > 1$, then there are $k-1$ preceding events at time t in h, namely $(i_\ell, s_\ell, (t,\ell)) \in h$,

[1] For instance, Bergin and MacLeod (1993) give an example of a continuum of outcome paths that are all *consistent* with a strategy profile, but not actually determined (or caused) by that profile.

[2] An alternative format employed by Bergin and MacLeod (1993) allows players to condition on the event history as well as the real time, thus formally producing a continuum of decision nodes. To tame the strategies, they introduce 'inertia' into the strategies. By contrast, Simon and Stinchcombe (1989) deliberately take the view of continuous time as very fine discrete time.

[3] We have been unable to find this approach elsewhere in the literature. An unrelated notion that also tries to deal with the inadequacies of the real time domain owes to Fudenberg and Tirole (1985). They introduce variable intensity atoms to handle entry in pre-emption games.

$\ell = 1, \ldots, k-1$; further, $s_\ell \in [0,1]$ for each odd ℓ, $s_\ell = N$ for each even ℓ, and $i_{2j-1} \neq i_{2j}$. If h contains an infinite sequence of events at real time t, then the first event after t (if one exists) must be an offer. Let $\mathcal{P}$ denote the set of paths.

C. Histories. For paths $h \in \mathcal{P}$, define $T_1(h) = \sup\{t \mid (i,s,(t,k)) \in h\}$ and $T_2(h) = \sup\{k \mid (i,s,(t,k)) \in h \text{ and } t = T_1(h)\}$. A *history* is a path that ends in some finite real time, and thereby belongs to $\mathcal{H} = \{h \in \mathcal{P} \mid T_1(h) < \infty\}$. A history $h \in \mathcal{H}$ has a *last event* if there exists an element $(i,s,(t,k)) \in h$ where $(t,k) = (T_1(h), T_2(h))$. We distinguish histories with 'last events' from those without one.

We partition $\mathcal{H}$ into five sets. First, histories in $\mathcal{H}_i$ have a standing offer by player i:

$$\mathcal{H}_i = \{h \in \mathcal{H} \mid (i,s,(T_1(h),T_2(h))) \in h \text{ and } T_2(h) < \infty \text{ is odd}\}.$$

By definition, when $T_2(h) < \infty$ is odd, the last event $(i,s,(T_1(h),T_2(h)))$ corresponds to an offer; i.e., $s \in [0,1]$. Similarly, when $T_2(h) < \infty$ is even, the last event corresponds to a response $s \in \{Y,N\}$. The set of all histories where the last offer has just been accepted is $\mathcal{H}_Y$. Similarly, $\mathcal{H}_N$ consists of the null history and all histories ending in a rejected offer. Finally, $\mathcal{H} \setminus \{\mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_Y \cup \mathcal{H}_N\}$ are all other histories with no last event. There are two types of such histories. First, histories $h \in \mathcal{H}_+$ have a *cluster point*, ending in a sequence of events $\{(i_n, s_n, (t_n, k_n)), n \in \mathbb{N}\}$, where $t_n \uparrow T_1(h) < \infty$. Second, histories $h \in \mathcal{H}_{++}$ 'end' in a *deadlock* at real time $T_1(h)$ if they have the property $T_2(h) = \infty$: there is a sequence of events $\{(i_n, s_n, (T_1(h), n)), n \in \mathbb{N}\}$, with $s_n = N$ for all n even.

A finite history uniquely identifies a decision node in the game tree. Here we also introduce an offer node "after" infinite histories associated with cluster points or deadlocks.

D. Actions and Sealed Envelope Instructions. As usual, a strategy profile $\sigma = (\sigma_1, \sigma_2)$ will map from $\mathcal{H}$ into an action set. We imagine that at each decision node, a player is called upon either to approve a tabled offer, or to submit his instructions (in an envelope to his bargaining agent) about when and what he will offer next. The *action set* $A_i(h)$ of player i at h described below reflects the facts that: (a) he cannot speak if he has just tendered an offer, or if the game has ended; (b) he must reply if an offer has just been tendered to him; (c) once an offer has been rejected, he must plan to propose an offer $x \in [0,1]$ after some *elapse time* in $[0,\infty]$, possibly immediately or never (zero or infinite elapse time); but (d) it is not feasible to propose an offer at the current moment (zero elapse time) after a deadlock, where there has been no last event. Altogether,

$$A_i(h) = \begin{cases} \emptyset & \text{if } h \in \mathcal{H}_i \cup \mathcal{H}_Y \\ \{Y,N\} & \text{if } h \in \mathcal{H}_j \\ [0,1] \times [0,\infty] & \text{if } h \in \mathcal{H}_N \cup \mathcal{H}_+ \\ [0,1] \times (0,\infty] & \text{if } h \in \mathcal{H}_{++}. \end{cases}$$

We call histories $h \in \mathcal{H}_j$ *reply nodes*, and histories $h \in \mathcal{H}_N \cup \mathcal{H}_+ \cup \mathcal{H}_{++}$ *offer nodes*. A *strategy* is a map σ on $\mathcal{H}$ such that $\sigma_i(h) \in A_i(h)$ for all $h \in \mathcal{H}$.

Hereafter, $\mathscr{B}(X)$ are the Borel measurable subsets of X (where the topology will be understood from the context), and $\Delta(X)$ the set of probability measures on $(X, \mathscr{B}(X))$. A *behaviour strategy* σ is a profile of mixtures $\sigma_i(h) \in \Delta(A_i(h))$ for all histories $h \in \mathcal{H}$.

E. Histories Generated from Strategies. We now define the outcome path $h^\sigma \in \mathcal{P}$ generated by a strategy profile σ. A naive method to construct $h(\sigma)$ is to start with the null history, and then (possibly forever) sequentially append the 'next event' associated with $\sigma(h)$ to the current history h. However, this may not be possible because it will never advance the real time past any deadlock or cluster point.

For any $(t,k),(\tau,\ell) \in \mathcal{T}$, define $(t,k) \oplus (\tau,\ell) = (t+\tau,\ell)$ if $\tau > 0$ and otherwise $(t,k) \oplus (0,\ell) = (t,k+\ell)$. For short, we write $(t,k) \oplus \tau$ instead of $(t,k) \oplus (\tau,1)$. For any path $h \in \mathcal{P}$, we define a *successor path* $\psi_\sigma(h)$ determined from h by the strategy profile σ. There are three cases:

- If $h \in \mathcal{H}_j$ then $\psi_\sigma(h) = h \cup \{(i, \sigma_i(h), (T_1(h), T_2(h)+1))\}$
- If $h \in \mathcal{H}_Y$ or $T_1(h) = \infty$ then $\psi_\sigma(h) = h$
- Assume $h \in \mathcal{H}_N \cup \mathcal{H}_+$ and $(x_i, t_i) = \sigma_i(h)$. If $t_1 = t_2 = \infty$, then $\psi_\sigma(h) = h$. Otherwise, $\psi_\sigma(h) = h \cup \{(i, x_i, T(h) \oplus t_i)\}$, where $i = 1$ if $t_2 \geq t_1$ and $i = 2$ if $t_2 < t_1$.

In the first case, there's a standing offer by $j \in \{1,2\}$, and i must respond at once. In the second case, the game has ended or bargaining lasts forever, since for any real time, there is always a next offer. The third case may be hardest to digest. It considers two possibilities with no offer on the table. First, both players may decide never to speak again. Alternatively, one or more players may plan to propose in finite time; if the offer is immediate, then only the artificial time advances. As an arbitrary tie-breaking rule, if both players speak simultaneously, we assume that only player 1 is heard.

For any $h \in \mathcal{P}$ and $(t,k) \in [0,\infty) \times \{1,\ldots,\infty\}$, let $h_{(t,k)} = \{\, (i,s,(\tau,\ell)) \in h \mid (\tau,\ell) \preceq (t,k) \,\}$. Observe that $h_{(t,k)} = \emptyset$ if h contains no events weakly before (t,k). Fix a pure strategy σ. An arbitrary path $h \in \mathcal{P}$ may be inconsistent with σ as it may contain events that are not generated by σ. For example, it may be that $\sigma(\emptyset) = ((x_1,10),(x_2,20))$, so that the offer by player 1 will arrive first. Then, any history h containing events before time 10 is incompatible with σ. Accordingly, let us define

$$H(\sigma) = \{\, h \in \mathcal{P} \mid \psi_\sigma(h_{(t,k)}) \subset h \;\; \forall\, (t,k) \prec T(h) \,\}.$$

The appendix proves that our extensive form results in a well-defined outcome profile:

Lemma 1 (Outcome Profiles) *The outcome path h^σ is well-defined by $h^\sigma = \inf\{h \in H(\sigma) | h = \psi_\sigma(h)\}$, where the infimum is taken w.r.t. set inclusion ($h^\sigma = \emptyset$ if $H(\sigma) = \emptyset$).*

F. Payoffs and Subgame Perfection. The payoff to a pure strategy profile σ with $h^\sigma \in \mathcal{H}_Y$ is the vector $\pi(\sigma) = (\pi_1(\sigma), \pi_2(\sigma))$, where $\pi_i(\sigma) = e^{-r_i t}(1-x)$ and $\pi_j(\sigma) = e^{-r_j t}x$ if player i made the final offer x at time $t = T_1(h^\sigma) < \infty$ and player j accepted it. Otherwise, if bargaining lasts forever, or no proposal occurs after some rejection, then $\pi(\sigma) = 0$. The payoff $\pi(\sigma)$ of a behavior strategy σ is defined by taking expectations.

A behavior strategy profile σ is a *subgame perfect equilibrium* (SPE) if for all $h \in \mathcal{H} \setminus (\mathcal{H}_i \cup \mathcal{H}_Y)$, any $(s,t) \in \text{supp}(\sigma_i(h))$ is a best reply to σ_j at h. This paper is in fact

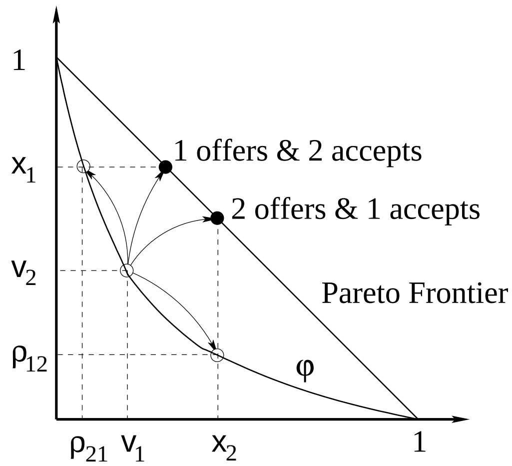

Figure 1: **Example Aspiration Set, Payoff Frontier, and Transitions.** Aspiration value pairs are denoted by ∘, and agreements by •. Lines indicate continuations following proposals. Proposals are sometimes turned down by his opponent who is indifferent.

entirely focused on a Markovian class of subgame perfect equilibria. To step back from the abstraction and fix ideas, we now give a simple example that we later revisit.

Example (Constant Acceptance Rate): In the example SPE, whenever one player's *aspiration value* (his expected payoff) is v_i, the other's will be $\varphi(v_i)$, where $\varphi : (0,1) \to (0,1)$ is the strictly decreasing function $\varphi(v_i) = (1 - v_i)/(1 + v_i)$. Note that $\varphi = \varphi^{-1}$ and so the graph $\mathscr{G}(\varphi)$ is symmetric, and that $v + \varphi(v) < 1$ for all $v \in (0,1)$.

Our mixed strategy equilibrium is Markovian with state space $\mathscr{G}(\varphi)$, and an arbitrary initial state in $\mathscr{G}(\varphi)$. At any $v \in \mathscr{G}(\varphi)$, each player i chooses a proposal $\bar{x}_i(v)$ and randomly chooses an elapse time from an exponential distribution with parameter $\lambda_i(v)$. If player 1's offer x_1 prevails, say, where possibly $x_1 \neq \bar{x}_1(v)$ (if player 1 deviates), then player 2 accepts it with fixed probability $\alpha \in (0,1)$ if $x_1 \geq \bar{x}_1(v)$, and rejects it otherwise. If the offer is rejected, the state moves to $(\varphi(x_1), x_1)$ if $x_1 \geq v_2$, and remains at v otherwise.

The functions λ_j and $\bar{x}_j$ $(j = 1,2)$ satisfy $v_i = (1-\alpha)\varphi(\bar{x}_i(v)) + \alpha(1 - \bar{x}_i(v))$. Hence, player i is indifferent about making the (equilibrium) offer $\bar{x}_i(v)$ or not, and

$$v_i = \int_0^\infty \lambda_j(v) e^{-[\lambda_j(v)+r_i]t} \bar{x}_j(v) dt = \frac{\lambda_j(v)}{\lambda_j(v) + r_i} \bar{x}_j(v), \tag{1}$$

so that player i is indifferent about proposing at any moment or waiting for his opponent to propose instead. It is easy to check that $v_i < (1-\alpha)\varphi(\bar{x}_i(v)) + \alpha(1-\bar{x}_i(v))$ when $x_i > \bar{x}_i(v)$; thus, player i does not want to make a disequilibrium offer. Notice that since $\bar{x}_i(v) > v_j$, the equilibrium offer is acceptable. Also, $\lambda_i(v) < \infty$ everywhere; thus, deadlocks almost surely do not occur. This also will be a general property of our aspirational equilibria.

Between offer events, the players engage in a waiting game to see who will make the next offer. Eventually, one player breaks down and proposes a pie split; that offer may be rejected. The bargaining state randomly transitions through $\mathscr{G}(\varphi)$ as offers are made and rejected until absorption on the Pareto frontier, when an offer is finally accepted.

3 The Aspirational Refinement

Any pie split at any time is an SPE of our continuous-time model. We now introduce an equilibrium refinement motivated by the earlier psychological study of Siegel and Fouraker (1960). We assume that players' bargaining aspirations govern their behaviour. We proceed via assumptions that ensure time-constant strategies and stationarity in payoffs.

