Frictional Matching Models[*]

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Abstract

This article offers a self-contained graduate lecture on the developments in frictional matching models 1990–2010, exploring how search frictions skew the matches that occur. This research succeeded by exploiting new tools from monotone methods under uncertainty. Seeing how this journey plays out is instructive in itself for economic theory. The article also fleshes out the different attack on the existence of equilibrium required for these search and matching models.

1 Introduction

Frictional matching models tell the story of the death of a salesman — specifically, the Walrasian auctioneer. In his stead is a mixture of chance and choice that jointly determine pairwise transactions. Stepping away from the free-market paradigm is compelling in this setting, for if *any* market lacks a centralized trading system, it is the one for match partners.

Earlier work in frictional matching models that continues to this day[1] took the viewpoint that these models offered more realistic foundations for the Walrasian predictions. As such, the convergence properties were explored as the frictions vanished. By contrast, this literature assumes that the frictions are non-vanishing, and themselves shed useful light on the matches that do happen.

Our focus is on matching among heterogenous individuals with match synergies. We compare how these search frictions may subvert these synergies, and how sorting still emerges, possibly with more powerful match complementarities. Another restriction imposed here is to study productive and social matching rather than matching for trade purposes. This is both

[*]Hector Chade and my advanced theory class at Wisconsin provided helpful feedback.
[1]For instance, Gale (1986), Rubinstein and Wolinsky (1990), and Satterthwaite and Shneyerov (2007).

a technically and an economically more cohesive class, but ignores a large macro literature on money, for instance. This class has a common forefather in two path-breaking "marriage" papers, Gale and Shapley (1962) and Becker (1973), who initiated the study of the pure matching model and the Walrasian matching paradigm.

In this setting, individuals are both consumer and consumption good alike. This makes the question of the mutually agreeable matches a much richer one. In the Gale and Shapley case of *nontransferable utility* (NTU), "the match partner is all you get". Once we allow matches to be greased by side payments in the *transferable utility* (TU), we find a richer class subsuming Becker's.[2] In a frictional setting, we will see that some surprising matching outcomes can emerge in each case.

At the heart of search and matching models is the importance of rents. The competitive models of the world seem at odds with so much of our life experiences — a fact which motivated Stigler's original foray into price search. In "real life", we are often faced with *positive surplus* of inside over outside options — the extent by which the currently available partner is better than a future best option. Understanding this wedge plays a critical role in all of our analysis.[3]

In this environment, of primary interest is the allocation question. The fundamental benchmark is *positive assortative matching* — the "best with the best", and so on. When does this compelling outcome obtain? It so happens that search frictions render this otherwise natural conclusion somewhat murkier. We will see that search frictions frustrate sorting, but show how it can be rescued as a prediction.

This article resolves this question, and in so doing develops and fleshes out insights from a relatively new toolkit for graduate economic theory on monotone methods under certainty and uncertainty.[4] As a result, the route taken is instructive lesson into how to apply these methods.

[2]In a nutshell, NTU presumes that there are match fixeties that cannot be assuaged by monetary transfers. This paradigm is appropriate whenever the desirability of match continuation is not mutual. If TU aptly described the world, then all divorces would be mutual — for a match either would or would not have surplus.

[3]By contrast to search frictions, the critical surplus with an informational friction is the excess of the complete information versus incomplete information value. Anderson and Smith (2010) show that this friction too engenders surprising matching patterns.

[4]Karlin (1968), Karlin and Rinott (1980), and the more specific results for the log-supermodular case (total positivity of order two) in Athey (2002).

payoffs	Y_1	Y_2	Y_3
X_1	$2, 23$	$4, 24$	$\mathbf{6, 25}$
X_2	$4, 16$	$\mathbf{8, 18}$	$12, 20$
X_3	$\mathbf{6, 9}$	$12, 12$	$18, 15$

sums	Y_1	Y_2	Y_3
X_1	**25**	28	31
X_2	20	**26**	32
X_3	15	24	**33**

Figure 1: **Match Payoffs and Payoff Sums.** We indicate the stable (blue) matching for the NTU match payoffs at left. Once monetary or utility transfers are allowed (the TU case at right), the corresponding payoff sums are the relevant benchmark, and the stable matching switches (now red).

2 Frictionless Matching Benchmarks

2.1 A Motivational Example

Let's consider a matching market between two sides of the market, called X's and Y's. The Gale-Shapley algorithm (1962) will discover the stable matching. Consider the payoffs (in utils) in Figure 1. If the X's do the proposing, then everyone first asks Y_3, since he is best for all. Then Y_3 accepts X_1, sealing that match. At the next round, X_2 and X_3 ask Y_2, who then accepts X_2. Finally, X_3 and Y_1 match. The final matching is (X_1, Y_3), (X_2, Y_2), (X_3, Y_1). Or, if instead Y's propose, each asks X_1, who accepts Y_3. Each Y then asks X_2, who chooses Y_2. Finally, X_3 and Y_1 match. The same matching arises in this example — and so there is a unique stable matching.

Next observe how these matches are inefficient. For instance, X_3 is willing to sever his match with Y_1, and pay Y_3 up to $18-6 = 12$ to join forces with him; furthermore, this deal sweetener exceeds the loss $25 - 15 = 10$ that Y_3 suffers from quitting his match with X_1. The above unique NTU matching is thus unstable once transfers are allowed. Intuitively, the total payoffs (at right in Figure 1) determine matching decisions once individuals can offer side-payments — and, as they say, money changes everything!

While side-payments have the ring of illegality, at issue here is the world with wages and other plain vanilla transfers. These are driven by *outside options* — namely, the least amount that someone can *guarantee* himself in the matching model by exercising his best other match. In this setting, the outside options are wages. Imagine that two agent firms (i, j) can costlessly form that pay out wages using the match payoff sums $f(i, j)$. Let the Walrasian auctioneer call out wages for everyone until all markets clear for the individuals. Call the wages of the X and Y sides w_i^X and

w_j^Y. Then the assortatively matched firms make nonnegative profits exactly when $w_i^X + w_i^Y \leq f(i,i)$ for $i = 1, 2, 3$. Meanwhile, the non-matched firms will not form only if wages over-exhaust output, namely, $w_i^X + w_j^Y \geq f(i,j)$ for all payoffs $f(i,j)$ $(i \neq j)$ at the right in Figure 1. One can show that these inequalities have a solution.[5]

2.2 Conditions for Assortative Matching

The more general matching model has two distinct sides of the market — at one end of the spectrum, men and women in a social setting, and at the other, workers and firms in an employment context. A simpler matching setting is the "unisex" model of symmetric partnerships.

Individuals can be of several observable types — the matching appeal of the social mate, the productivity of the worker or the excellence of the firm. Any individual of type x matches like another other such individual, and thus, we can simply refer to the types. Types are simply scalars $x \in [0,1]$, and have an atomless distribution with cdf $L(x)$ and finite density $\ell(x) \equiv L'(x)$. We assume that a type x earns $f(x,y)$ when matched with a type y. Throughout, the function $f : \mathbb{R}^2 \mapsto \mathbb{R}_+$ is a nonnegative and for simplicity, continuous and twice differentiable on $\mathbb{R}^2$.

This is in many ways a ridiculously simplified setting. Firstly, beauty is not in the eye of the beholder: There is no disagreement about a worker's productivity across jobs, nor a firm's appeals to different workers. Second, there is no role for choice, such as effort, apart from the matching decision. Relaxing these restrictions is an important avenue for future research.