3.1 Exponential Offer Times

The proper subgame after an offer node h is formally equivalent to the original game, after resetting the clock. To flesh this out, we need some notation. Introduce a forward time-shift operator $\Upsilon_{(t,k)}$ on histories. For all $h' \in \mathcal{H}$ and vector times (τ, ℓ), let

$$\Upsilon_{(\tau,\ell)}(h') \equiv \{(i, s, (\tau, \ell) \oplus (t, k)) | (i, s, (t, k)) \in h'\}.$$

Let h be an offer node and put $(\tau, \ell) = T(h)$. We now define the history: h followed by $h' \neq \emptyset$. If $\ell = \infty$, the first event in h' must occur after a positive elapse time. Then $h \cup \Upsilon_{(\tau,\ell)}(h')$ concatenates the prior history h with the new history h'. Thus for any offer node h, and any $h' \in \mathcal{H}$, define $\sigma_i|_h(h') = \sigma_i(h \cup \Upsilon_{(\tau,\ell)}(h'))$. Then $\pi_i(\sigma|_h)$ is player i's expected continuation value after h, discounted from time $T_1(h)$.

Given an offer node h and $t > 0$, let $\pi(\sigma|h, t)$ denote the expected payoff vector of following σ after history h (discounted from time $T_1(h) + t$), given that no intervening event has occurred after h in $[T_1(h), T_1(h) + t)$. Notice that $\pi(\sigma|h, 0) = \pi(\sigma|_h)$.

The first three properties of an SPE that we assume here are:

A1. *Action-time independence*: For all offer nodes h, we have $\sigma_i(h) = \sigma_i^x(h) \times \sigma_i^t(h)$, where $\sigma_i^x(h) \in \Delta([0, 1])$ and $\sigma_i^t(h) \in \Delta([0, \infty])$ are independent mixtures over offers and elapse times. Also, for all offer nodes h, $x \in [0, 1]$ and $t \geq 0$, the probability $\sigma_i(h \cup \{(j, x, T(h) \oplus t)\})$ that i accepts j's offer x does not depend on the real time t.

A2. *Payoff-time independence*: For all offer nodes h, the expected value $\pi_i(\sigma|h, t)$ is independent of $t \in \text{co}(\text{supp}(\sigma_j^t(h)))$.

A3. *Meaningful offers*: Equilibrium offers are accepted with strictly positive probability.

Action-time independence A1 asserts an independent randomization over offers and time. For while we have chosen a countably discrete extensive form to represent the game, we still wish to admit the possibility that players may reassess their strategies at any point in real time. Consider a discrete time bargaining game where in every period a player randomizes over silence and proposing, assuming the other player doesn't propose first. That this randomization is independent across periods is the analogue of our time stationarity assumption. The standard difficulties with assuming a continuum of independent randomizations in real time was another reason for our choice of extensive form.

While A1 precludes a drift in expected payoffs *between offer events*, it does not restrict the continuation values at offer events. Payoff-time independence A2 asserts that expected values can only be affected by proposal events. This precludes time as a coordination device, and thereby constrains continuation values. We thus focus on simple stationary strategies: Player i's expected value remains constant as long as it is possible that j makes an offer (the support restriction). To understand the convex hull proviso of A2, we have in mind, following A1, strategies where player j randomizes between making an offer or not at *every* moment.

Assuming meaningful offers A3 precludes offers as either cheap talk, payoff-irrelevant babbling, or equilibrium protocol. It cannot be common knowledge, for instance, that the first offer must be made and ignored. With our assumption in force, every offer is serious, and we are later able to make falsifiable predictions based on the offers made.

Lemma 2 *Let σ be an SPE satisfying* A1–A3. *Then for all offer nodes h, the mixture over offer times has an exponential distribution, say $F_{\lambda_i}(\tau) = 1 - e^{-\lambda_i \tau}$, where $\lambda_i \in [0, \infty]$.*

Proof: For any offer node h, let $\pi_1(\sigma|_{h\cup\{(2,x,T(h)\oplus t)})$ be player 1's expected value once player 2 makes the offer $(x, 1-x)$ at t units of elapsed time after history h. If this is an equilibrium offer, then by A3, there is a positive probability that player 1 will accept it. Since accepting the proposal is an optimal response, $x = \pi_1(\sigma|_{h\cup\{(2,x,T(h)\oplus t)\}})$. Using A1,

$$\bar{\pi}_1(t) = \textstyle\int_0^1 \pi_1(\sigma|_{h\cup\{(2,x,T(h)\oplus t)\}})\sigma_2^x(h)(dx) = \int_0^1 x\sigma_2^x(h)(dx)$$

is player 1's expected value immediately after receiving an (equilibrium) offer from player 2 at time $T_1(h) + t$, but before having seen its content. Clearly, $\bar{\pi}_1(t)$ does not depend on t, and thus it will be denoted simply by $\bar{\pi}_1$. By assumption A3, $\bar{\pi}_1 > 0$.

Let $G_2(\tau) = \sigma_2^t(h)([0,\tau])$ be the chance that player 2 makes an offer in the time interval $[T_1(h), T_1(h) + \tau]$. We first claim that either $G_2(t) = 1$ for all $t \geq 0$, or G_2 is absolutely continuous, i.e. having no atoms. The case $G_2 \equiv 1$ is clear for then $\text{supp}(G_2) = \{0\}$. Hereafter, we assume $G_2 \not\equiv 1$, and argue that G_2 is absolutely continuous. Now,

$$\pi_1(\sigma|h,t) = \bar{\pi}_1 \int_t^\infty e^{-r_1(\tau-t)} \left[\frac{dG_2(\tau)}{1 - G_2(t-)}\right] \equiv \bar{\pi}_1\phi(t),$$

where $G_2(t-) = \lim_{\epsilon\downarrow 0} G_2(t-\epsilon)$ for $t > 0$ and $G_2(0-) = 0$. But A2 then implies that $\phi(t)$ is constant on $\text{co}(\text{supp}(G_2))$. This is impossible if G_2 has an atom at any $t \geq 0$, for then $\phi(t) - \lim_{\tau\uparrow t}\phi(\tau) = \bar{\pi}_1[G_2(t) - G_2(t-)]/[1 - G_2(t-)] > 0$.

Let $g_2(t) = G_2'(t)$ be the density function, and $\lambda_2(t) = g_2(t)/[1 - G_2(t)]$ the hazard-rate function. Payoff-time stationarity says that $\pi(\sigma|h,t) = \hat{\pi}_1$ does not depend on t. So:

$$0 = -\bar{\pi}_1\lambda_2(t) + [r_1 + \lambda_2(t)] \int_t^\infty \bar{\pi}_1 e^{-r_1(\tau-t)} \left[\frac{g_2(\tau)}{1 - G_2(t)}\right] d\tau = -\bar{\pi}_1\lambda_2(t) + [r_1 + \lambda_2(t)]\hat{\pi}_1.$$

Hence $\lambda_2(t) = r_1\hat{\pi}_1/(\bar{\pi}_1 - \hat{\pi}_1) = \lambda_2$ is constant. Since $\lambda_2 \geq 0$, $\bar{\pi}_1 \geq \hat{\pi}_1$. This implies that $G_2(t) = 1 - c \cdot e^{-\lambda_2 t}$, where $c \in (0, 1]$. But, if $c < 1$, then $G_2(0) = 1 - c > 0$, so that G_2 has an atom at 0. Hence, $c = 1$. Therefore, $G_2 = 1 - e^{-\lambda_2 t}$ whether G_2 is absolutely continuous or places all probability mass at 0 (in which case we make $\lambda_2 = \infty$). □

3.2 Bargaining as a Continuous Time Markov Process

By action-time independence, offers are made at a constant rate, and from a time-invariant offer distribution. We next impose that after any event history, the players' behaviour depends exclusively on their current expected values.

A4. *Action-payoff stationarity*: $\pi(\sigma|_h)=\pi(\sigma|_{h'}) \Rightarrow \sigma|_h=\sigma|_{h'}$, for all offer nodes h, h'.

Action-payoff stationarity is the primary basis for our aspirational refinement of SPE. It asserts that the players' strategies are functions of their aspirations while bargaining.

We now show that an SPE obeying A1–A4 admits a simple Markovian structure, summarized by a state space $\mathscr{A}$, and a quintuple $(v^0, \lambda, \mu, \alpha, \rho)$, where

- $v^0 \in \mathscr{A} \equiv \{(v_1, v_2) \in \mathbb{R}^2_+ \mid v_1 + v_2 \leq 1\}$
- $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_i : \mathscr{A} \mapsto \mathbb{R}_+ \cup \{\infty\}$
- $\mu = (\mu_1, \mu_2)$ with $\mu_i : \mathscr{A} \mapsto \Delta([0, 1])$
- $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i : \mathscr{A} \times [0, 1] \mapsto [0, 1]$
- $\rho = (\rho_1, \rho_2)$ with $\rho_i = (\rho_{i1}, \rho_{i2}) : \mathscr{A} \times [0, 1] \mapsto \mathscr{A}$

and where (for well-defined expectations) λ_i, α_i and ρ_i are assumed (Borel) measurable functions, and $\mu_i(B|v) \equiv \mu_i(v)(B)$ is a measurable function of v for each $B \in \mathscr{B}([0, 1])$.

Such a quintuple recursively specifies an *aspirational profile* (AP), hereby denoted $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$, as follows: Starting at stage $n = 0$ with the initial state $v^0 \in \mathscr{A}$, each player i randomly and independently chooses a time t_i from the distribution $F_{\lambda_i(v^n)}$ (possibly ∞) to propose an offer x_i from the distribution $\mu_i(\cdot|v^n)$. If $t_1 \leq t_2$, then player 1's offer x_1 is made, and player 2 accepts it with chance $\alpha_2(v^n, x_1)$. If $t_1 > t_2$, then player 2's offer x_2 is made, and player 1 accepts it with chance $\alpha_1(v^n, x_2)$. If player i rejects the offer x_j (possibly not in $\text{supp}(\mu_j(v^n))$) then $v^{n+1} = \rho_i(v^n, x_j)$. Finally, increment n by 1.

An AP $\sigma = \bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ has a simple recursive structure which we now exploit. Let $\bar{\mathcal{H}}_N \equiv \{h \in \mathcal{H}_N \mid h \text{ is finite}\}$. After any history $h \in \bar{\mathcal{H}}_N$, the continuation strategy profile $\sigma|_h$ is another AP $\bar{\sigma}(v, \lambda, \mu, \alpha, \rho)$ with the same structure but a different initial state $v \in \mathscr{A}$. So any history $h = \{(i_1, x_1, t_1), (j_1, N, t_1), \ldots, (i_n, x_n, t_n), (j_n, N, t_n)\} \in \bar{\mathcal{H}}_N$, with rejected offers at times $t_1 \leq t_2 \leq \cdots \leq t_n$, generates a state vector $V_\rho(h|v^0)$ by repeatedly applying the function ρ. Let $v^{\ell+1} = \rho_j(v^\ell, x_\ell)$, where j is i_ℓ's opponent, $\ell = 0, \ldots, n-1$; then $V_\rho(h|v^0) = v^n$. Moreover, as noted above, $\sigma|_h = \bar{\sigma}(V_\rho(h|v^0), \lambda, \mu, \alpha, \rho)$.

So far, our definition of an AP $\sigma = \bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ only specifies players' actions after *finite* histories. We still must specify actions after infinite histories (i.e., cluster points and deadlocks). For our purposes, we can prescribe arbitrary behavior after any infinite histories that occur with probability 0 in an SPE σ. In that case we can pick an arbitrary $v^* \in \mathscr{A}$ and define $\sigma|_h = \bar{\sigma}(v^*, \lambda, \mu, \alpha, \rho)$ for all $h \in \mathcal{H}_+ \cup \mathcal{H}_{++}$. But if σ produces cluster points with positive probability, the extension to infinite histories cannot be arbitrary.

Definitions A strategy profile σ *coincides* with an AP $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ if for any offer node h, $\sigma|_h = \bar{\sigma}(\pi(\sigma|_h), \lambda, \mu, \alpha, \rho)$. An *aspirational equilibrium* (AE) is an SPE σ that coincides with an AP $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$.

Observe that if σ coincides with an AP $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ then after any offer node h, the state of the AP coincides with the value of $\sigma|_h$.

Theorem 1 *Let σ be an SPE satisfying* A1–A4*. Then σ is an AE $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$.*

Proof: Let σ be an SPE obeying A1–A4. We first define $(v^0, \lambda, \mu, \alpha, \rho)$. For each offer history h, let $v = \pi(\sigma|_h)$. Given A1–A3, Lemma 2 asserts that after h, players' offer times follow exponential distributions, with parameters $\bar{\lambda}_1, \bar{\lambda}_2$. Put $\lambda_i(v) = \bar{\lambda}_i$, for $i = 1, 2$. By assumption A1, the offer distributions $\sigma_i^x(h)$ and $\sigma_i^t(h)$ are independent; let $\mu_i(v) = \sigma_i^x(h)$. Given the atomless exponential distribution of proposal times, in equilibrium, ties and instantaneous offers almost surely do not occur (and so do not affect expectations).

Assume that after the offer history h, player i proposes first. Player j accepts an offer x_i at time $T_1(h) + t$ with chance $\alpha_j(v, x_i) = \sigma_j(h \cup \{(i, x_i, (T_1(h) + t, 1))\}) = \gamma_j(\{(i, x_i, (t, 1))\})$, where $\gamma = \sigma|_h$. By A4, γ depends on h only through $v = \pi(\sigma|_h)$, and by A1, $\gamma_j(\{(i, x_i, (t, 1))\})$ does not depend on t, so that α_j is well defined. For such an offer, define the tail history $h' = \{(i, x_i, (t, 1)), (j, N, (t, 2))\}$, and let $\rho_j(v, x_i) = \pi(\gamma|_{h'})$. Again, ρ_j is well defined because γ only depends on v.