In the symmetric unisex case, the production function obeys $f(x,y) \equiv f(y,x)$. For as in the earlier example, this arises when matched individuals can share outputs. *Positive assortative matching* (PAM) occurs when types sort into matches according to their quantile. In this uni-sex environment, each type $x \in [0,1]$ matches with another type x. By contrast, with *negatively assortative matching* (NAM), each type x matches with the "opposite percentile" type $y(x)$ — namely, the one for whom $L(y(x)) \equiv 1 - L(x)$.

A passing result in Becker 1973 considered the NTU setting — namely,

[5] For we know on the one hand, that there is solution to the social planner's problem, and on the other hand, that it solves the Kuhn-Tucker optimization conditions. If we let each type's wage be its shadow value to the planner (i.e., the Lagrange multiplier), then the complementary slackness conditions yield the desired inequalities.

the stable matches determined by the Gale-Shapley algorithm.[6] Consistent with the earlier example, Becker found out that the "market outcome" in the NTU case is PAM exactly when preferences over partners are increasing: $f_2(x, y) > 0$. Essentially, the market clears "top to bottom" in this case.

Becker (1973) focused on the transferable utility (TU) world, allowing utility side payments. He then deduced that PAM is both efficient and a competitive equilibrium when types are productive complements, i.e. having a positive cross partial derivative $f_{12}(x, y) > 0$. Economists have also seized on the lattice-theoretic label *supermodular* for this property (Topkis, 1998).

To see Becker's efficiency claim,[7] let's consider any pair of types $x < y$. Integrating $f_{12} > 0$ on any rectangle $[x, y] \times [x, y]$ yields a positive discrete cross partial difference $f(y, y) - f(x, y) > f(y, x) - f(x, x)$. In other words, $f(x, x) + f(y, y) > 2f(y, x)$. This implies that mixed matches (x, y) are inefficient: For a mass $\epsilon > 0$ of such mixed matches near (x, y) can create equal $\epsilon/2$ masses of matches near (x, x) and (y, y) with higher total output.

We now prove Becker's claim that PAM is a competitive equilibrium. Let's introduce the *match surplus function* $s(x, y)$ for easier comparison later:

$$s(x, y) = f(x, y) - w(x) - w(y) \tag{1}$$

This measures the amount by which the inside match option exceeds the sum of the two best outside options. Since every type x will find the best match y, we know that he solves the first order condition $s_2(x, y) = 0$. Then this "ideal partner", say $y(x)$, is rising in x precisely because $s_{12}(x, y) \equiv f_{12}(x, y) > 0$. In the unisex matching world, the only increasing function that clears the market is $y(x) = x$. So try the candidate wage profile $w(x) \equiv f(x, x)/2$, namely equal sharing of the output, which is obviously feasible. Second, given these wages, no other matches are profitable:

$$w(x) + w(y) \equiv f(x, x)/2 + f(y, y)/2 > f(x, y) \tag{2}$$

This follows because of symmetry and supermodularity of $f(x, y)$.

Conversely, consider *submodular* production, with $f_{12} < 0$. Then inequality (2) reverses, and the equilibrium (and efficient) outcome now in-

[6] Eeckhout (2000) formally proves this, among other things.

[7] As so often happens, mathematicians got there long before economists, and with far greater generality, but without the essential economic context. In this case, Lorentz (1953) deduced the formal content of Becker's Pareto claim.

volves NAM. Reversing the order on types, define the production function $g(x,y) \equiv f(x,1-y)$. Then $g_{12} > 0$, and therefore PAM is optimal for the production function $g(x,y)$. Logically, NAM is optimal for $f(x,y)$.

The distinction between the TU and NTU allocation predictions seen in the motivational example emerges for any production functions that are submodular and increasing, or supermodular and decreasing. In the first case, NTU predicts PAM and TU predicts NAM; in the second case, the opposite holds. For instance, types in the example were $1, 2, 3$, and the X-payoffs $2XY$, and the Y-payoffs $XY - 8X + 30$. Easily, the own payoffs were falling in the partner types, but the shared payoff sums were supermodular.

The importance of Becker's focus on complementarity was underscored in Kremer (1993). He pointed out that production functions where success depends multiplicatively on all types — such as when both parties must successfully perform a task — were automatically complementary; thus, Becker's sorting theorem applies in these settings. Kremer and Maskin (1996) instead motivate non-complementary cases with examples where the "strongest link" matters — capturing matches with a defined "second banana" role. This setting yields some interesting discontinuous efficient matching patterns. Adding more structure, Ackerberg and Botticini (2002) explore matching in a contractual setting between principals (landowners) whose projects vary in their riskiness and agents (tenants) of varying risk tolerances. Serfes (2005) explores a more general setting, where either PAM or NAM emerges. Here, positive sorting means that poor tenants (and so very risk averse) cultivated risky vines, while wealthier tenants took charge of safer cereal production.

3 Matching with Search Frictions

Becker's frictionless matching world predicts that the long side of the market can get capriciously shafted. For instance, with 10^9 women and $10^9 + 1$ men, if all matches yield payoff 1 and being unmatched yields nothing, then men earn a zero wage. But if one man dies, as so often happens, then a 50-50 split may emerge. The search world that we now explore does not suffer from this somewhat counterfactual fragility.

3.1 Anonymous Search for Partners

Anonymous search is a natural formulation of search frictions that builds on the story underlying the consumer price search literature (Lippman and McCall, 1976). Proceeding in continuous time, we imagine that unmatched individuals find it hard to locate partners. The simplest story might be the *linear search technology*. It assumes that potential partners arrive at some exogenous and fixed "rendezvous" rate $\rho > 0$ — and so with chance $\rho\ dt$ in any infinitesimal length dt interval. These partners are randomly and representatively drawn from the pool of unmatched individuals on the other side of the market. But it is more tractable to analyze the *quadratic search technology*, in which ρ is scaled by the mass of unmatched partners. Or equivalently, one can imagine that randomly-chosen potential partners for any unmatched individual should arrive at rate ρ, but if that partner happens to be matched, then he misses the meeting.

Since there is a continuum of atomless (negligible) individuals, we can intuitively ignore mixed strategies, and simply assume that the strategies of all individuals assume the form of *acceptance sets* $A(x) \subseteq [0, 1]$. Having in mind that everyone is both consumer and consumption good, we must keep track of the inverse *opportunity set* $\Omega(x) \equiv \{y \in [0, 1] | x \in A(y)\}$ of each type x. So the sets $A(x)$ and $\Omega(x)$ correspond to the preferences and opportunities of type x. The *matching set* consists of all mutually agreeable matches, namely, $M(x) = A(x) \cap \Omega(x)$. Such matches in this intersection solve the double coincidence of wants.

In a further simplification, let us explore a world without on-the-job search. Namely, we venture that everyone is either matched and thus unavailable, or unmatched and searching. In this way, the opportunity cost of matching is that new partners cease to arrive; this intuitively affords individuals an incentive to *decline* some matches, and avoid trivialities.

Almost all successful research on equilibrium search and matching has assumed a steady-state model. For even the simplest of nonstationary environments can be notoriously intractable, and should only be attacked if the underlying theoretical exercise really turns on the nonstationarity.[8] We follow this lesson, which conveniently obviates any time subscripts.

There are two primary ways to secure a steady-state. First, we could

[8] Shimer and Smith (2001), still in progress, underscores how hard even the simplest nonstationary heterogeneous agent search model can be to analyze.

venture that matches dissolve at some fixed rate $\delta > 0$ — or equivalently, with chance approaching δdt in any small dt interval, where $\delta > 0$. Second, we could assume eternal matches (so that $\delta = 0$), and then posit a steady inflow of unmatched individuals that replaces the newly matched.

We assume that $f(x, y)$ now describes the flow payoffs, discounted at some interest rate $r > 0$. We can then define two interrelated values: First, the expected present (Bellman) value $V(x)$ of payoffs to x when initially unmatched, assuming optimal behavior; second, the expected present value $V(x|y)$ of payoffs to x when initially matched with y. Since the unmatched value is a pure option on getting matched, its return is the expected arrival rate of the expected surplus $s(x|y) = V(x|y) - V(x)$ from matching with an acceptable type $y \in M(x)$. If we let $U(y)$ denote the stationary cumulative distribution function of unmatched individuals, then we arrive at a natural expression for the return on the unmatched value: $rV(x) = \rho \int_{y \in M(x)} [V(x|y) - V(x)]U'(y)dy$.

The analysis simplifies if we use the *average* present *unmatched value* $v(x) = rV(x)$ and the *average matched value* $v(x|y) = rV(x|y)$, since they share the same flow payoff units. The earlier value accounting expression is then:

$$v(x) = (\rho/r) \int_{y \in M(x)} [v(x|y) - v(x)]U'(y)dy \tag{3}$$

Next, the average present value $v(x|y)$ of type x from matching with y includes a flow payoff $f(x, y)$, as well as a δ arrival rate of a capital loss of $v(x) - v(x|y)$. This yields the implicit equation

$$v(x|y) = f(x, y) + (\delta/r)[v(x) - v(x|y)] \tag{4}$$

3.2 Assortative Matching

As with Becker (1973), we ask what assumptions on productive interaction yield PAM for all levels of search frictions, and for all type distributions. More to the point, since possible mates are drawn from an atomless continuum, matching sets can no longer be point-valued. In this case, what exactly does PAM mean? We now explore a natural definition of PAM in this setting, that subsumes the PAM definition in the Walrasian setting. It should have the same econometric properties — that the expected partner of type x must be weakly increasing in x. We find a mathematically elegant

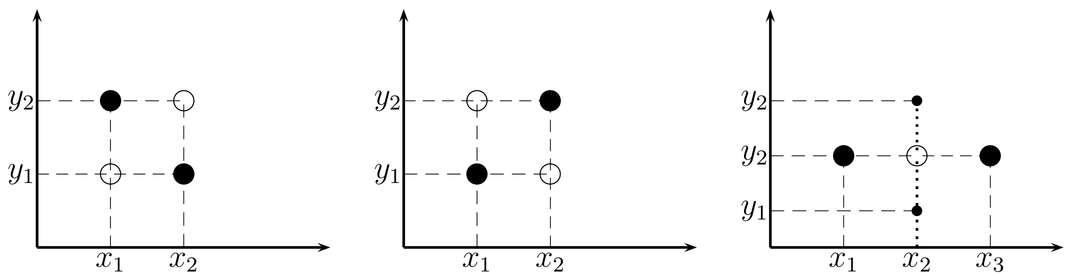

Figure 2: **Assortative Matching.** The left two panels respectively depict the definitions of PAM and NAM: If the pairs indicated by filled dots match, then the pairs indicated by hollow dots match as well. The right panel depicts the proof of convexity. If low and high types, x_1 and x_3, match with $y_2 \in (y_1, y_3)$, then so must a middle type $x_2 \in (x_1, x_3)$, given PAM or NAM.

lattice definition that achieves this. This definition is necessary and sufficient for the above econometric property to hold for all type distributions.

Let us reconsider the basic insight that PAM means that high types should match with other high types, and low types with other low types. One simple way of formulating this is that if any mixed high and low types are matched, namely the pairs (x_1, y_2) and (x_2, y_1), with $x_1 < x_2$ and $y_1 < y_2$, then so are the higher and lower types in each match, i.e., (x_1, y_1) and (x_2, y_2). (Equivalently, the matching set is increasing in the "strong set order".) Figure 2 illustrates this and the parallel definition of NAM.[9]

A useful property of this definition is that given PAM or NAM, every type has a convex matching set. This proof is graphically obvious, as seen in Figure 2. This implies that the matching set can be written $M(x) = [\theta(x), \psi(x)]$, with lower and upper bounds $\theta(x)$ and $\psi(x)$. Next, since the matching sets of the types on the vertical axis are convex, the lower bound function $\theta(x)$ is quasiconvex, and the upper function $\psi(x)$ is quasiconcave. This in turn offers an easier attack on the assortative matching characterization. For *if matching sets are convex, then PAM arises exactly when the lower and upper bounds are weakly increasing* — as this ensures the lattice definition of PAM in Figure 2. And conversely, PAM clearly fails when the upper or lower bound ever locally dip. This suggest an immediate definition that *if matching sets are convex, and the lower and upper bounds are strictly increasing whenever possible, then matching obeys strict PAM.*

[9] At a formal level, we ask that the set of mutually agreeable matches be a *lattice* for PAM: For all pairs of partners z, z' in $\mathbb{R}^2$, if z and z' are each in the matching set, so is the meet $z \wedge z'$ and join $z \vee z'$ (respectively, the vector-max and vector-min).

4 NTU Matching

Let us start in the world of NTU matching, as this will prove a springboard into the harder TU matching exercise. We will assume for now that an equilibrium exists, and describe who matches with whom. In one simple case, we will en route essentially construct an equilibrium in the process.

For simplicity, let's assume that preferences over match partners are monotone, with $f_2 > 0$. In this case, the acceptance set is $A(x) = [\theta(x), 1]$, for some cut-off partner $\theta(x)$ reminiscent of a reservation wage. Then the opportunity set is $\Omega(x) = \{y \in [0,1] | x \geq \theta(y)\}$. Thus, by our earlier discussion, PAM arises exactly when $\theta(x)$ is nondecreasing — so that higher types are choosier. Optimality is then captured by the coincidence of inside and outside options, or $f(x, \theta(x)) = v(x)$. In other words, one should match with all types that provide at least the unmatched value.

4.1 Block Segregation

There is one low-hanging fruit in this domain — so low-hanging, that it was repeatedly re-discovered in a "great minds thinking alike" rush of research in the 1990s.[10] Indeed, suppose that we can express output multiplicatively, as $f(x, y) = h_1(x)h_2(y)$. For instance, everyone might just care about his partner's type, $f(x, y) = y$. This implies that all types x share the same cardinal preferences over match partners y. Namely, we can create an affine transformation of match payoffs $f(x', y) = [h_1(x')/h_1(x)]f(x, y)$ — that includes the unmatched option, whose flow payoff is zero.[11] And in a world of uncertainty, cardinal preferences govern risky choices, such as the decision to accept a match, or to press on and search. As such, any two individuals with the same opportunity sets must make the same choices.

So motivated, consider the highest type $x = 1$. He is blessed by the matching affections of all types, which wonderfully simplifies his decision problem. Faced with search frictions, his optimal *reservation partner* or *threshold partner* is $\theta(1) < 1$. Then everyone in the interval $[\theta(1), 1]$ shares

[10] Roughly chronologically, it was found by Collins and McNamara, then me, then Bloch and Ryder, and then in some order by Coles and Burdett, Chade, and Eeckhout — the latter two impressively in their PhD theses.