Fix a history h, and let t^* be the last time of any cluster point or deadlock in h, if any, or set $t^* = 0$. Decompose $h = h^* \cup \Upsilon_{(t^*,1)}(h')$, where $h' \in \bar{\mathcal{H}}_N$, and let $v^* = \pi(\sigma|_{h^*})$ and $v = \pi(\sigma|_h)$. By definition, $\sigma|_h = \bar{\sigma}(v^*, \lambda, \mu, \alpha, \rho)|_{h'} = \bar{\sigma}(v, \lambda, \mu, \alpha, \rho)$ since $v = V_\rho(h'|v^*)$. Hence, $v = \pi(\bar{\sigma}(v, \lambda, \mu, \alpha, \rho))$. □

3.3 IC Condition Characterization of Aspirational Equilibrium

For any strategy profile σ, let $\mathscr{A}(\sigma)$ denote its set of continuation values (on and off the outcome path implied by σ). Formally,

$$\mathscr{A}(\sigma) = \{\pi(\sigma|_h) \mid h \text{ is an offer node}\} \subset \mathscr{A}.$$

We now identify an intuitive weak *(mutual) individual rationality* (IR) requirement for offers — it must provide both players with nonnegative surplus over their current values.

Lemma 3 *If σ is an AE $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$, then supp$(\mu_j(v)) \subset [v_i, 1 - v_j]$ for all $v \in \mathscr{A}(\sigma)$.*

Proof: After an offer $x_j \in \text{supp}(\mu_j(v))$ is made, player i's continuation value is x_j, and since $\alpha_i(v, x_j) > 0$, $x_j \geq v_i$. We claim that $x_j \leq 1 - v_j$. By contradiction, assume $x_j > 1 - v_j$. If i accepts this offer, j gets $1 - x_j < v_j$. Alternatively, if $\alpha_i(v, x_j) < 1$ and i rejects this offer, then i's continuation value is $\rho_{ii}(v, x_j) = x_j$. Since $\rho_i(v, x_j) \in \mathscr{A}$, we have $\rho_{ii}(v, x_j) + \rho_{ij}(v, x_j) \leq 1$ and $\rho_{ij}(v, x_j) \leq 1 - x_j$. So j's expected value after offering x_j equals $\alpha_i(v, x_j)(1 - x_j) + (1 - \alpha_i(v, x_j))\rho_{ij}(v, x_j) < v_j$. In either case player j's continuation value is less than v_j, a contradiction. Hence, $\text{supp}(\mu_j(v)) \subset [v_i, 1 - v_j]$. □

The players essentially bargain over the remaining *surplus* $1 - v_1 - v_2$. Since, by Lemma 3, any equilibrium offer x_j must obey $v_i \leq x_j \leq 1 - v_j$ at the aspiration pair (v_1, v_2), it must concede a fraction of the surplus. The *surplus concession fraction* $\kappa_j \in [0, 1]$ obeys

$$x_j = v_i + \kappa_j(1 - v_1 - v_2). \qquad (2)$$

For any AP $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$, we denote player i's *expected offer* by $\bar{x}_i(v) = \int_0^1 x\mu_i(v)(dx)$, for $v \in \mathscr{A}$, and the corresponding *expected surplus concession fraction* by $\bar{\kappa}_i(v) = [\bar{x}_i(v) - v_j]/[1 - v_1 - v_2]$. We now give the *incentive compatibility* conditions for an AE.

Theorem 2 (IC Conditions) *Assume σ coincides with an AP $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$. Then, σ is an AE iff for all $v \in \mathscr{A}(\sigma)$ and offers $x \in [0,1]$, v_i satisfies the* waiting IC equation (3)*, as well as the* offer IC equation (4)*, and the* acceptance IC equation (5) *below:*

$$v_i \geq \frac{\lambda_j(v)}{\lambda_j(v) + r_i}\bar{x}_j(v) \text{ with equality if } \lambda_i(v) < \infty \tag{3}$$

$$v_j \geq \alpha_i(v,x)(1-x) + (1-\alpha_i(v,x))\rho_{ij}(v,x) \text{ with equality if } x \in supp(\mu_j(v)) \tag{4}$$

$$\rho_{ii}(v,x) \begin{cases} = x & \text{if } \alpha_i(v,x) \in (0,1) \\ \leq x & \text{if } \alpha_i(v,x) = 1 \\ \geq x & \text{if } \alpha_i(v,x) = 0. \end{cases} \tag{5}$$

Proof: Let h be an offer node such that $v = \pi(\sigma|_h)$ satisfies $\lambda_i(v) < \infty$ and $\lambda_j(v) > 0$. After any such history, the equilibrium conditions are equivalent to (a) player i is willing to wait for j's offer; (b) player j is indifferent among his equilibrium offers, and cannot improve his expected value by deviating; (c) after rejecting x, player i expects at least x if he is required to reject x with positive probability, and no more than x if he is required to accept x with positive probability. Condition (a) was the original logic behind (1), which yields (3) with equality. Next, any offer x is accepted by j with probability $\alpha_j(v,x)$, and when it is rejected, i's aspiration value drops to $\rho_{ji}(v,x)$. Condition (b) is equivalent to (4). Finally, (5) follows from (c), where $\rho_{ii}(v,x) > x$ only if x is an off-path offer.

Next assume $\lambda_i(v) = \infty$, so that player i makes an offer immediately. In this case, he need not be willing to wait and (3) may hold with strict inequality. Similarly, if $\lambda_j(v) = 0$, player j makes no offers (even though by definition of σ he still must choose an offer for time ∞). In this case, (4) need not be satisfied with equality when $x \in \text{supp}(\mu_j(v))$. □

Suppose that $\lambda_1(v), \lambda_2(v) < \infty$ for all $v \in \mathscr{A}(\sigma)$. Then, $\bar{x}_i(v) > v_1$ by (3), while (4) asserts that i's expected continuation value upon offering is his current value v_i. In other words, the payoff to waiting exceeds that of offering. Players are thus engaged in a sequence of *wars of attrition* until agreement. Put differently, there is an endogenous *proposee advantage*, unlike the hard-wired proposer advantage with temporal monopoly.

It is easy to construct AE's using Theorem 2. We will build on the earlier symmetric aspiration function $\varphi(x) = (1-x)/(1+x)$. Observe that even with pure offers, the IC conditions allow two degrees of freedom for each player's choice of $(\lambda, \mu, \alpha, \rho)$. So fixing the aspiration set (and thereby ρ) still leaves one degree of freedom in choosing (λ, μ, α). Earlier, we also assumed a constant acceptance rate α. We now consider a constant surplus concession. In section 3.4, we shall see that a constant offer rate λ is impossible.

Example (Constant Surplus Concession): In a *κ-concession rule*, each player concedes a fixed surplus fraction $\kappa \in (0,1]$ whenever he makes an offer. By (3) and (4),

$$\lambda_i(v) = \frac{r_j v_j}{\kappa(1 - v_1 - v_2)} \quad \text{and} \quad \alpha_j(v,x) = \frac{v_i - \varphi(x)}{1 - x - \varphi(x)}.$$

(The definition of $\alpha_j(v,x)$ can be modified to strictly punish player i for deviant offers, if so desired.) Then one can easily verify that $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ is an AE for all $v^0 \in \mathscr{G}(\varphi)$.

Finally, an AE cannot have both a constant acceptance rate *and* constant conceded surplus fraction κ. For a constant κ is equivalent to a constant angle of incline from the aspiration vector v to the offer $(1-\bar{x}_1, \bar{x}_1)$. A constant acceptance rate then forces the aspiration set $\mathscr{A}(\sigma)$ to lie inside and parallel to the Pareto frontier. This contradicts Lemma 6 below, that $\mathscr{A}(\sigma)$ must approach the Pareto frontier at the extremes.

We can now rigorously establish one of our claimed 'intuitive' properties of an AE.

Corollary 1 *If j's equilibrium offer x_j is rejected with strictly positive chance, and rejection incurs more delay, then rejection strictly harms j, and acceptance strictly helps j.*

Proof: Let $v \in \mathscr{A}(\sigma)$ be the current value of the AE σ. Delay after rejection implies $\rho_{ii}(v, x_j) + \rho_{ij}(v, x_j) < 1$. By assumption, $0 < \alpha_i(v) < 1$, and therefore $\rho_{ii}(v, x_j) = x_j$ and $\rho_{ij}(v, x_j) < 1 - \rho_{ii}(v, x_j) = 1 - x_j$. Finally, by (4), v_i is the strict convex combination $\alpha_i(v, x_j)(1 - x_j) + (1 - \alpha(v, x_j))\rho_{ij}(v, x_j)$, and thus $\rho_{ij}(v, x_j) < v_j < 1 - x_j$. □

3.4 The Proposal Rate

A. Positive Aspirations. The aspiration value set $\mathscr{A}(\sigma)$ includes values that can only be reached after deviations from σ. For much of our analysis, only aspiration values that can be reached on the equilibrium path matter. Accordingly, let $\mathscr{A}_{IR}(\sigma) \subset \mathscr{A}(\sigma)$ be all value vectors that can be reached after histories with only IR offers. $\mathscr{A}_{IR}(\sigma)$ includes all equilibrium values and possibly some non equilibrium values, but it excludes, for example, values reachable only if one player's overly generous offer were rejected. (Assumptions will now be enumerated B1, B2, etc., since they instead constitute refinements of AE.)

B1. *No Extreme Offers*: No player i ever accepts the offer $x_j = 0$, and is never the first to make a full concession offer $x_i = 1$.

Assumption B1 does not rule out the possibility that the offer $x_i = 1$ be required in equilibrium after some history h. But, if $x_i = 1$ is required, it must be that along h, player i has already made the offer $x_i = 1$ (out of equilibrium) at some point.

Lemma 4 *If σ is an AE $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ satisfying B1, and $v \in \mathscr{A}(\sigma)$, then $v > 0$.*

Proof: (Part 1) By contradiction, suppose first that $(0, v_2) \in \mathscr{A}(\sigma)$ for some $v_2 \in (0, 1]$. If player 1 offers $x_1 = 1$ and player 2 rejects it, the continuation value is $(0, 1)$. Since $v_2 < 1$, player 1 never offered $x_1 = 1$ along the history leading to $(0, v_2)$. But starting at $(0, v_2)$, the continuation strategy must deliver the pie split $(0, 1)$. Thus, in equilibrium either 1 offers $x_1 = 1$ or 2 offers $x_2 = 0$ and player 1 (eventually) accepts it, contrary to B1.

(Part 2) Finally, we show that $(0, 0) \notin \mathscr{A}(\sigma)$. If $(0, 0) \in \mathscr{A}(\sigma)$, then starting at $v = (0, 0)$, no one proposes again. Consider the out of equilibrium offer $x_1 \in (0, 1)$ by player 1. If player 2 accepts this offer, then 1 gets $1 - x_1 > 0$, violating (4). Hence, $\alpha_2(v, x_1) = 0$. But for player 2 to be willing to reject x_1, we must have $\rho_2(v, x_1) = (0, v_2') \in \mathscr{A}(\sigma)$ for some $v_2' \geq x_1$, contradicting Part 1. □

B. Infinite or Zero Proposal Rates. The element of our model absent from temporal monopoly models is the offer rate. We first must understand when player i never offers, or insistently does so — i.e. when the rate λ_i implied by an AE is zero or infinity.

Lemma 5 *Assume σ is an AE $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ satisfying B1. For any $v \in \mathscr{A}(\sigma)$: (a) $\lambda_i(v) = 0$ implies that $\lambda_j(v) > 0$; (b) $\lambda_i(v) = \infty$ for some player i iff $v_1 + v_2 = 1$, in which case the game ends immediately with the split v; (c) deadlocks almost surely do not occur.*

Proof: Let $v = \pi(\sigma|_h) \in \mathscr{A}(\sigma)$ for the offer node h. Assume $\lambda_i(v) = 0$. If $\lambda_j(v) = 0$, then bargaining stops and expected values are $\pi(\sigma|h) = (0, 0)$. This contradicts Lemma 4.

For (b), assume that $\lambda_i(v) = \infty$. Then after history h, player i offers x_i immediately. From Lemma 3, $\text{supp}(\mu_i(v)) \subset [v_j, 1 - v_i]$. If $x_i > v_j$, then player j earns at least x_i by accepting, so that $\pi_j(\sigma|h) > v_j$. Contradiction. Hence, $\mu_i(v)$ is a point mass at $x_i = v_j$. But since $\alpha_j(v, x_i) > 0$ by A3 and rejecting x_i moves the state back to v, player j accepts v_j immediately in real time (eventually in artificial time). This implies that i's expected value is $v_i = \pi_i(\sigma|h) = 1 - v_j$. Thus, $v_1 + v_2 = 1$. Conversely, any delay causes inefficiency. So if $v_1 + v_2 = 1$, at least one of the two players i must offer immediately, that is, $\lambda_i(v) = \infty$.

Finally, (c) follows from (b) and A3 (since the values are fixed if ever $\lambda_i(v) = \infty$). $\square$

If $v_1^0 + v_2^0 < 1$, the state vector v is never on the Pareto frontier (until agreement), and hence, infinite offer rates are never observed on the equilibrium path. But the state v can land on the Pareto frontier after an off-path offer. For example, player i may make an out of equilibrium offer, which calls for player j to reject with probability 1. After rejection, the continuation value vector is arbitrary and can be chosen to be on the Pareto frontier.

We now address the opposite case where $\lambda_i = \infty$, which forces $v_1 + v_2 = 1$.

Corollary 2 *Immediate agreement on any efficient v with $v_1, v_2 \in (0, 1)$ is an AE outcome.*

In light of this corollary, the interesting implications obtain for AE with delay. We thus *hereafter assume $v_1^0 + v_2^0 < 1$,* and so we have delayed agreement, as $\lambda_1, \lambda_2 < \infty$.

C. Exploding Proposal Rates on the Boundary. We now consider the boundary behavior of the proposal rates λ_i. Work by Perry and Reny (1993), Sakovics (1993), and Stahl (1993) showed that imposing a boundedly positive delay time between offers confers an offerer advantage — for then declining burns a boundedly positive fraction of the pie. It's instructive to see that this is not a feature of our model. Indeed, offer rates must explode in some subgames. To see this, we argue that $\mathscr{A}_{IR}(\sigma)$ touches or approaches the Pareto frontier, and that the proposal rate blows up at these aspiration vectors.