[11] With explicit flow search costs, this fails, since it is not true that $-c = [h_1(x')/h_1(x)](-c)$ for all x, x'. Thus, block segregation would not arise. See Chade (2001) for more discussion.

the opportunity set of type 1, as well as the same cardinal preferences. *Ipso facto*, they will choose the same cut-off partner $\theta(1)$. Thus, the type interval $[\theta(1), 1]$ constitutes a closed matching set. But now this logic can be iterated, considering the preferences of individuals just slightly below $\theta(1)$. They too share a threshold partner $\theta(2) < \theta(1)$, and so on. What unfolds is a unique partition of $[0, 1]$ with class boundaries $\theta(1) > \theta(2) > \cdots$, and an equilibrium perhaps best described as "block segregation". It is easy to see that there will be only finitely many such boundaries exactly when $f(0, 0) > 0$. For as the threshold partner vanishes, the chance of lesser types finding a willing partner vanishes too. If the bottom type yields strictly positive payoff, then it is best to accept him and end the search.

Understanding the match allocation sheds insights into wages and values. For instance, block segregation induces upward jumps in the value function.

4.2 Strictly Increasing Matching Sets

One's curiosity is naturally piqued whenever continuous fundamentals lead to a discontinuous economic outcome. In this case, preferences and the search technology are quite smooth, while block-segregation sees similar agents making radically different match decisions. This striking matching outcome provides an equilibrium economic rationale for Victorian classist fables — attested by the clever "marriage and class" title of Burdett and Coles (1997). But the stark perfectly non-communicating nature of these classes almost demands a larger picture.

Let us ponder what is special about block segregation payoff functions of the form $f(x, y) = h_1(x)h_2(y)$. While motivated from decision theory, might some mathematical structure be relevant here? In particular, the function $\log f(x, y)$ is additively separable in x, y, or equivalently has a zero cross-partial derivative. Any such function is also known as log-modular. Could it be that stepping just outside this class offers the missing bigger picture? What if we consider strictly log-supermodular functions, for which $\log f(x, y)$ has a positive cross partial derivative?

To answer this question, let's consider two equivalent expressions for the value function. The first one relates it to the *inside option.* Inspired by the log-supermodularity insight, we work with logarithms whenever possible. Assume a differentiable threshold $\theta(x)$. Differentiating the log-optimality condition $\log v(x) \equiv \log f(x, \theta(x))$ in the type x yields:

$$\frac{v'(x)}{v(x)} = \frac{f_1(x, \theta(x))}{f(x, \theta(x))} + \theta'(x)\frac{f_2(x, \theta(x))}{f(x, \theta(x))} \tag{5}$$

Next, we proceed likewise from the purely accounting definition of the values as an *outside option* — namely, the present value of all future match output. It helps to ignore deaths, putting $\delta = 0$, so that the matched value in (4) is $v(x|y) = f(x, y)$. We could then re-write the Markov recursive equation (3). Instead, focus on the highest types for whom the world is their oyster, since everyone wishes to match with them. They obviously will employ a cut-off partner θ. Looking in the class of such matching rules, the resulting *policy value* $v_\theta(x)$ solves an analogous equation to (3):

$$v_\theta(x) = (\rho/r) \int_{y \geq \theta} [f(x, y) - v_\theta(x)]U'(y)dy = \frac{\rho \int_{y \geq \theta} f(x, y)U'(y)dy}{r + \rho[1 - U(\theta)]} \tag{6}$$

Since the best cut-off is the optimal threshold $\theta(x)$, the partial derivative of the policy value $v_\theta(x)$ in θ vanishes at $\theta = \theta(x)$. If we differentiate $\log v(x) \equiv \log v_{\theta(x)}(x)$ in x, then we arrive at an Envelope Theorem implication:

$$\frac{v'(x)}{v(x)} = \frac{\int_{y \geq \theta(x)} f_1(x, y)U'(y)dy}{\int_{y \geq \theta(x)} f(x, y)U'(y)dy} \tag{7}$$

Jointly, the optimization and accounting lessons (5) and (7) imply

$$\frac{f_1(x, \theta(x))}{f(x, \theta(x))} + \theta'(x)\frac{f_2(x, \theta(x))}{f(x, \theta(x))} = \frac{\int_{y \geq \theta(x)} f_1(x, y)U'(y)dy}{\int_{Y \geq \theta(x)} f(x, y)U'(y)dy} \tag{8}$$

We now finish the argument with an elementary insight about ratios. Note that the inequality $3/4 < 5/6$ implies $3/4 < (3 + 4)/(5 + 6) < 5/6$. This logic underlies the easy proof that if $a(t), b(t) > 0$ are smooth functions, and $[a(t)/b(t)]' > 0$, then we have the hopefully intuitive inequalities:

$$\frac{a(t_0)}{b(t_0)} < \frac{\int_{t_0}^{t_1} a(t)dt}{\int_{t_0}^{t_1} b(t)dt} < \frac{a(t_1)}{b(t_1)} \qquad \forall\, t_0 < t_1 \tag{9}$$

If f_1/f is strictly increasing in y, then this inequality implies that the right side of (8) exceeds the first term on its left side. From this, we deduce that

$\theta'(x) > 0$. This says that higher agents are more selective. This argument is tight in the following sense: For if f is strictly log-submodular, then past some point, even higher types are willing to accept lower types: $\theta'(x) < 0$.

The argument that the lower threshold rises in the type can be finished by inverting the matching set, and deducing that the upper bound $\psi(x)$ falls as the type x falls because $\theta(x)$ is increasing. One can show that this shifting upper bound reinforces the logic above.

Theorem 1 (PAM and NTU) *Assume x earns $f(x, y) > 0$ in a match with y, where $f_2(x, y) > 0$. Then the equilibrium NTU matching is block segregation if f is log-modular, and strict PAM if f is strictly log-supermodular.*

The left panel of Figure 3 illustrates the theorem, whereas the right panel shows the necessity of the log-supermodularity assumption. So quite unlike the frictionless NTU result in Becker (1973), monotonicity alone does not deliver PAM. It is not enough that one prefers that higher types are preferred. In fact, even Becker's supermodular condition for PAM in the Walrasian setting is not enough: That higher types benefit more from better partners is still too weak. Rather, the payoff of matching up must rise *proportionately* faster at higher types. For as an outside option, the value is the price of the agent's time. Then log-supermodularity alone[12] guarantees that the value of an agent's time rises faster than the value of matching with a fixed reservation partner; hence, the only way to equalize the inside and outside options is for the reservation partner to rise.

5 TU Matching

5.1 Values

Transferable utility models are richer as they do not exogenously specify the payoff of each agent; instead, this is determined by the strength of outside options. This demands that we understand the accounting of shared surplus. Since matches dissolve at rate δ, we have the parallel formula of (4):

$$v(x|y) + v(y|x) \equiv f(x, y) + (\delta/r)[v(x) + v(y) - v(x|y) - v(y|x)]$$

[12]Log-supermodularity is stronger than supermodularity if f is increasing in both types. In the smooth case, log-supermodular production says $(\log f)_{12} > 0$, and thus $f_{12}f > f_1 f_2$. So if $f_1, f_2 > 0$, then $f_{12} > 0$, and so production is supermodular.