Lemma 6 *Fix an AE σ. If $\bar{v}_i = \sup\{v_i \mid v \in \mathscr{A}_{IR}(\sigma)\}$, then $[(\bar{v}_1, 1 - \bar{v}_1) + \mathbb{B}_\varepsilon] \cap \mathscr{A}_{IR}(\sigma) \neq \emptyset$ and $[(1 - \bar{v}_2, \bar{v}_2) + \mathbb{B}_\varepsilon] \cap \mathscr{A}_{IR}(\sigma) \neq \emptyset$ for all $\varepsilon > 0$, where $\mathbb{B}_\varepsilon$ denotes the ε ball in $\mathbb{R}^2$.*

Proof: WLOG, let $i = 2$. By contradiction, assume $[(1 - \bar{v}_2, \bar{v}_2) + \mathbb{B}_\varepsilon] \cap \mathscr{A}_{IR}(\sigma) = \emptyset$ for some $\epsilon > 0$. Consider the triangle Δ with vertices $(1 - \bar{v}_2, \bar{v}_2)$, $(1 - \bar{v}_2, 0)$ and $(1, 0)$. Note that v_2 is uniformly bounded away from $\bar{v}_2$ in the region $\Delta \setminus [(1 - \bar{v}_2, \bar{v}_2) + \mathbb{B}_\varepsilon]$. Therefore, by the definition of $\bar{v}_2$, there exists $v \in \mathscr{A}_{IR}(\sigma)$ with $v_1 < 1 - \bar{v}_2$ and $v_2 < \bar{v}_2$. Suppose that player 1 then offers $x \in (\bar{v}_2, 1 - v_1)$ (possibly off-path). If $\alpha_2(v, x) = 1$, then his final payoff is $1 - x > v_1$, and offering x immediately lifts his payoff, a contradiction. But if $\alpha_2(v, x) < 1$, then player 2 must expect a continuation value of at least $x > \bar{v}_2$. So there exists $w \in \mathscr{A}_{IR}(\sigma)$ with $w_2 > \bar{v}_2$, contrary to the definition of $\bar{v}_2$. □

Theorem 3 (Boundary Proposal Rates) *Let σ be an AE $\bar{\sigma}(v^0, \lambda, \pi, \alpha, \rho)$. Assume $w = (y, 1 - y)$ is a limit aspiration vector, where $y \in (0, 1)$. Then $\lim_{v \to w} \lambda_i(v) = \infty$, $i = 1, 2$. If $y = 0$, then $\lim_{v \to w} \lambda_1(v) = \infty$, and if $y = 1$, then $\lim_{v \to w} \lambda_2(v) = \infty$.*

Proof: Since $\text{supp}(\mu_1(v)) \subset [v_2, 1 - v_1]$, as $v \in \mathscr{A}_{IR}(\sigma)$ approaches $w = (y, 1 - y)$, $\mu_1(v)$ converges in probability to a point mass at $1 - y$. By condition (3) of an AE (in Theorem 2)

$$1 - y = \lim_{v \to w} \bar{x}_1(v) = \lim_{v \to w} \left[1 + \frac{r_2}{\lambda_1(v)}\right] v_2.$$

If $1 - y \neq 0$, then $\lim_{v \to w} \lambda_1(v) = +\infty$. Similarly, (3) yields $\lim_{v \to w} \lambda_2(v) = +\infty$ if $y \neq 0$:

$$y = \lim_{v \to w} \bar{x}_2(v) = \lim_{v \to w} \left[1 + \frac{r_1}{\lambda_2(v)}\right] v_1. \qquad \square$$

That is, as a player's aspiration value converges to the lowest possible payoff he will ever get in any AE, he makes offers at an increasingly unbounded pace. If $y \in (0, 1)$, when the aspiration vector v is near the Pareto frontier and the surplus $1 - v_1 - v_2$ is close to 0, then both players make offers very often. When $y = 0$ (resp. $y = 1$), the conclusion only applies to player 1 (resp. 2). Recall that for the AE σ specified by a κ-concession rule and the function $\varphi(x) = [1 - x]/[1 + x]$, we have that $\lambda_i(v) = r_j/[\kappa v_i]$. Thus, this σ provides an example where $\lambda_2(v)$ but *not* $\lambda_1(v)$ tends to ∞ as $v \to (1, 0)$.

Example (Battle of the Sexes): Suppose that in equilibrium, players are engaged in a war of attrition to first propose their less favoured pie split among $(2/3, 1/3)$ and $(1/3, 2/3)$; such an offer is always accepted. (Think of this as a method of deciding which pure equilibrium to play in a Battle of the Sexes.) This is an AE with $v^0 = (1/3, 1/3)$ provided $\lambda_1 = r_2$ and $\lambda_2 = r_1$ (to satisfy the IC conditions). But $\mathscr{A}(\sigma)$ must contain more than just v^0, as player i can always make offers like $x = 1/2$. The simplest aspiration set is $\mathscr{A}(\sigma) = \{(1/3, 2/3), (1/3, 1/3), (2/3, 1/3)\}$ — where after any offer by i, the value reverts at once to j's favoured outcome. For $v \in \{(1/3, 2/3), (2/3, 1/3)\}$, at least one offer rate must be infinite for immediate agreement.

Notice that the conclusion of Corollary 1 fails for this example. In fact, if the offer is rejected (off-path), then agreement is still immediate at that proposal. No harm is done.

D. Exploding Proposal Rates in the Interior. We now ask whether an AE can have a proposal rate explosion at an aspiration vector v that is not on the Pareto frontier. We now show this can occur, and likewise cluster points can occur with positive probability. We then provide a simple intuitive condition that rules it out.

Example (Exploding Proposal Rates): Assume that $r_1 = r_2 = 1$ and $v^0 = (3/8, 3/8)$. Consider the function $\varphi : (1/4, 1/2) \to (1/4, 1/2)$ given by $\varphi(x) = 3/4 - x$, with graph $\mathscr{G}(\varphi)$. We construct σ so that $\mathscr{A}(\sigma) = \mathscr{G}(\varphi) \cup \{(\frac{1}{4}, \frac{3}{4}), (\frac{1}{2}, \frac{1}{2}), (\frac{3}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{4})\}$.

For all $v \in \mathscr{G}(\varphi)$, let $\rho_{ij}(v, x) = \varphi(x)$ and $\rho_{ii}(v, x) = x$ if $x \geq v_i$, and $\rho_i(v, x) = v$ if $x < v_i$. For any $v \in \mathscr{G}(\varphi)$, let $\mu_i(v)$ be a degenerate distribution at $\bar{x}_i(v)$, where

$$\bar{x}_i(v) = \frac{1}{2}\left[\frac{1}{2} + v_j\right], \quad \lambda_i(v) = \frac{4v_j}{1 - 2v_j}, \quad \alpha_j(v, x) = \begin{cases} 0 & \text{if } x < \bar{x}_i(v) \\ 2v_i - 1/2 & \text{if } x \geq \bar{x}_i(v). \end{cases}$$

When $v \in \{(1/4, 1/2), (1/2, 1/4)\}$, we continue as in the 'Battle of Sexes' example, to $(1/4, 3/4)$ or $(1/2, 1/2)$ from $(1/4, 1/2)$, and to $(1/2, 1/2)$ or $(3/4, 1/4)$ from $(1/2, 1/4)$.

As $v_i \downarrow 1/4$ and $v_j \uparrow 1/2$, we have $\lambda_i(v) \uparrow \infty$, $\kappa_i(v) \downarrow 0$, and $\alpha_j(v, \bar{x}_i(v)) \downarrow 0$. Thus, the AP $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ may generate a cluster point in finite time. In particular, for example, starting at $v^0 = (3/8, 3/8)$, consider an infinite history h where player 1 makes all the offers $\bar{x}_1(v^k)$ that player 2 always rejects, where $v^{k+1} = (\varphi(\bar{x}_1(v^k)), \bar{x}_1(v^k))$ for all $k \geq 0$. The set of such histories has probability $\Pi_{k=0}^{\infty}(1 - \alpha_2(v^k, \bar{x}_1(v^k)))\lambda_1(v^k)/[\lambda_1(v^k) + \lambda_2(v^k)] > 0$ (proof omitted). Along any such history h, offers arrive increasingly rapidly, becoming stingier and more likely to be rejected. The expected elapse time of the k-th offer by player 1 is $t_k = 1/\lambda_1(v^k) = 2^{-k}/[2(4 - 2^{-k})]$; therefore, the expected time of the cluster point generated by such a history is $\sum t_k \leq \sum 2^{-k}/6 = 1/3$. Let σ be the AE that coincides with $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ for any finite history. For an infinite history $h \in \mathcal{H}_+$ ending with player i making a sequence of offers converging to 1/2 in finite time (a cluster point), put $\sigma = \bar{\sigma}(v^i, \lambda, \mu, \alpha, \rho)$, where $v_i^i = 1/4$ and $v_j^i = 1/2$.

We noted in §3.2 that our AP does not fully describe strategies with cluster points. It is natural to ask how to rule out this possibility. In fact, the example works because surplus concessions κ_i and acceptance chances α_j both vanish as we approach $(v_i, v_j) = (1/2, 1/4)$. The next result shows that if we rule out this possibility, cluster points cannot obtain.

Lemma 7 *Let σ be an AE $\bar{\sigma}(v^0, \lambda, \pi, \alpha, \rho)$ such that for any closed subset A in the interior of $\mathscr{A}$, there exists $\underline{\kappa} > 0$ such that $\bar{\kappa}_1(v), \bar{\kappa}_2(v) \geq \underline{\kappa} > 0$ for all $v \in A$. Then proposal rates $\lambda_1(v), \lambda_2(v)$ are bounded in the set A. Hence, cluster points (and infinite histories, if B1 holds) happen with probability zero, except possibly at the last moment of bargaining.*

Proof: The result follows from rewriting the delay IC equation (3) as

$$\lambda_j \bar{\kappa}_j = v_i r_i / (1 - v_1 - v_2). \tag{6}$$

4 Properties of Aspirational Equilibrium

4.1 Bargaining Duration via Aspirations as a Submartingale

Denote by $\tau(v^0)$ the random time to agreement when the initial aspiration vector is $v(0) = v^0$. When the players follow an AE $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$, at any time t until $\tau(v^0)$, each player has a well-defined aspiration level $v_i(t)$ between offer events. Moreover, if an offer transpires at time t, then let $v_i(t)$ be the expected value of player i immediately after the proposal is tendered, but before its content is heard, and $\bar{v}_i(t)$ just after the content is heard and the reply is given. (Define $\bar{v}_i(t) = v_i(t)$ at all other times.)

We now assert that the process $e^{-r_i t} v_i(t)$ is a martingale 'until' the players reach an agreement at time $\tau(v^0)$. We eliminate the termination date in the following artificial stochastic process, where we "freeze" the terminal value:

$$z_i(t) = \begin{cases} e^{-r_i t} v_i(t) & \text{for } t \leq \tau(v^0) \\ e^{-r_i \tau(v^0)} \bar{v}_i(\tau(v^0)) & \text{for } t > \tau(v^0). \end{cases}$$

The process $\bar{z}_i(t)$ is likewise defined, using $\bar{v}_i(t)$ for $t \leq \tau(v^0)$ instead.

Theorem 4 *Let σ be an AE $\bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$. The stochastic processes $z_i(t)$ and $\bar{z}_i(t)$ are martingales. The aspiration process $v_i(t)$ is a strict submartingale until agreement time τ.*

For an intuition about the martingale, either player is indifferent about offering, since he expects his value back by waiting and never offering. Of course, the final agreement time may owe to either player proposing, and so conditional on getting proposed to, the expected discounted value exceeds the current value. The proof shows that this surplus over the martingale exactly balances the loss in the event that a player ends up as proposer.

Proof of Theorem 4: Fix $t \in (0, \infty)$. First consider the $z_i(t)$ process. Let $\theta_i \geq 0$ be the time to the first (hypothetical) offer by player i (i.e., the realization of $\sigma_i^t(\emptyset)$). Let $\tau_k \geq 0$ be the time to the k-th offer event. Defining $a \wedge b \equiv \min\{a, b\}$, we thus have $\tau_1 = \theta_1 \wedge \theta_2$. Almost surely, the number of offers in $[0, t]$ is finite, and hence $t = \lim_{k \to \infty} t \wedge \tau_k$. Then by the Dominated Convergence Theorem, $\lim_{k \to \infty} E[z(t \wedge \tau_k) | z(0) = v^0] = E[z(t) | z(0) = v^0]$. We will prove that $v^0 = E[z(t \wedge \tau_k) | z(0) = v^0]$ for all $k < \infty$ and so $v^0 = E[z(t) | z(0) = v^0]$.

We first show that $z(0) = E[z(t \wedge \tau_1) | z(0)]$. Then by the strong Markov property,

$$E\left[z(t \wedge \tau_{k+1}) | z(0)\right] = E\left[E\left[z(t \wedge \tau_{k+1}) | z(t \wedge \tau_k)\right] | z(0)\right] = E\left[z(t \wedge \tau_k) | z(0)\right],$$

and thus by induction, $z(0) = E[z(t \wedge \tau_k) | z(0)]$ for all $k \geq 1$. Consider the events $B_1 = [t > \tau_1 \text{ and } \theta_1 < \theta_2]$, $B_2 = [t > \tau_1 \text{ and } \theta_1 \geq \theta_2]$ and $B_3 = [t \leq \tau_1]$. Clearly $\{B_1, B_2, B_3\}$ is a partition of the states of the world. Since θ_1 and θ_2 are independent exponential random variables, $P[\theta_i \leq s | \theta_i < \theta_j] = 1 - e^{\lambda_0 s}$, where $\lambda_0 = \lambda_1 + \lambda_2$. Therefore

$$E[e^{-r_1 \theta_1} | B_1] = \int_0^t e^{-r_1 s} \frac{\lambda_0 e^{-\lambda_0 s}}{1 - e^{-\lambda_0 t}} ds = \left[\frac{\lambda_0}{r_1 + \lambda_0}\right] \left[\frac{1 - e^{-(r_1 + \lambda_0) t}}{1 - e^{-\lambda_0 t}}\right] = E[e^{-r_1 \theta_1} | B_2],$$

Further, we have the conditional expectations

- $E[z_1(t \wedge \tau_1) | B_1] = v_1(0) E[e^{-r_1 \theta_1} | B_1]$ by (4) of Theorem 2
- $E[z_1(t \wedge \tau_1) | B_2] = \bar{x}_2(v^0) E[e^{-r_1 \theta_1} | B_2]$ since the AE σ obeys $\sigma_2^x(\emptyset) = \mu_2(v^0)$
- $E[z_1(t \wedge \tau_1) | B_3] = v_1(0) e^{-r_1 t}$ since the aspiration value is constant on $[0, t]$ if $t < \tau_1$.