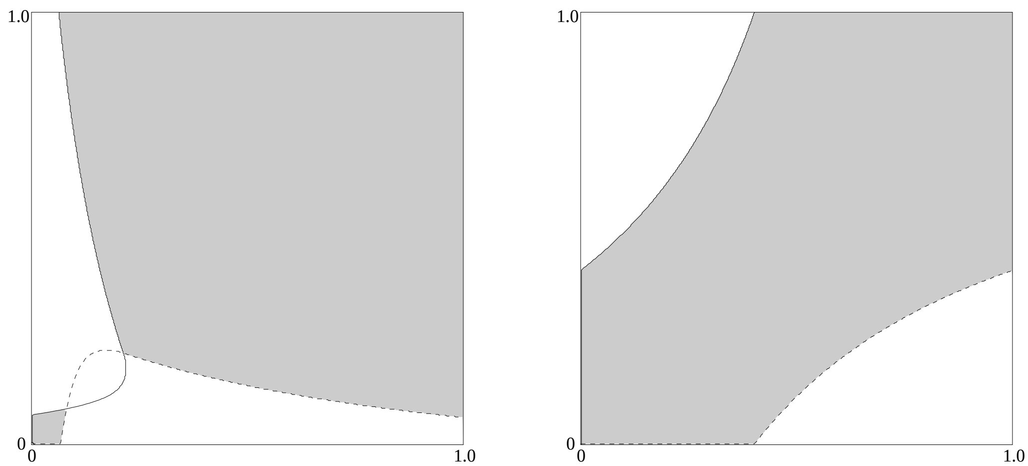

Figure 3: **NTU Matching with Anonymous Search.** At left is the graph of the matching set with payoff function $f(x,y) = x + xy + y$. Preferences are both increasing in partners, as the frictionless result requires, and supermodular ($f_{12} > 0$), and yet PAM fails. At right is the graph of the matching set with a log-supermodular $f(x,y) = e^{xy}$. This secures PAM.

This implies that the average total match surplus for both parties equals

$$[v(x|y) - v(x)] + [v(y|x) - v(y)] = \frac{r}{\delta + r}[f(x,y) - v(x) - v(y)] \qquad (10)$$

Since side payments are allowed in a TU model, parties agree to match precisely when there is non-negative surplus. Analogous to (1), we define the average *surplus function* $s(x,y)$ and the *matching set* $M(x)$ as follows:

$$s(x,y) \equiv f(x,y) - v(x) - v(y) \geq 0 \quad \Leftrightarrow \quad y \in M(x) \qquad (11)$$

The unmatched value serves a similar outside options role in this search model with transferable utility as the wage in Becker's model: Individuals agree to match exactly when their joint match value weakly exceeds the sum of their values. But in a search setting, this surplus of inside over outside options will generally be strict. How match surplus is split is therefore critical. Equal division is a neutral benchmark:

$$v(y|x) - v(x) = v(x|y) - v(y) \qquad (12)$$

This is often dubbed the *Nash bargaining solution*; but since no bargaining is modeled, it is more natural to appeal to simplicity. Any other split would surely need motivation, justifying why one class of agents is stronger than another.

Modifying the NTU value expression (3) in light of the TU assumption of a Nash split (12), we find:

$$v(x) = \frac{1}{2}\frac{\rho}{\delta + r}\int_{y \in M(x)} (f(x,y) - v(x) - v(y))\, U'(y)dy \tag{13}$$

This resembles the optimality equation (3) for NTU, but with a fraction 1/2 scaling the surplus, and matches instead discounted by $r + \delta$, given match dissolutions $\delta > 0$. It helps to abbreviate $2\beta \equiv \rho/(r+\rho)$.

Towards a fundamental equation that lies at the heart of the TU search-matching paradigm, let us rewrite (13) without the matching sets as:

$$v(x) = \beta \int \max\langle 0, f(x,y) - v(x) - v(y)\rangle U'(y)dy \tag{14}$$

We can see that the value function resembles a potential (see Hart and Mas-Colell, 1989), and is uniquely defined given the production function $f(x,y)$ and density $U'(y)$. Understanding such potentials is a recurrent and interesting open problem in dynamic economic theory.[13]

The value function solving equation (14) is continuous — and in this respect, the TU model critically differs from the NTU model. As a result, *the match surplus vanishes around the boundary of the matching set.*[14] This intuitively means that marginal changes in the matching set have zero partial effect on match value (3), since those matches yield vanishingly little surplus. The value derivative therefore admits the simple expression:

$$\begin{aligned} v'(x) &= \beta \int (\max\langle 0, f_1(x,y) - v'(x)\rangle)\, U'(y)dy \\ &= \frac{\beta \int_{y \in M(x)} f_1(x,y) U'(y)dy}{1 + \beta \int_{y \in M(x)} U'(y)dy} \end{aligned} \tag{15}$$

Given (15), any value function solving (14) is increasing and differentiable.

Equation (15) resembles the NTU value equation (6), except that the marginal value v' replaces the value v, and the marginal product f_1 substi-

[13] Another potential underlies the search-theoretic trade model of Smith (1995).

[14] By contrast, the own surplus $s(x|y) \equiv f(x,y) - v(x)$ in an NTU match rises as the partner y approaches the top of the matching interval $M(x)$. Just as well, this too follows from the Envelope Theorem logic, the matching set is jointly optimal.

tutes for the production function f. This observation originally inspired the PAM analysis in Shimer and Smith (2000) for the TU model. For it suggests that the own-marginal product f_1 intuitively should be log-supermodular to ensure PAM. Let's keep this thought in mind as we analyze the problem.

In light of our earlier discovery that PAM follows if matching sets are convex and have monotone bounds, our plan of attack is:

A. We show that if all matching sets are convex, then supermodularity implies PAM *provided*[15] the marginal product of 0 vanishes: $f_2(0, y) \equiv 0$.

B. We argue that matching sets are convex when own marginal products are log-supermodular, and an even stronger form of supermodularity obtains.

The Nash split underlies the analysis here, and is the source of much simplicity. But stepping away from this solution can also shed light on discrimination. For an asymmetric surplus split is a compelling way of seeing how discriminatory wages can be economically self-sustaining. It may well be that some observable but payoff-irrelevant trait of individuals (like their color) acts as a flag for a different surplus split. In this case, one could imagine that discrimination persists in equilibrium without any explicit monetary taste for it. This was originally pointed out by Akerlov (1985), and later revisited by Mailath, Samuelson, and Shaked (2000).

5.2 Convex Matching and Supermodularity $\Rightarrow$ PAM

We proceed by contradiction, since that gives us an extra ingredient in the proof to work with. Suppose that matching sets are all convex, production is supermodular, but that PAM fails. There are two ways this can occur.

First, since PAM fails, some matching set $M(x_1)$ has a higher upper bound than another set $M(x_2)$, with $x_2 > x_1$, say $y_1 = \psi(x_1) > \psi(x_2) = y_2$. In this case, the value function would have to fall: $v(x_1) > v(x_2)$. For production is supermodular exactly when match surplus is supermodular, as $s_{12} = f_{12} > 0$. So type x_1 sees his match surplus rise more slowly in his parter's type than does type x_2. Also, type x_1 has a higher upper partner (yielding zero surplus) than type x_2. These two effects together mean that integrating $s_2(x, y)$ down from y_1, the surplus of type x_1 is lower with every partner than the surplus of type x_2. Hence, $v(x_1) > v(x_2)$.

[15] We show why this is needed in Figure 4.

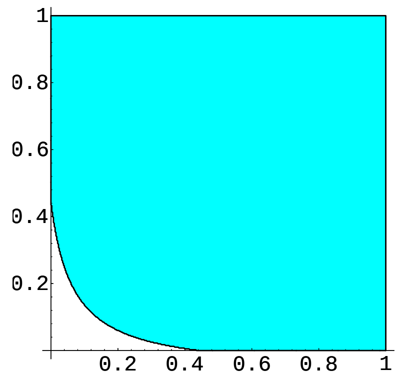

Figure 4: **Complementarity and Nonsorting.** This depicts matching sets for $f(x,y) = x + y + xy$, $\rho = 50r$, $\delta = r/2$, and $L(x) = x$ on $[0,1]$. Although production is supermodular and matching sets convex, NAM and not PAM arises — since $f(0,0) = 0$ and $f(0,y) > 0$ for $y > 0$ force $0 \notin M(0)$.

Second, PAM may fail if the lower bound on the matching set $M(x)$ dips in x. For since the value function is increasing, we have $s_2(0,y) = f_2(0,y) - v'(y) = 0 - v'(y) < 0$. Thus, the highest surplus partner of type 0 is 0. And since match payoffs are nonnegative, his matching set includes 0. But as remarked, convex matching sets have a quasi-convex lower bound function $\theta(x)$. Since $\theta(x)$ is initially weakly increasing, it is always weakly increasing. This completes the proof of PAM.

5.3 Convex Matching Sets

The natural way to establish convex matching sets is to prove that the surplus function is quasi-concave in the partner's type. To this end, toss aside an algebraic complexity, and assume that all types match; this means that changes in the matching set can be ignored.[16] This reduces the implicit equation (15) for the marginal value to $v'(x) = \gamma E_Y f_1(x,Y)$, for a constant $\gamma < 1$ rising in the meeting rate ρ and falling in the interest rate r.[17]

A. Convexity for High Types. The match surplus of type x has slope $s_2(x,y) = f_2(x,y) - v'(y)$ in his partner's type y. Given symmetry $f(x,y) \equiv$

[16] And let's not be troubled by the fact that in this case, matching sets are trivially convex, since this difficulty is just an annoyance in the proof in Shimer and Smith (2000).

[17] Specifically, $\gamma = \rho U(1)/[r + \rho U(1)]$, which clearly lies between 0 and 1.

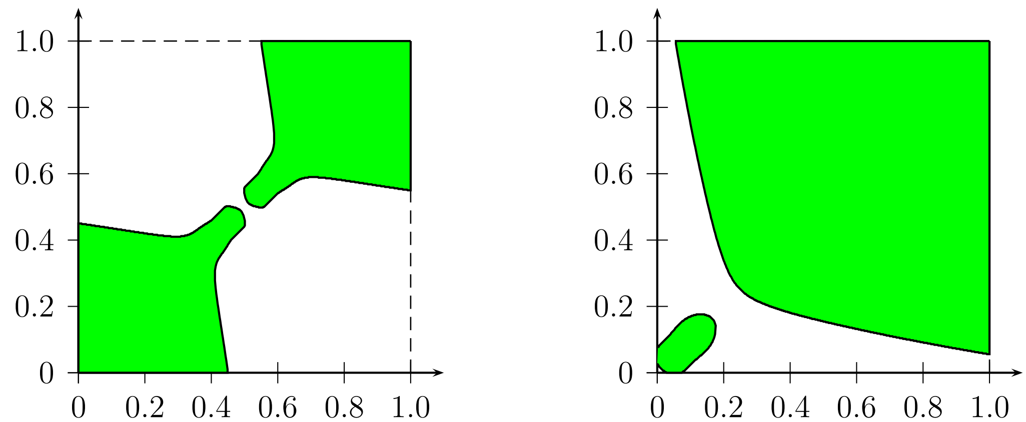

Figure 5: **Non-Convex Matching** The left and right panels respectively depict matching sets for the SPM production functions $f(x,y) = (x+y-1)^2$ and $f(x,y) = (x+y)^2$ with $\delta = r$ and a uniform type distribution. Matching is easier at left than at right, with a meeting rate $\rho = 100r$ versus $\rho = 35r$.

$f(y,x)$, the marginal value is also $v'(y) \equiv \gamma E_Y f_2(X,y)$. Hence:

$$s_2(x,y) = f_2(x,y) - \gamma E_X f_2(X,y) > E_X[f_2(x,y) - f_2(X,y)] \tag{16}$$

since $\gamma < 1$. An easy implication of supermodularity is that $f_2(1,y) - f_2(x',y) > 0$ whenever $x' < 1$, and so $s_2(1,y) > 0$. Given continuity of the marginal surplus, we have $s_2(x,y) > 0$ for high enough types $x < 1$. Namely, *highest types see their match surplus rising in their partner's type y, when production is supermodular.* In this case, the matching set of any high type is a convex upper set in $[0,1]$.

Figure 5 shows how matching sets of lower types may fail to be convex. The right panel considers $f(x,y) = (x+y-1)^2$. Types near $x = 1/2$ might not even match with peer types in one case — for $x = 1/2$ produces nothing when matched with her own type, and obviously receives no transfer. The right panel finds a similar convexity failure with the monotonic function $f(x,y) = (x+y)^2$. In this case, the matching sets of types $x \in [0.06, 0.18]$ include low and high types, but not middle types on the diagonal.

B. Convexity for Low Types. The match surplus of lower types need not always rise in their partner's type: Very high types may require too much compensation relative to the payoff gain they confer. Convexity can only be deduced from a quasiconcave surplus function. Since the surplus function is smooth, an easy sufficient condition for quasiconcavity is that its derivative obey a *single-crossing property*: If $s_2(x, \bar{y}) = 0$ for some $\bar{y}$, then $s_2(x, y) < 0$ for all $y > \bar{y}$. Since $s_2(x, y) = f_2(x, y) - \gamma E_X[f_2(X, y)]$ by (16), this is equivalent to a *level crossing property* for a known function:[18]

$$\frac{E_X[f_2(X, y)]}{f_2(x, y)} \gtrless 1/\gamma \qquad \text{for } y \gtrless \bar{y} \tag{17}$$

The ideal partner y of type x is the one with whom match surplus is maximal, and thus for whom the marginal value equals the own marginal product: $v'(y) = f_2(x, y)$. Our log-supermodularity assumption guarantees a unique ideal partner, since higher partners yield lower match surplus.

We must prove inequality (17) for all types x lower than those included in part A. We can prove it for the lowest type $x = 0$, showing that the left side of (17) rises in y. Emulating the logic used in the NTU model after the critical equation (8), this level crossing property holds when $f_2(x, y)/f_2(0, y)$ increases in y. Since $X > 0$ with probability one, it suffices that the marginal product f_2 be log-supermodular. And by continuity, this holds for types near $x = 0$. So far, we have seen that *if the marginal product of production is log-supermodular, then the least types have convex matching sets.*

Immediately, we see the problem with the sorting failures in Figure 5 — for when $f(x, y) = (x + y)^2$, the ratio $f_2(x, y)/f_2(0, y) = 1 + x/y$ falls in the partner's type y, since $\log f(x, y) = 2 \log(x + y)$ is strictly submodular. Not surprisingly, we found a failure of matching set convexity for type $x = 0$.

C. A Single-Crossing Property for Gambles. To finish the argument, we must take a detour. We formulate and prove a basic building block lemma for the preservation of monotonicity under expectations. Notably, this turns out to be a famous result in the economics of uncertainty.