Since $P[B_i] = \lambda_i \left[1 - e^{-\lambda_0 t}\right] / (\lambda_1 + \lambda_2)$ for $i = 1, 2,$ and $P[B_3] = e^{-\lambda_0 t}$, by equation (3) of Theorem 2, we have $E[z_1(t \wedge \tau_1)|z_1(0)]$ equal to

$$= [P[B_1]v_1(0) + P[B_2]\bar{x}_2(v^0)] \left[\frac{\lambda_0}{r_1 + \lambda_0}\right]\left[\frac{1 - e^{-(r_1+\lambda_0)t}}{1 - e^{-\lambda_0 t}}\right] + P[B_3]v_1(0)e^{-r_1 t}$$
$$= v_1(0) \left[1 - e^{-(r_1+\lambda_0)t}\right] + v_1(0)e^{-r_1 t}e^{\lambda_0 t} = v_1(0).$$

Finally, for the process $\bar{z}_i(t)$ we only need to observe that by (3) and (4) of Theorem 2, $E[\bar{z}(t)|z(t)] = z(t)$, and therefore $E[\bar{z}(t)|\bar{z}(0) = v^0] = v^0$ as well.

Finally, the discounted martingale implies the un-discounted submartingale. □

By the martingale convergence theorem, since values are a bounded submartingale, they almost surely converge. But being generated by an AP, they are also a Markov process; so any limit value v must be stationary under the dynamic. But the only stationary values are on the Pareto frontier, where the real delay time is zero, by Lemma 5. Hence,

Corollary 3 (Bargaining Finiteness) *Bargaining ends almost surely in finite time.*

Not only is the bargaining duration $\tau(v^0)$ a.s. finite, but we can bound it below too:

Corollary 4 (*a*) *In an AE, we have* $E[\tau(v^0)] \geq -\log[v_1^0 + v_2^0]/\max(r_1, r_2)$. *Hence,* (*b*) *if players* i *and* j *made the the last two offers,* x_i *and* x_j*, and both were rejected, then the expected time until the bargaining ends is at least* $-\log(x_1 + x_2)/\max(r_1, r_2)$.

Proof: Denote $\bar{r} = \max(r_1, r_2)$ and $\bar{\tau} = E[\tau(v^0)]$. By the Optional Stopping Theorem,

$$v_i^0 = E[\bar{z}_i(\tau(v^0))|\bar{z}_i(0)=v_i^0] = E[e^{-r_i\tau(v^0)}\bar{v}_i(\tau(v^0))|v_i^0] \geq E[e^{-\bar{r}\tau(v^0)}\bar{v}_i(\tau(v^0))|v_i^0]$$

since $\bar{z}_i(t)$ is a martingale process, and $r_i \leq \bar{r}$. Because $(\bar{v}_1(\tau(v^0)), \bar{v}_2(\tau(v^0)))$ is the final agreed pie split, it lies on the Pareto frontier, or $\bar{v}_1(\tau(v^0)) + \bar{v}_2(\tau(v^0)) = 1$. Finally, since $E[e^{-\bar{r}\tau(v^0)}] \geq e^{-\bar{r}\bar{\tau}(v^0)}$ by Jensen's inequality, we have $v_1^0 + v_2^0 \geq e^{-\bar{r}\bar{\tau}}$, as required in ($a$).

For part (b), after the first offer is rejected, $v_2 = x_1$. After the second offer, $v_1 = x_2$ and v_2 drops below x_1 (Corollary 1). Thus, $v_1 + v_2 \leq x_1 + x_2$. Now apply part (a). □

4.2 Relating Offer Rates, Offers, and Acceptance Rates

We now relate our three strategic choices (λ, μ, α) — suppressing arguments where clear.

A. Offers and Timing. The war of attrition aspect of bargaining yields a simple testable implication about rejected offers alone, consistent with Siegel and Fouraker (1960).

Corollary 5 (Consecutive Offers) *If a player makes consecutive equilibrium offers, then his second offer is strictly more generous to the other player. If players* i, j *make sequential offers* x_i *and* x_j*, then player* j *offers* i *less than* i *offered himself:* $x_j \leq 1 - x_i$.

Proof: After player i makes a proposal that is rejected, the new aspiration value shifts down for player i, and up to x_i for player j. Now apply Lemma 3. □

The IC conditions yield stronger implications about offer timing and acceptance rates. By the new delay IC (6):

$$v_i = \frac{r_j \lambda_j \bar{\kappa}_j}{r_i r_j + r_i \lambda_i \bar{\kappa}_i + r_j \lambda_j \bar{\kappa}_j} \tag{7}$$

From this formula, as the chance that j offers vanishes, i's value vanishes. By contrast, in a temporal monopoly, as the chance that j gets to offer vanishes, j's value vanishes.

Equation (7) also betrays a crucial separability between the delay and offer IC equations (6) and (4), evident in our examples and theorems. Surplus concessions κ interact with offer rates λ alone to determine current values in (6); the aspiration set is only relevant in the linkage in (4) between surplus concessions κ and acceptance rates α.

Corollary 6 Ceteris paribus, *the rate λ_j that player j offers varies inversely with the expected fraction of surplus in the pie $\bar{\kappa}_j$ that he concedes.*

Corollary 6 captures the inherent trade-off between offer timing and content: *Ceteris paribus* (holding values fixed), if a player anticipates a more generous offer, then he should expect to wait longer. This is consistent with Corollary 4, that less generous alternating offers foretell a greater lower bound on the expected bargaining duration.

Equation (7) also provides insights into the nature of bargaining power. With temporal monopoly, there are two exogenous sources of asymmetry: relative impatience levels, and the offering order. In our aspirational paradigm, offering order is not an issue, but there is a intuitive new strategic component of bargaining power: Parties gain strength from their *refusal* to make offers. As is so often true in social bargaining, one can hurt the other party by according him the "silent treatment", forcing him to make all overtures.

B. Offers and Acceptance Rates. Our theory is much richer when we move beyond AE like the Battle of Sexes where players simply "make an offer that can't be refused." We now relate the content of offers to the acceptance rates. By using coarse bounds from the IC conditions, we find two necessary inequalities jointly satisfied by all parameters $(\lambda, \kappa, \alpha)$. They are especially useful as they obtain *for any aspiration set,* and any offer.

Corollary 7 *Consider an offer x_i by player i that might be rejected ($\alpha_j < 1$). Then*

$$\frac{\alpha_j}{1-\alpha_j}(1-\kappa_i) r_i < \lambda_j \bar{\kappa}_j \tag{8}$$

Proof: Using (2), rewrite the offer IC equation (4) as

$$(1-\alpha_j)(v_i - \rho_{ji}(v, x_j)) \geq \alpha_j (1-\kappa_i)(1 - v_1 - v_2)$$

Since rejection values are positive (by Lemma 4), and $\alpha_j < 1$, (6) yields the inequality. □

Corollary 7 captures the trade-off between the acceptance rate and surplus concession of a given offer: *Ceteris paribus,* if an offer concedes too little surplus, it must be accepted with a small chance. Corollary 7 shows that incentive constraints inherently force lower bounds on surplus concession fractions and offer rates, and upper bounds on acceptance rates. Eg., if offers are accepted at least half the time, and concede half the surplus ($\kappa_1 = \kappa_2 = 1/2$), then i's proposal rate must exceed j's interest rate. Only in extreme cases like the Battle of Sexes, which concede all the surplus, can offers almost surely be accepted.

4.3 Simple Aspiration Sets

A. Decreasing Aspiration Sets. We now give plausible assumptions that guarantee a *continuously decreasing* IR aspiration set $\mathscr{A}_{IR}(\sigma)$: There exists an open interval $W = (\underline{w}, \bar{w}) \subset [0,1]$ (with closure $\bar{W} = [\underline{w}, \bar{w}]$) and a continuous, strictly decreasing function $\varphi : \bar{W} \to [0,1]$ with: $\varphi(\underline{w}) = 1-\underline{w}$, $\varphi(\bar{w}) = 1-\bar{w}$, and $\mathscr{A}_{IR}(\sigma) \cup \{(\underline{w}, 1-\underline{w}), (\bar{w}, 1-\bar{w})\} = \mathscr{G}(\varphi)$. Depending on σ, $(\underline{w}, 1-\underline{w})$ and $(\bar{w}, 1-\bar{w})$ may or may not be in $\mathscr{G}(\varphi)$.[4] The simplicity of the constant α and κ examples owed to this property of the aspiration sets.

We assume that as soon as a player, say 2, hears an IR offer x_1, the past is forgotten, and his behaviour only depends on the new *interim* aspiration value vector (v_1, x_1).

B2. *Interim Stationarity.* The offerer's continuation value $\rho_{ij}(v, x_j)$ is independent of the rejecter's value v_i, and $\rho_{ii}(v, x_j) = x_j$ for all $v \in \mathscr{A}(\sigma)$ and IR offers x_j. So if $v, v' \in \mathscr{A}(\sigma)$ satisfy $v_j = v'_j$ and $x_j \geq \max(v_i, v'_i)$, then $\rho_{ij}(v, x_j) = \rho_{ij}(v', x_j)$.

$B2$ obtains if the acceptance chance obeys interim stationarity (so $\alpha_i(v, x_j)$ is constant in v_i for $v_i \in (0, x_j]$). We next assume in B3 that the same offer to a player with a greater value be accepted with a lower chance; we also assume away in B4 examples like the Battle of Sexes. Not conceding all surplus is in the same spirit as the second part of B1.

B3. *Strict Acceptance Monotonicity.* Fix v_j. The chance $\alpha_i(v, x_j)$ that i accepts j's offer x_j is strictly decreasing in the proposee's value v_i for all $v \in \mathscr{A}$ and $x_j \in [v_i, 1-v_j]$.

B4. *No Full Concessions.* For every $v \in \mathscr{A}_{IR}(\sigma)$, $\bar{\kappa}_i(v) < 1$, $i = 1, 2$.

Assumptions $B2$–$B4$ are satisfied by the constant κ and α examples (φ being 1-1); $B2$ and $B4$ are violated by the Battle of Sexes example. Unlike earlier assumptions, $B2$–$B3$ place restrictions both on and off the equilibrium path. Eg., $B2$ says that the proposer's continuation value is the same after any IR (possibly out-of-equilibrium) offer is rejected.

Lemma 8 *Given assumptions $B2$–$B4$, the IR aspiration set $\mathscr{A}_{IR}(\sigma)$ is decreasing.*

Proof: Assume that $v, v' \in \mathscr{A}(\sigma)$ with $v_j = v'_j$ and $v_i < v'_i$. Let $x_j \in \text{supp}(\mu_j(v'))$ such that $x_j < 1-v_j$ (guaranteed by $B4$). Then $v'_j = \alpha_i(v', x_j)(1-x_j) + (1-\alpha_i(v', x_j))\rho_{ij}(v', x_j)$, and since $\alpha_i(v', x_j) > 0$ (by A3) and $1 - x_j > v'_j$, we must have $\rho_{ij}(v', x_j) < v'_j$. Therefore, if player j makes the same offer x_j at v, he expects $\alpha_i(v, x_j)(1-x_j) + (1-\alpha_i(v, x_j))\rho_{ij}(v, x_j) > v'_j = v_j$ since $\alpha_i(v, x_j) > \alpha_i(v', x_j)$ by $B3$, and $\rho_{ij}(v, x_j) = \rho_{ij}(v', x_j)$ by $B2$. Hence, player j can raise his expected value v_j by offering x_j, a contradiction.

For each $w \in \mathscr{A}$ with $w_1 + w_2 < 1$, let $L(w)$ denote the right triangle with vertices w, $(w_1, 1-w_1)$ and $(1-w_2, w_2)$. Clearly if $L(w)$ and $L(w')$ contain $\mathscr{A}_{IR}(\sigma)$, then $\mathscr{A}_{IR}(\sigma) \subset$

[4] If $(1-x, x) \in \mathscr{A}_{IR}(\sigma)$ were reached from $v \in \mathscr{A}_{IR}(\sigma)$ after player 1, say, made an offer x and player 2 rejected, then $w \leq (1-x, x)$ (because the offer x is IR). That is, $1 - x = \rho_{21}(v, x)$. By (4), $v_1 = 1 - x$, for otherwise offering x strictly increases player 1's expected value, contradicting B4 below. Hence, $(1-x, x) \in \mathscr{A}_{IR}(\sigma)$ can only be reached in the limit after a cluster point. Thus, $(\underline{w}, \varphi(\underline{w})) \in \mathscr{A}_{IR}(\sigma)$ only if σ can produce a cluster there.