Lemma 1 (SCP for Gambles) *Define $h(x, y) > 0$ on $[0, 1]^2$, and let the partial derivative $h_1(x, y) > 0$ be log-supermodular. Fix an arbitrary density for x on $[0, 1]$. For any $y \in [0, 1]$, if $\bar{x}$ is the "certainty equivalent" of X solving $h(\bar{x}, y) = E_X[h(X, y)]$, then $h(\bar{x}, z) \geq E_X[h(X, z)]$ for all $z \geq y$. Further, the opposite inequality follows if $h_1(x, y)$ is log-submodular.*

[18] When the level to be crossed through is zero, then this is known as a single-crossing property (SCP), and is commonly exploited in comparative statics analysis in economics. Any SCP is an ordinal property — and holds under weaker conditions. We will eventually need a SCP to finish our proof, but for now, we require this stronger cardinal property.

For a classic application, consider Diamond and Stiglitz's (1974) analysis of increasing risk and risk aversion. Their main result, Theorem 3, proved that if a (thrice smooth) utility function $U(x, y)$ is increasing in wealth x, and if the Arrow-Pratt coefficient of risk aversion $-U_{xx}/U_x = -(\log(U_x))_x$ increases in a parameter y, then the certainty equivalent $\bar{x}$ of any given wealth gamble X rises in the parameter y.

To see how this follows from Lemma 1, observe that the coefficient of risk aversion can be written as $-(\log(U_x))_x$. So it weakly increases in y iff $-(\log(U_x))_{xy} \geq 0$. This says that the marginal utility of income U_x is log-submodular in (x, y). In words, decrements of the marginal utility of income grow proportionately larger at greater levels of risk aversion. This is the intuitive reason why the certainty equivalent of any gamble rises in y, since the utility gains from more favorable gamble outcomes are worth less.

Because of its importance, and the instructive way that its proof turns on monotone methods under uncertainty, we now prove Lemma 1. Define

$$H(z|y) \equiv E_X\left(\phi(X|z, y)[h(X, y) - h(\bar{x}, y)]\right)$$

where

$$\phi(x|z, y) \equiv \frac{h(x, z) - h(\bar{x}, z)}{h(x, y) - h(\bar{x}, y)} = \frac{\int_{\bar{x}}^{x} h_1(t, z)dt}{\int_{\bar{x}}^{x} h_1(t, y)dt}$$

Now, $H(y|y) = h(\bar{x}, y) - E_X[h(X, y)] = 0$ since $\phi(x|y, y) \equiv 1$. It then suffices to show that $H(z|y) = E_X[h(X, z)] - h(\bar{x}, z) \leq 0$ for all $z \geq y$. To do this, we return to the ratio inequality in (9). Intuitively, the ratio in (9) was increasing in t when $[a(t)/b(t)]' > 0$ and decreasing when $[a(t)/b(t)]' < 0$ (indeed, we can just differentiate). Since $z \geq y$, this generalized inequality yields $\phi'(x|y, z) \geq 0$ when h_1 is log-supermodular and $\phi'(x|y, z) \leq 0$ when h_1 is log-submodular. As the covariance of two weakly increasing functions $x \mapsto \phi(x|y, z)$ and $x \mapsto h(x, z) - h(\bar{x}, z)$ cannot be negative, this yields:

$$H(z|y) \leq E_X[\phi(X|y, z)]E_X[h(X, y) - h(\bar{x}, y)] = 0$$

when h_1 is log-supermodular, using the premise of the Lemma.

A different proof is possible that appeals to the fact that the single-crossing property of a function is preserved under any expectation with a log-supermodular kernel (like ϕ).[19]

D. Convexity for All Types. Our applications of monotone methods have fallen short of deducing a convex matching set for all types. Is it true that every type is either high enough for supermodular production to yield a rising surplus function, or low enough for log-supermodular marginal products to suffice? We now finish the proof of convexity, empowered by the single-crossing property for gambles.

Let's recall the earlier sufficient condition for a quasi-concave function. We need to show that for all types $x \in [0, 1]$:

$$f_2(x, y) - v'(y) = 0 \Rightarrow f_2(x, z) - v'(z) \leq 0 \qquad \text{whenever } z > y \qquad (18)$$

We reinterpret our earlier analysis as checking assertions (a) and (b) below.

(a) The premise of (18) fails for high types $x < 1$, since surplus is rising.

(b) The implication (18) holds for low types $x > 0$.

(c) Every type $x \in (0, 1)$ is "low" or "high".

Now, since $f_{12} > 0$, there is a unique cut-off $\bar{x}$ so that $f_2(\bar{x}, y) = E_X f_2(X, y)$. Define the partial derivative function $h \equiv f_2$. Then $h_1 = f_{12} > 0$ by strict supermodularity. We now introduce one final assumption on production — that the cross partial f_{12} is log-supermodular. In this case, Lemma 1 asserts $f_2(\bar{x}, z) \geq E_X f_2(X, z)$ for all $z \geq y$. So if $x = \bar{x}$, then:

$$\frac{f_2(x, z)}{f_2(x, y)} \leq \frac{v'(z)}{v'(y)} = \frac{E_X f_2(X, z)}{E_X f_2(X, y)} \qquad \text{whenever} \quad z \geq y \qquad (19)$$

We now show that the threshold $\bar{x}$ is the critical type separating the high types in case (a) for whom the surplus function rises in the partner's type, and the low types in case (b) with single-peaked preferences. For if $x \geq \bar{x}$, then by (16), the surplus $s(x, y)$ has strictly positive slope in y, and thus (18) is a valid syllogism. On the other hand, the left side of (19) rises in x since the marginal product f_2 is log-supermodular. Thus, the inequality (19) holds if $x < \bar{x}$. This proves that all types have convex

[19] This is an example of the variation diminishing property in Karlin (1968). Athey (2002) explores this result and a class of related comparative statics tools.

matching sets. Barring our simplifying assumption that all types match, this proves the following assortative matching characterization:

Theorem 2 (PAM with TU) *If x and y jointly produce symmetric output $f(x,y) > 0$, then the equilibrium TU matching obeys strict PAM if f is supermodular, f_1 and f_{12} are log-supermodular, and $f_2(0,y) \equiv 0$ for all y.*

An interesting open problem is to prove that log-supermodularity of f_{12} is needed, or to dispense with this proof ingredient altogether. It clearly plays a subtle role in the proof — essentially acting as a SCP that itself sends us to one of two proof cases. In principle, if it fails, we should find for some search frictions, a hole in the middle of the search set. Shimer and Smith (2000) were not able to produce such an example.

An interesting benchmark to consider is the the simplest model with two types of agents, low and high, and thus match payoffs $c > b > a$, as described below:

	L	H
H	b	c
L	a	b

The frictionless Walrasian world explored by Becker (1973) yields PAM whenever the supermodular inequality holds: $a + c > 2b$. Burdett and Coles (1999) found that in the TU search model, PAM occurs when this holds, and as long as a match between two low types produces at least 2/3 as much as a high-low match: $a > 2b/3$. Deducing this offers some nontrivial practice with the search analysis.

This restriction on the two type setting can only be understood in light of the boundary condition in Theorem 2. For the log-supermodularity of the marginal products only binds on a model with three or more types — for which there is a 2×2 array of marginal products (output differences). Writing the log-supermodularity condition in its product form, it requires that this matrix have a non-negative determinant. By the same token, the log-supermodular cross partial derivative is only restrictive in a model with four or more types.