$L(w) \cap L(w')$. Also, there exists w'' such that $L(w) \cap L(w') = L(w'')$. Let L^* be the intersection of all triangles $L(w)$ containing $\mathscr{A}_{IR}(\sigma)$. Then $L^* \neq \emptyset$ since $v^0 \in L^*$, and clearly there exists w^* such that $L^* = L(w^*)$. Let $\underline{w} = w_1^*$ and $\bar{w} = 1 - w_2^*$. We now show that $\mathscr{A}_{IR}(\sigma)$ intersects every vertical and horizontal line through L^* once and only once. Suppose, eg., the vertical line at some level $v_1 \in (\underline{w}, \bar{w})$ does not intersect $\mathscr{A}_{IR}(\sigma)$: i.e. $\{v_1\} \times [0,1] \cap \mathscr{A}_{IR}(\sigma) = \emptyset$. Then $[0, v_1] \times [0, 1 - v_1] \cap \mathscr{A}_{IR}(\sigma) = \emptyset$. For if w were in this intersection and player 2 offered $x_2 = v_1$ at w, then $\rho_1(w, x_2) \in \{v_1\} \times [0,1] \cap \mathscr{A}_{IR}(\sigma)$ by $B2$. So $\mathscr{A}_{IR}(\sigma) \subset L(\underline{w}, 1 - v_1) \cup L(v_1, 1 - \bar{w})$. But for any $w \in L(\underline{w}, 1 - v_1)$, say, and any IR offer x_i at w, $\rho_j(w, x_i) \in L(\underline{w}, 1 - v_1)$. Thus, $\mathscr{A}_{IR}(\sigma) \subset L(\underline{w}, 1 - v_1)$ or $\mathscr{A}_{IR}(\sigma) \subset L(v_1, 1 - \bar{w})$, depending on which contains v^0, contrary to the definition of L^*.

If $\mathscr{A}_{IR}(\sigma)$ is not decreasing, there must exist two Pareto-ranked points $v, v' \in \mathscr{A}_{IR}(\sigma)$, say $v < v'$. Since v' is the only point in $\mathscr{A}_{IR}(\sigma)$ on the horizontal line through v', if player 1 offers $x_1 = v_2'$ at v, he gets $1 - x_1 \geq v_1' > v_1$ if the offer is accepted and v_1' otherwise. His expected value is thus strictly higher than v_1 after making that offer, a contradiction. So no two points in $\mathscr{A}_{IR}(\sigma)$ can be Pareto ranked, and $\mathscr{A}_{IR}(\sigma) \cup \{(\underline{w}, 1 - \underline{w}), (\bar{w}, 1 - \bar{w})\}$ is the graph of a decreasing function $\varphi : \bar{W} \to [0,1]$. Since $\mathscr{A}_{IR}(\sigma)$ intersects every vertical and horizontal line through L^*, φ is continuous, and $\varphi(\underline{w}) = 1 - \underline{w}$ and $\varphi(\bar{w}) = 1 - \bar{w}$. □

Suppose that a mixture over offers is entertained. Casual empiricism suggests that the reason a proposer might consider sweetening an offer is to ensure a greater chance of acceptance. In his classic text, eg., Zeuthen (1930) simply assumes it as a behavioural postulate of bargaining. In fact, this fails in our context unless the aspiration set is well-behaved: Perhaps a less generous offer is rewarded by a better continuation value.

Corollary 8 *Assume B2–B4 (so that $\mathscr{A}_{IR}(\sigma)$ is continuously decreasing). If $x' > x$ are two IR offers made by player i at the value $v \in \mathscr{A}_{IR}(\sigma)$, then $\alpha_j(v, x') > \alpha_j(v, x)$.*

Proof: We have $v_i = \alpha_j(v, z)(1 - z) + (1 - \alpha_j(v, z))\rho_{ji}(v, z)$ for $z = x$ and $z = x'$. Since $\mathscr{A}(\sigma)$ is decreasing, $\rho_{ji}(v, x) > \rho_{ji}(v, x')$, and $\alpha_j(v, x) < \alpha(v, x')$. □

We can finally establish another of our claimed intuitive properties of an AE: strictly IR offers. Corollary 7, without assumptions $B2$–$B4$, directly implies that the expected offer is strictly IR: $\bar{\kappa} > 0$. Using $\mathscr{A}_{IR}(\sigma)$ decreasing, we now deduce this for all AE offers.

Corollary 9 *Assume B2–B4. If $v \in \mathscr{A}_{IR}(\sigma)$ with $v_1 + v_2 < 1$ and x_i is an AE offer at v, then x_i strictly increases player j's aspiration: $x_i > v_j$.*

Proof: By Lemma 3, suppose instead $x_i = v_j$. Since $\mathscr{A}_{IR}(\sigma)$ is decreasing, player i gets $1 - x_i > v_i$ if the offer is accepted (which happens with positive chance by A3) and v_i otherwise. This yields player i an immediate positive expected gain. Contradiction. □

B. Convex Rejections and Consecutive Offers. We introduce a stronger property of the IR aspiration set: $\mathscr{A}_{IR}(\sigma)$ has *convex rejections* if for all $v \in \mathscr{A}_{IR}(\sigma)$ and IR offers x_1, x_2,

$$\frac{1 - \rho_{12}(v, x_2)}{\rho_{11}(v, x_2)} < \frac{1 - v_2}{v_1} < \frac{1 - \rho_{22}(v, x_1)}{\rho_{21}(v, x_1)} \tag{9}$$

This property obtains if φ is a convex function, and was satisfied by our constant α and κ examples. It will permit more powerful and intuitive conclusions about consecutive offers.

Lemma 9 (Convexity) *The aspiration set* $\mathscr{A}_{IR}(\sigma)$ *has convex rejections, iff for* $i = 1, 2$,

$$\frac{\alpha_j}{1-\alpha_j} > \frac{\kappa_i}{1-\kappa_i} \cdot \frac{\lambda_j \bar{\kappa}_j}{r_i + \lambda_j \bar{\kappa}_j} \tag{10}$$

In particular, it suffices that $\alpha_1 > \kappa_2$ *and* $\alpha_2 > \kappa_1$.

Proof: Using (8) and then (7), the right inequality of (9) holds iff

$$1 - \frac{\alpha_2(1-\kappa_1)r_1}{(1-\alpha_2)\lambda_2\bar{\kappa}_2} = \frac{\rho_{21}(v, x_2)}{v_1} < \frac{1-\rho_{22}(v, x_1)}{1-v_2} = 1 - \frac{\kappa_1 r_1 r_2}{r_1 r_2 + r_2 \lambda_2 \bar{\kappa}_2}$$

since $\rho_{22}(v, x_1) = x_1 = v_2 - \kappa_1(1 - v_1 - v_2)$ by (2). This yields (10) for $i = 1$. □

So the convex rejections property obtains provided players' offers do not concede too much surplus, or are not too likely to be rejected. A consequence of Lemma 9 and (6) is:

Corollary 10 (Consecutive Offers, Encore) *Assume* (10)*, or more simply* $\alpha_1 > \kappa_2$ *and* $\alpha_2 > \kappa_1$ *(so that* $\mathscr{A}_{IR}(\sigma)$ *has convex rejections). Then* $\lambda_j \bar{\kappa}_j$ *rises after player* j*'s proposal is rejected; that is, he either must concede more surplus or offer more rapidly.*

Suppose that player j's offer has been rejected. We knew from Corollary 5 that his next offer must be more generous: His 'absolute' bargaining position has worsened. We now know that $\lambda_j \bar{\kappa}_j$ is lower, and he expects a smaller share of the remaining surplus. He must now either accelerate his proposal rate, or sweeten the surplus his offers concede.

5 Sensitivity Analysis

We wish to ask what happens if one party becomes more impatient. Since our bargaining model admits a multiplicity of aspirational equilibria, it is not immediately obvious how to perform such an exercise. There are many ways in which an AE may adjust to changes in the interest rates $r = (r_1, r_2)$. The Rubinstein bargaining model makes precise predictions for this adjustment: when player 1, say, becomes more impatience, his share of the pie decreases. We posit that in our model, the opposite may happen. The idea is simple: to restore equilibrium when player 1 becomes more impatient, it is enough to increase the offer rates of player 2. More precisely, by (3) of Theorem 2, $\sigma = \bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ is an AE when the interest rates are $r = (r_1, r_2)$ iff $\hat{\sigma} = \bar{\sigma}(v^0, \hat{\lambda}, \mu, \alpha, \rho)$ is an AE for the interest rates $\hat{r} = (\hat{r}_1, \hat{r}_2)$, where $\hat{\lambda}$ are the corresponding offer rates $\hat{\lambda}_i = (\hat{r}_j / r_j)\lambda_i$, $i = 1, 2$. So, for example, if player 1 becomes twice as impatient ($\hat{r}_1 = 2r_1$ and $\hat{r}_2 = r_2$), and player 2 doubles the Poisson rates of his offers in σ, then the new strategy $\hat{\sigma}$ is an AE.

For our next result we need to strengthen the war of attrition property and the assumption A3 on meaningful offers.

B3. *Strong War of Attrition.* For any $v \in \mathscr{A}_\rho(v^0)$, expected offers obey $\bar{x}_1(v) + \bar{x}_2(v) \geq 1$.

B4. *Acceptance Positivity*: Equilibrium offers are accepted with boundedly positive probability $\underline{\alpha} > 0$.

As previously noted, the players are engaged in a war of attrition in terms of expected continuation values; assumption B3 requires that the war of attrition holds (weakly) in terms of the immediate expected offers (i.e. ignoring rejection values). Written in the form $\bar{x}_i(v) \geq 1 - \bar{x}_j(v)$, it has the following interpretation. At any $v \in \mathscr{A}_\rho(v^0)$, the offer by i to j is more generous to j than what j would offer himself. That is, in terms of the next offer alone, it is always better to let your opponent speak first.

Assumption A3 requires that equilibrium offers be always accepted with positive probability. Even though $v_1 + v_2 < 1$ for all $v \in \mathscr{A}_\rho(v^0)$, it is still possible that there exist sequences $\{v^n\} \subset \mathscr{A}_\rho(v^0)$ and $\{x_j^n\} \in \text{supp}(\mu_j(v^n))$ such that $v_1^n + v_2^n \to 1$ and $\alpha_i(v^n, x_j^n) \to 0$, even if we assume that $\alpha_i(v,x)$ is continuous in (v,x). Assumption B4 precludes this possibility.

Theorem 5 *Assume that for the interest rates $r = (r_1, r_2)$, $\sigma = \bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ is an AE satisfying B1–B4. Let $\hat{\lambda}$ be the offer rates $\hat{\lambda}_i(v) = (\hat{r}_j/r_j)\lambda_i(v)$, $i = 1,2$, $v \in \mathscr{A}$, so that $\hat{\sigma} = \bar{\sigma}(v^0, \hat{\lambda}, \mu, \alpha, \rho)$ is an AE for the interest rates $\hat{r} = (\hat{r}_1, \hat{r}_2)$. If $\hat{r}_i/r_i > \hat{r}_j/r_j$, then player i's expected pie share is higher with $\hat{\sigma}$ than with σ.*

There are (at least) two other ways in which an AE can be adjusted to accomodate changes in the interest rates. In the discussion that follows, for simplicity assume that $\hat{r}_1 > r_1$ and $\hat{r}_2 = r_2$, and that $\sigma = \bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ is an AE for r. Then, define $\hat{\mu}_1 = \mu_1$ and $\hat{\mu}_2$ so that for each $v \in \mathscr{A}$, the expected offer of player 2 is $[(\lambda_2(v)+\hat{r}_1)/(\lambda_2(v)+r_1)]\bar{x}_2(v)$, where $\bar{x}_2(v)$ denotes his corresponding expected offer in the AE σ. That is, $\hat{\mu}$ maintains player 1's offers while it makes player 2's offers more generous (each time he makes an offer). Again, by (3) of Theorem 2, it is easy to see that $\sigma = \bar{\sigma}(v^0, \lambda, \mu, \alpha, \rho)$ is an AE when the interest rates are $r = (r_1, r_2)$ iff $\hat{\sigma} = \bar{\sigma}(v^0, \lambda, \hat{\mu}, \alpha, \rho)$ is an AE for the interest rates $\hat{r} = (\hat{r}_1, \hat{r}_2)$. It is very intuitive (though we have not proven this formally) that this adjustment yields the same conclusion of Theorem 5: as player 1 becomes more impatient, his expected pie share *increases*. More generally, as player 1 becomes more impatient, we can restore equilibrium by a combination of the adjustments of μ and λ proposed above. That is, let $\hat{\lambda}_1 = \lambda_1$ and $\hat{\mu}_1 = \mu_1$, and define $\hat{\lambda}_2 \geq \lambda_2$ and $\hat{\mu}_2 \geq \mu_2$ so that for each $v \in \mathscr{A}$, the expected offer of player 2 is $[(\hat{\lambda}_2(v) + \hat{r}_1)/(\hat{\lambda}_2(v) + r_1)]\bar{x}_2(v)$. The conclusions of Theorem 5 should not change in this case either.

Finally, we can also restore equilibrium by an adjustment involving the function ρ. It is easier to describe this adjustment in terms of the function φ (or φ^{-1}) instead. In the AE σ, along the equilibrium path, before agreement, when player 2's expected value is w, player 1's expected value is $\varphi^{-1}(w)$. In the AE $\hat{\sigma} = \bar{\sigma}(v^0, \hat{\lambda}, \hat{\mu}, \hat{\alpha}, \hat{\rho})$ for the interest rates $\hat{r}$, $\mathscr{A}_{IR}(\hat{\sigma})$ is the graph of the function $\hat{\varphi}$ specified below. For each $w \in (0,1)$, let $v = (\varphi^{-1}(w), w)$, and define $\hat{\varphi}^{-1}(w) = [\lambda_2(v)/(\lambda_2(v) + \hat{r}_1)]\bar{x}_2(v)$, $\hat{v} = (\hat{\varphi}^{-1}(w), w)$, $\hat{\lambda}_2(\hat{v}) = \lambda_2(v)$, and $\hat{\mu}_2(\hat{v}) = \mu_2(v)$. That is, when player 2 holds an aspiration value w, he acts in $\hat{\sigma}$ and σ exactly the same way. When player 1 is more impatient, of course this means that for any aspiration value w for player 2, player 1's aspiration value is now smaller ($\hat{\varphi}^{-1}(w) < \varphi^{-1}(w)$). We treat player 1 symmetrically. For each $w \in (0,1)$, now let $v = (w, \varphi(w))$ and $\hat{v} = (w, \hat{\varphi}(w))$. Then define $\hat{\lambda}_1(\hat{v}) = \lambda_1(v)$ and $\hat{\mu}_1(\hat{v}) = \mu_1(v)$. To complete the specification of $\hat{\sigma}$, we now define $\hat{\alpha}$. For any v in the graph of the function $\hat{\varphi}$, define $\hat{\alpha}$ by the equations

$$\begin{aligned} v_1 &= \hat{\alpha}_2(v,x)(1-x) + (1-\hat{\alpha}_2(v,x))\hat{\varphi}^{-1}(x) \quad x \in (v_2, 1-v_1) \\ v_2 &= \hat{\alpha}_1(v,x)(1-x) + (1-\hat{\alpha}_2(v,x))\hat{\varphi}(x) \qquad x \in (v_1, 1-v_2). \end{aligned}$$

This equations define $\hat{\alpha}_j$ only partially, for aspiration vectors $v \in \mathscr{A}_{IR}(\hat{\sigma})$ and 'rational' offers $x \in (v_i, 1-v_j)$. As usual, there is flexibility on how to specify $\hat{\alpha}$ (and indeed $\hat{\sigma}$) elsewhere.

It is clear that player 1 is worse off with $(\hat{\sigma}, \hat{r})$ than with (σ, r) because for each aspiration value v_2, player 2's offer rates and offers in σ and $\hat{\sigma}$ are the same. This is reflected in the fact that the graph of $\hat{\varphi}$ is to the left of the graph of φ. But, alternatively, we could also view the graph of $\hat{\varphi}$ as being below the graph of φ. So player 2 is also worse off. Therefore, comparing the expected value of the pie split produced by σ and $\hat{\sigma}$ is not trivial, and we do not have a clear prediction in this case.

6 Conclusion

Summary. We have analyzed the complete information bargaining problem truly without procedures. Our approach therefore restricts the equilibrium concept rather than the action space. Our refinement of SPE is behaviourally-based, and inspired by research on the importance of players' aspirations in decision-making. It also yields a very tractable theory of bargaining. While not unique, all AE have the same intuitive elements of bargaining absent from the temporal monopoly theory: wars of attrition endogenously arise; offers are concessions; offers may be declined, disappointing the proposer, or accepted, strictly pleasing him. Offer timing and content are also entwined. For instance, an embedded martingale structure yields immediate bounds on the duration of bargaining. Further, a player's second consecutive offer must be more generous that his first. We have shown more generally how acceptance rates, surplus conceded, and offer rates are related.

Relative to temporal monopoly, our refinement inverts the strategic role of the proposer — no longer strength but weakness. This changes many results, and casts doubt on the standard impatience comparative static. We show that for a natural mapping between AE for different discount rates, increased impatience skews the final pie split in one's favour. Our methodology, showing how the expected absorbing state of a Markov process changes as the initial state changes, is new and should be applicable elsewhere.

En route, we have given a new formalization of continuous time games in the spirit of standard extensive forms, specialized to our complete information bargaining context.

Literature Reprise. Temporal monopoly has been the foundation of most if not all dynamic noncooperative bargaining papers written since 1982. Even behavioural bargaining papers have adopted this paradigm. For instance, Yildiz (2001) admits different priors over *who has the temporal monopoly* each period (the 'recognition' process). Our behaviourally-founded bargaining paper escapes this paradigm. We do, however, omit an obviously important aspect of bargaining, by avoiding incomplete information. That complication is the natural next hurdle to surmount.

This paper returns to some early influential views of bargaining. Edgeworth (1881), for instance, thought [§II, p. 51] "there are an indefinite number of arrangements à priori possible, towards one of which the system is urged by ... the *Art of Bargaining* — higgling dodges and designing obstinacy, and other incalculable and often disreputable accidents." By avoiding temporal monopoly, we have given just such a stochastic story of bargaining, despite complete information. Our randomness instead owes to mixed strategies and imperfect information. Harsanyi (1956) even wrote that random elements may be the source of bargaining strength: "Of course, information on the two parties' ... strength alone may not suffice: the outcome may depend significantly on such 'accidental' factors as ... bargaining skill." This view is also well-captured by our model: Having randomly offered and been rejected first, perhaps many times, hurts one's bargaining position.

There is also a long-standing view of bargaining as an irreversible concession game, which has naturally led to war of attrition analyses.[5] Osborne (1985) offers a simple complete information war of attrition bargaining game, but unlike our completely unrestricted offers, assumes fixed offers.[6] Abreu and Gul (2000) is a recent incomplete information war of attrition. The war of attrition in Abreu and Pearce (2001) even owes to a concession game. Our bargaining *is* a war of attrition, and yet *is not* a concession game. To see this distinction, assume players each have two units in Abreu and Pearce's five unit bargaining problem. In their pure concession game, a player's concession of the last unit cannot be rejected. Just as in Rubinstein (1982), we have instead assumed exploding offers — i.e. implying no future commitment: Bygones are bygones.[7]

Exploding offers not only ensures an analysis that is stationary in aspiration space, but also captures the realistic inherent risk of declining any offer: One may ultimately offer or accept a strictly worse outcome. Accepting an offer is like an irreversible decision to exercise an option, and spurning one like a risky decision not to sell an asset. If offers implied irrevocable commitments, some sort of third party commitment technology would be required to enforce equilibria where the players gradually concede the pie.

A Appendix: Omitted Proofs

A.1 Outcome Profiles are Well-Defined: Proof of Lemma 1

Claim 1 *If* $\{ h^n \mid n \in \mathbb{N} \} \subset H(\sigma)$, *then* $h^\infty = \bigcup_{n\in\mathbb{N}} h^n \in H(\sigma)$.

Proof: History h^∞ is a countable set, being a countable union of countable sets. Hence, $h^\infty \in \mathcal{P}$. Now for any $(t,k) \in [0,\infty) \times \{0,\ldots,\infty\}$, there exists $n \in \mathbb{N}$ with $h^\infty_{(t,k)} \subset h^n$. If

[5] For a striking example, the pre-Ståhl frontier bargaining textbook Coddington (1968) amazingly formalizes the bargaining problem as "represented quite generally by ... (1) a pair of variables q_1, q_2 representing the demands of the bargainers at any point in time." This view held at least since Hicks (1932) and Zeuthen (1930), whose concession games unfolded in one instantaneous mental flash.

[6] Our Battle of Sexes with fixed offers has this flavour for on-path, but not off-path, offers.

[7] eg. Fershtman and Seidmann (1993) assume players can't accept worse offers than they have rejected.

$T(h^n) = (t,k)$ and $h^\ell \subset h^n$ for all $\ell \neq n$, then $h^\infty = h^n$ and $T(h^\infty) = (t,k)$. In this case, $(t,k) \not\prec T(h^\infty)$, and we're done. Otherwise, choose n so that $h^\infty_{(t,k)} \subset h^n$ and $T(h^n) \succ (t,k)$. Then $h^\infty_{(t,k)} = h^n_{(t,k)}$, and $\psi_\sigma(h^\infty_{(t,k)}) \subset h^n \subset h^\infty$ since $h^n \in H(\sigma)$. So $h^\infty \in H(\sigma)$. □

For any time $t_1 \in \mathbb{R}_+$ let

$$H^{t_1}(\sigma) = \{\, h \in H(\sigma) \mid T_1(h) \leq t_1 \,\} \quad \text{and} \quad \theta(t_1) = \sup_\succ \{\, T(h) \mid h \in H^{t_1}(\sigma) \,\}.$$

The supremum in the definition of $\theta(t_1)$ is always attained. For example, if there is a cluster point in real time at $\theta_1(t_1)$, then there exists a sequence $\{\hat{h}^n\} \subset H^{t_1}(\sigma)$ with $T_1(\hat{h}^n) \uparrow \theta_1(t_1)$. In this case, the supremum is attained at $\bigcup_{n\in\mathbb{N}} \hat{h}^n \in H^{t_1}(\sigma)$. And if there is a deadlock at $\theta_1(t_1)$, then there is a sequence $\{\hat{h}^n\} \subset H^{t_1}(\sigma)$ such that $T(\hat{h}^n) = (\theta_1(t_1), n)$, and again the supremum is attained at $\bigcup_{n\in\mathbb{N}} \hat{h}^n$.

For each $n \in \mathbb{N}$, let $h^n \in H^n(\sigma)$ be such that $T(h^n) = \theta(n)$.

Claim 2 *For every $h \in H(\sigma)$ with $T_1(h) < +\infty$, there exists $n \in \mathbb{N}$ such that $h \subset h^n$.*

The claim asserts that $\{\, h \in H(\sigma) \mid T_1(h) < +\infty \,\} = \bigcup_{n\in\mathbb{N}} H^n(\sigma)$.

Proof: Let $h \in H(\sigma)$ be such that $T_1(h) < \infty$. By definition, $h \subset h^n$ for all $n \geq T_1(h)$. If $\psi_\sigma(h) \neq h$, then $\bar{\sigma}(h) = (i, s, (t,k))$ where $t < \infty$. Therefore, $T_1(\psi_\sigma(h)) = t < \infty$, and $\psi_\sigma(h) \subset h^n$ for all $n \geq T_1(\psi_\sigma(h))$. □

Claim 3 *The map $\sigma \mapsto h^\sigma \in \mathcal{H}$ is well-defined by $h^\sigma = \bigcup_{n\in\mathbb{N}} h^n$.*

Proof: Denote $h^\infty = \bigcup_{n\in\mathbb{N}} h^n$. Claim 1 implies $h^\infty \in H(\sigma)$. We first show that $\psi_\sigma(h^\infty) = h^\infty$. If $T_1(h^\infty) < +\infty$, Claim 2 yields $h^\infty \subset h^n$ for some n. In this case, $h^\infty = h^n = h^\ell$ for all $\ell > n$, and $\psi_\sigma(h^\infty) = \psi_\sigma(h^n) = h^n$. (This case occurs when σ leads to an offer that is accepted, or to a juncture where no player speaks again.) Next, if $T_1(h^\infty) = \infty$, then $\psi_\sigma(h^\infty) = h^\infty$, by definition.

We finally show that if $\bar{h} \in H(\sigma)$ is such that $\psi_\sigma(\bar{h}) = \bar{h}$, then $h^\infty \subset \bar{h}$. It is enough to show that $h \subset \bar{h}$ for all $h \in H(\sigma)$. By contradiction, assume this is not the case. Let

$$(\bar{t}, \bar{k}) = \sup_\succ \{\, T(h) \mid h \in H(\sigma) \text{ and } h \subset \bar{h} \,\}.$$

Assume first that $\bar{t} < \infty$. Then the supremum is attained (the argument is similar to that in Claim 1). Then there exists $h \in H(\sigma)$ such that $T(h) = (\bar{t}, \bar{k})$. If $\psi_\sigma(h) = h$, then $h^\infty = h \subset \bar{h}$, a contradiction. If $\psi_\sigma(h) \neq h$, then, since $T(\psi_\sigma(h)) \succ T(h) = (\bar{t}, \bar{k})$, $\psi_\sigma(h) \not\subset \bar{h}$, contradicting the fact that $\bar{h} \in H(\sigma)$. Now assume that $\bar{t} = +\infty$. In this case, $\bar{h} \supset h^n$ for all n, and thus $\bar{h} \supset h^\infty$

Thus, h^∞ is the smallest element of $H(\sigma)$ with $h^\infty = \psi_\sigma(h^\infty)$. Hence, $h^\infty = h^\sigma$. □

A.2 Sensitivity Analysis: Proof of Theorem 5

We will expand the set of states and view the bargaining process on the equilibrium path as a stochastic process, where agreement corresponds to absorption on the Pareto frontier. With each $w \in W = (0,1)$ we identify two states, w^A and w^B. The first represents the *absorbing* (or *agreement*) state $(w, 1-w)$ on the Pareto frontier, and the second the *transient* (or *bargaining*) state $(w, \varphi(w))$. Abusing notation, denote $\mu_i(w, \varphi(w))$ by $\mu_i(w)$.

For every $v \in \mathscr{A}$, let $\gamma(v) = \lambda_1(v)/[\lambda_1(v) + \lambda_2(v)]$ be the chance that in equilibrium player 1 makes the next offer at the aspiration vector v. Define $e(w) = 1 - w$, $w \in [0,1]$.

Below, we will exploit the fact that our bargaining process is really a mixture of the two artificial processes where only one of the players $i = 1,2$ makes all the offers. Indeed, let $M_i^N(w,B)$ ($M_i^Y(w,B)$, resp.) be the chance that from an initial state $w \in W$, the next state (after the next offer is accepted or rejected) is a transient (absorbing) state in $B \in \mathscr{B}(W)$, when player i makes all the offers (that is, when we set $\lambda_j \equiv 0$). Then,

$$M_1^N(w,B) = \int_{\varphi(B)} (1 - \alpha_2(w,x))\mu_1(dx|w), \;\; M_1^Y(w,B) = \int_{e(B)} \alpha_2(w,x)\mu_1(dx|w),$$
$$M_2^N(w,B) = \int_B (1 - \alpha_1(w,x))\mu_2(dx|w), \;\; M_2^Y(w,B) = \int_B \alpha_1(w,x)\mu_2(dx|w)$$

for all $w \in W$ and $B \in \mathscr{B}(W)$. The bargaining stochastic process is described by the following transient-to-transient and transient-to-absorbing transition probability functions $M^N, M^Y : [0,1] \times \mathscr{B} \to [0,1]$:

$$\begin{aligned} M^N(w,B) &= \gamma(w)M_1^N(w,B) + (1-\gamma(w))M_2^N(w,B) \\ M^Y(w,B) &= \gamma(w)M_1^Y(w,B) + (1-\gamma(w))M_2^Y(w,B). \end{aligned}$$

Thus, $M^N(w,B)$ is the mixture of the chances $M_1^N(w,B)$ and $M_2^N(w,B)$, i.e. the probability that from an initial state $w \in W$, the next state is a transient state in B in our actual bargaining process.