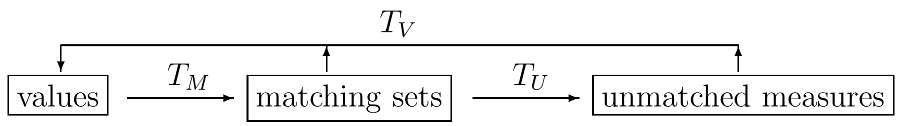

Figure 6: **Big Picture of Search Existence Proof Logic.** Search equilibrium requires that the three maps T_M (optimality), T_U (implied unmatched density), and T_V (value accounting) be both well-defined and continuous.

6 Existence of Search Equilibrium

A Walrasian equilibrium is a *pair* of *prices* and *allocations*, such that the allocations are optimal given prices, and prices clear the market allocations. In the frictional world, an equilibrium is conceptually and technically a more complicated object. A *search equilibrium* is a *triple* (v, M, u) — namely, the value function v, the matching set function M, and the new equilibrium object is the unmatched measure u (simply, here a density). So the value function acts like a wage function (the "price"), since it represents the outside options, while the matching set $M(\cdot)$ is the allocation. The unmatched density u captures the friction.

Existence no longer follows so straightforwardly. For we must keep track not only of the mass of realized trades, but also of the mass of unmatched traders. We have so far suppressed the unmatched density $u(x)$ from all integrals, since its exact form did not affect any proofs. For this existence proof discussion, we now introduce it. The unmatched density $u(x) \leq \ell(x)$ obeys the implicit equation:

$$\delta[\ell(x) - u(x)] = \rho u(x) \int_{M(x)} u(y)dy \tag{20}$$

So motivated, search equilibrium turns on three maps T_M, T_V, T_U. First, given any value function v, write the optimal matching set as $M = T_M(v)$, as defined by (11) in the TU setting, for example. Next, the matching set M yields a density of unmatched agents $u = T_U(M)$, implied by (20). Finally, the triple of unmatched density u, matching set function M, and value function v yields a new value function $v = T_V(v, M, u)$. Each map must be well-defined and continuous in some fashion for us to apply a suitable fixed point theorem, and secure a fixed point of the composite map $v \mapsto T_V(v, T_M(v), T_U(T_M(v)))$ — as schematically depicted in Figure 6.

Showing that T_M is continuous — namely, nearby value functions lead to nearby matching sets is not too hard in the TU model — since the matching set is an upper contour set of the surplus function, via (11). It is somewhat trickier in the NTU case, since the value functions can be discontinuous.[20] Equally well, the map T_V yielding continuation value functions is intuitively continuous, since it is defined by an integral. That $u = T_U(M)$ exists and is continuous is absent from the Walrasian analysis, and is the hardest step. Shimer and Smith (2000) thus call it the "fundamental matching lemma".[21]

For this reason, Bloch and Ryder (2000) made a clever and obviously false assumption that matched pairs are replaced instantaneously by unmatched "clones". Many papers have since made this assumption, and it is obviously an innocuous sin in any paper whose sole purpose is to understand results that hold for all unmatched densities.[22]

Since it is novel to the frictional matching world, we now explore why T_U is continuous in our search setting. Consider the finite n-type case, with a symmetric matrix of matching chances $M = [m_{ij}]$. With $n = 2$ types, we can see that the steady-state equation (20) reduces to the vector equation $\delta\ell = \delta u + \rho A(m, u)$, where

$$A(m, u) = \begin{bmatrix} m_{11}u_1^2 + m_{12}u_1u_2 \\ m_{21}u_1u_2 + m_{22}u_2^2 \end{bmatrix}$$

We want to invert this, and express u as a function of M. Differentiating, we discover the directional derivative:

$$D_uA(m, u) = \begin{bmatrix} 2m_{11}u_1 + m_{12} & m_{12}u_1 \\ m_{21}u_2 & m_{21}u_1 + 2m_{22}u_2 \end{bmatrix}$$

Towards inverting this, observe that $D_uA(m, u)$ is a positive definite matrix.

[20]Smith (2006) uses a bounded variation norm on this richer value function space.

[21]In an amazing piece of sleuthing (personal communication), Georg Noldeke has since found that this math also arises in chemical reaction networks. Noldeke claims that chemist Martin Feinberg has lecture notes from 1979 which cover what economists call the quadratic search technology.

[22]Collins and McNamara (1990) assume a stationary distributions from which partners are drawn but they do not explain carefully how this is sustained. For this reason, their paper is not properly an equilibrium model. But their analysis can be justified with such a cloning assumption.

For if we consider any $x' = (x_1, x_2) \in \mathbb{R}^2$, we have:

$$x' D_u A(m, u) x = (2u_1 x_1^2) m_{11} + (u_2 x_1^2 + (u_1 + u_2) x_1 x_2 + u_1 x_2^2) m_{12} + (2u_2 x_2^2) m_{22}$$

since $m_{21} = m_{12}$. This is non-negative, because $m, u \geq 0$, as well as:

$$u_2 x_1^2 + (u_1 + u_2) x_1 x_2 + u_1 x_2^2 = (\sqrt{u_2} x_1 + \sqrt{u_1} x_2)^2 + 4 x_1 x_2 (\sqrt{u_2} - \sqrt{u_1})^2 \geq 0$$

So the derivative $D_u A(m, u)$ is positive definite, and thus so too is the sum $\delta I + \rho D_u A(m, u)$. Finally, any positive definite matrix is invertible. Then by the Implicit Function Theorem, the steady-state equation analogous to (20), written $B(m, u) = 0$, implicitly well-defines a smooth function $u = T_U(M)$ since the derivative $B_u(m, u) \equiv \delta I + \rho D_u A(m, u)$ is smooth and invertible. This completes a two-type distillation of the much more involved proof in Shimer and Smith (2000) that works for the continuum type world.

The above existence proof logic suggests the algorithm for computing the equilibrium — use tatonnement via value adjustment, analogous to Walrasian wage adjustment. This method is used to produce the simulations in Shimer and Smith (2000) as well as Smith (2006).

7 Explicitly Costly Search

There are two traditional search friction stories that one might entertain. On the one hand, one might imagine that search takes a while, and thus the cost of search is foregone match payoffs. This is what we have explored — the *time cost of search* story. But the original price search model imagined that the act of search was critical cost, as happens when searching for a low gas price. This is *explicitly costly search.* It is best modeled in discrete time, and arguably not the most plausible description of search for match partners. This is fundamentally an easier model to analyze, since the cost of search is the same for all partners.

Still, it is natural to inquire about the explicitly costly search. In this world, Morgan (1995) finds in the NTU world that if higher partners are better, and types are productively complementary, then higher individuals have a greater benefit from holding out for better partners. Faced with the same search cost, they hold out for better partners, and PAM arises.

The analysis is quite straightforward. Atakan (2006) likewise shows that productively complementary types alone guarantees PAM in the TU setting. Table 1 summarizes the larger picture of sorting in frictionless and frictional matching markets.

Table 1: **Summary of the Assortative Matching Literature.** The left columns owe to Becker (1973), where TU means that wages are competitively set. The top middle entry is found in Morgan (1995), while Alp (2006) is the middle bottom entry. The right bottom entry is Shimer and Smith (2000). Smith (2006) derives the top right entry.

	No Search	Fixed Cost Search	Opportunity Time Cost Search
NTU	$f_2 > 0$	$f_2 > 0, f_{12} > 0$	$f_2 > 0, (\log f)_{12} > 0$
TU	$f_{12} > 0$	$f_{12} > 0$	$f_{12} > 0, (\log f_1)_{12} > 0, (\log f_{12})_{12} > 0$

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