Let $\mathscr{L}$ be the space of functions $K : W \times \mathscr{B}(W) \to \mathbb{R}$ such that for each $w \in W$, $K(w,\cdot)$ is a signed Borel measure on W, and for each $B \in \mathscr{B}(W)$, $K(\cdot, B)$ is a (Borel) measurable function.

Let $\mathcal{K} \subset \mathscr{L}$ be the set of stochastic kernels K, where for each $w \in W$, $K(w,\cdot) \in \Delta(W)$. A stochastic kernel K represents the transition probabilities from transient to absorbing states for some stochastic process. For each $w \in W$ and $B \in \mathscr{B}(W)$, $K(w,B)$ is the probability that when the process starts at the transient state w, the first absorbing state it visits is an element of B. For any kernel $K \in \mathcal{K}$, let $\bar{K}(w) = \int_W xK(w,dx)$ be the expected absorbing state when the process starts at the bargaining state $w \in W$. For any two kernels $K_1, K_2 \in \mathcal{K}$, write $K_1 \preceq K_2$ if $\bar{K}_1(w) \le \bar{K}_2(w)$ for all $w \in W$.

We assume $\hat{r}_1/r_1 > \hat{r}_2/r_2$, so that $\hat{\gamma}(w) > \gamma(w)$ for all $w \in W$. That is, player 1 is relatively more impatient when interest rates are $(\hat{r}_1, \hat{r}_2)$ than when they are (r_1, r_2). Let Q (resp. $\hat{Q}$) be the kernel corresponding to the stochastic process generated by the AE σ (resp. $\hat{\sigma}$). We now show that player 1's expected pie split is *higher* when the interest rates are $\hat{r}$ (and the players follow $\hat{\sigma}$) than when they are r (and the players follow σ).

Define $\Psi : \mathscr{L} \to \mathscr{L}$ by

$$\Psi(K)(w, B) = \gamma(w)\Psi_1(K)(w, B) + (1 - \gamma(w))\Psi_2(K)(w, B)$$

for each $K \in \mathscr{L}$, where

$$\Psi_i(K)(w, B) = \int_W K(x, B) M_i^N(w, dx) + M_i^Y(w, B)$$

for each $w \in W$ and $B \in \mathscr{B}(W)$. The stochastic kernel Q satisfies the standard equation $Q(w, B) = \Psi(Q)(w, B)$ for all $w \in W$ and $B \in \mathscr{B}$. Similarly, $\hat{Q}$ is a fixed point of the map $\hat{\Psi}$ defined using $\hat{\gamma}$ in place of γ. Clearly, $\Psi(\mathcal{K}) \subset \mathcal{K}$. We will show that for an appropriate norm on $\mathscr{L}$, Ψ ($\hat{\Psi}$) is a contraction and therefore Q ($\hat{Q}$) is the unique fixed point.

Step 1 *There exists a positive measure ξ on $(W, \mathscr{B}(W))$ and $\beta \in (0, 1)$ such that*

$$\int_W M^N(x, B) d\xi(x) \leq \beta\xi(B) \quad \textit{for all } B \in \mathscr{B}(W),$$

and $\xi(B) > m(B)$ for all $B \in \mathscr{B}(W)$, where m denotes the Lebesgue measure.

Proof of Step 1: Consider the modified Markov process with states $\hat{W} = (0, 1]$ and stochastic kernel $\hat{M}$ defined below. In this modified process, the state 1 replaces the absorbing states of the bargaining stochastic process, but it is not an isolated absorbing state. Instead, state 1 acts as a 'revolving door', sending the process back to a 'transient state' each time the process visits state 1. The modified process has only one absorbing class that contains all the states $\hat{W}$. Let $1 - \underline{\alpha} < \beta < 1$, and for any subset B, denote by χ_B the indicator function: $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \notin B$. The kernel $\hat{M}$ is a perturbation of the kernel M. For each $x \in W = (0, 1)$ and Borel set $B \subset W$, the kernel $\hat{M}$ amplifies the probability $M(x, B)$ of going from x to B by β^{-1}, and makes the probability of moving from state x to state 1 equal to $\hat{M}(x, \{1\}) = 1 - \hat{M}(x, W)$. That is, for each $B \in \mathscr{B}(\hat{W})$ and $x \in W$,

$$\hat{M}(x, B) = M^N(x, B)/\beta + [1 - M^N(x, W)/\beta]\chi_B(1),$$

and $\hat{M}(1, B) = m(B)$. Let $\hat{\xi}$ be an invariant measure of $\hat{M}$. Then

$$\hat{\xi}(B) = \frac{1}{\beta} \int_W M^N(x, B)\hat{\xi}(dx) + \hat{\xi}_1 m(B \quad \text{for all } B \in \mathscr{B}(\hat{W}),$$

where $\hat{\xi}_1 \equiv \hat{\xi}(\{1\})$. Since $M^N(x, W) \leq 1 - \underline{\alpha} < \beta$ for all $x \in W$,

$$\hat{\xi}(W) = \frac{1}{\beta} \int_W M^N(x, W)\hat{\xi}(dx) + \hat{\xi}_1 < \int_W \hat{\xi}(dx) + \xi_1 = \hat{\xi}(W) + \hat{\xi}_1.$$

Hence, $\hat{\xi}_1 > 0$. Let ξ be the measure defined by $\xi(B) = \hat{\xi}(B)/\hat{\xi}_1$ for all $B \in \mathscr{B}(W)$. □

Endow $\mathscr{L}$ with the norm

$$\|K\| = \int_W \int_W |K(w, dx)| d\xi(w) = \int_W |K(w, \cdot)|(W) d\xi(w),$$

where $|K(w, \cdot)|$ is the positive measure $K_+(w, \cdot) + K_-(w, \cdot)$. Then, $\mathscr{L}$ is a Banach space.

Step 2 Ψ *is a contraction mapping and has a unique fixed point* Q.

Proof of Step 2: Pick $K, K' \in \mathcal{L}$ and let $D(w,B) = |K(w,\cdot) - K'(w,\cdot)|(B)$ for all $w \in W$ and $B \in \mathscr{B}(W)$. For each $B \in \mathscr{B}(W)$, we have that

$$\Psi(K)(w,B) - \Psi(K')(w,B) = \int_W [K(x,B) - K'(x,B)] M^N(w,dx).$$

Hence

$$\begin{aligned} \|\Psi(K) - \Psi(K')\| &\leq \int_W \int_W D(x,W) M^N(w,dx) d\xi(w) \\ &\leq \beta \int_W D(w,W) d\xi(w) = \beta \|K - K'\|. \end{aligned}$$ □

Since $\mathcal{K}$ is a closed set in $\mathscr{L}$ and $\Psi(\mathcal{K}) \subset \mathcal{K}$, the unique fixed point satisfies $Q \in \mathcal{K}$.

So far, we have used player 1's expectation as the state variable of our bargaining stochastic process. Alternatively, we could have used player 2's expectation as the state variable, and for each $w \in W$ identify the transient state $(\varphi^{-1}(w), w)$ and the absorbing state $(e(w), w)$. For each kernel $K \in \mathcal{K}$ there corresponds a 'dual' kernel K^*, defined by $K^*(w,B) = K(\varphi^{-1}(w), e(B))$ for each $w \in W$ and $B \subset \mathscr{B}(W)$. When player 2 has an initial aspiration value w, player 1 has the initial aspiration value $\varphi^{-1}(w)$, and player 2's final share is in B iff player 1's share is in $e(B)$. Thus, $K^*(w,B)$ represents the probability that when player 2's expected value is w, his share in the final split is in the set B.

Step 3 *The fixed point* Q *of* Ψ *satisfies: (i)* $\bar{Q}(w) \geq w$ *for all* $w \in W$*; (ii)* $\bar{Q}^*(w) \geq w$ *for all* $w \in W^*$.

Proof of Step 3: We only to prove (*i*); part (*ii*) is the same as (*i*) with the players' roles reversed. Recall the definition of $\tau(v)$ in Section 4. Let $\tilde{\tau}(w,x) = E[\tau(w,\varphi(w))|\text{final allocation is } (x, 1-x)]$. Then

$$w = \int_W x e^{-\tilde{\tau}(w,x)} Q(w,dx) < \int_W x Q(w,dx) = \bar{Q}(w).$$ □

Step 4 $\Psi_1(Q) \preceq \Psi_2(Q)$.

Proof of Step 4: Denote $J_i = \Psi_i(Q)$, $i = 1,2$. Since the sum of the players' expected final shares cannot exceed 1, player 2's expected share starting at $w \in \bar{W}$ cannot exceed $1 - \bar{K}(w)$. By (*ii*), this expected share is at least $\varphi(w)$ (since the aspiration value of w for player 1 corresponds to the aspiration value $\varphi(w)$ for player 2). So $1 - \bar{K}(w) \geq \varphi(w)$ or $\bar{K}(w) \leq 1 - \varphi(w)$. In particular, $\bar{K}(\varphi^{-1}(w)) \leq 1 - w$. Hence

$$\begin{aligned} \bar{J}_1(w) &= \int_{\varphi(W)} [(1-\alpha_2(w,x))\bar{Q}(\varphi^{-1}(x)) + \alpha_2(w,x)(1-x)]\mu_1(w)(dx) \\ &\leq \int_{\varphi(W)} [(1-\alpha_2(w,x))(1-x) + \alpha_2(w,x)(1-x)]\mu_1(w)(dx) = 1 - \bar{x}_1(w)) \\ \bar{J}_2(w) &= \int_W [(1-\alpha_1(w,x))\bar{Q}(x) + \alpha_1(w,x)x]\mu_2(w)(dx) \\ &\geq \int_W [(1-\alpha_1(w,x))x + \alpha_1(w,x)x]\mu_2(w)(dx) = \bar{x}_2(w)). \end{aligned}$$

By B3, $\bar{x}_2(w) \geq 1 - \bar{x}_1(w)$, and thus $\Psi_1(Q) = J_1 \preceq J_2 = \Psi_2(Q)$. □

We finally complete the proof of Theorem 5. Since $\gamma(w) < \hat{\gamma}(w)$ for each $w \in W$,

$$\bar{Q}(w) = \gamma(w)\bar{J}_1(w) + (1 - \gamma(w))\bar{J}_2(w) \leq \hat{\gamma}(w)\bar{J}_1(w) + (1 - \hat{\gamma}(w))\bar{J}_2(w)$$

for all $w \in W$. This implies that $Q = \Psi(Q) \preceq \hat{\Psi}(Q)$. It is also easy to see that if K_1 and K_2 are two kernels such that $K_1 \preceq K_2$, then $\hat{\Psi}(K_1) \preceq \hat{\Psi}(K_2)$. Therefore $\hat{\Psi}(Q) \preceq \hat{\Psi}(\hat{\Psi}(Q)) = \hat{\Psi}^{(2)}(Q)$. By induction, $Q \preceq \hat{\Psi}^{(n)}(Q) \to \hat{Q}$, so $Q \preceq \hat{Q}$ as required. □

References

Abreu, D., and F. Gul (2000): "Bargaining and Reputation," *Econometrica*, 68(1), 85–117.

Abreu, D., and D. Pearce (2001): "A Behavioral Model of Bargaining with Endogenous Types," Princeton mimeo, May.

Bergin, J., and W. B. MacLeod (1993): "Continuous Time Repeated Games," *International Economic Review*, 34(1), 21–37.

Coddington, A. (1968): *Theories of the Bargaining Process*. Aldine Publishing Company, Chicago.

Edgeworth, F. (1881): *Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences*. C. Kegan & Company, London.

Fershtman, C., and D. Seidmann (1993): "Deadline Effects and Inefficient Delay in Bargaining with Endogenous Commitments," *Journal of Economic Theory*, 60(1), 306–321.

Fudenberg, D., and J. Tirole (1985): "Pre-emption and Rent Equalization in the Adoption of New Technology," *Review of Economic Studies*, 52(3), 383–401.

Gilboa, I., and D. Schmeidler (1994): "Reaction to Price Changes and Aspiration Level Adjustments," KGSM-MEDS mimeo, June.

Harsanyi, J. (1956): "Approaches to the Bargaining Problem Before and After the Theory of Games," *Econometrica*, 24, 144–157.

Hicks, J. (1932): *The Theory of Wages*. Macmillan, London.

Karandikar, R., D. Mookherjee, and D. Ray (1998): "Evolving Aspirations and Cooperation," *Journal of Economic Theory*, 80, 292–331.

Osborne, M. (1985): "The Role of Risk Aversion in a Simple Bargaining Model," in *Game-Theoretic Models of Bargaining*, ed. by A. Roth, chap. 9. Cambridge University Press, Cambridge.

Perry, M., and P. Reny (1993): "A Non-cooperative Bargaining Model with Strategically Timed Offers," *Journal of Economic Theory*, 59(1), 50–77.

Rubinstein, A. (1982): "Perfect Equilibrium in a Bargaining Model," *Econometrica*, 50(1), 97–109.

Sakovics, J. (1993): "Delay in Bargaining Games with Complete Information," *Journal of Economic Theory*, 59(1), 78–95.

Siegel, S. (1957): "Level of Aspiration and Decision Making," *Psychological Review*, 64, 253–262.

Siegel, S., and L. Fouraker (1960): *Bargaining and Group Decision Making.* McGraw Hill, New York.

Simon, L., and M. Stinchcombe (1989): "Extensive Form Games in Continuous Time: Pure Strategies," *Econometrica*, 57(5), 1171–1214.

Stahl, D. (1993): "Strategic Choice of Waiting and Reaction Times in Bargaining," *Economics Letters*, 43(1), 31–34.

Ståhl, I. (1972): *Bargaining Theory.* Stockholm Research Institute, Stockholm.

Yildiz, M. (2001): "Sequential Bargaining without a Common Prior on the Recognition Process," MIT working paper 01-43.

Zeuthen, F. (1930): *Problems of Monopoly and Economic Warfare.* Routledge and Sons, London.