Econometrica, Vol. 62, No. 4 (July, 1994), 939–948

THE FOLK THEOREM FOR REPEATED GAMES: A NEU CONDITION[1]

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1. INTRODUCTION

WE ARE CONCERNED here with perfect "folk theorems" for infinitely repeated games with complete information. Folk theorems assert that *any* feasible and individually rational payoff vector of the stage game is a (subgame perfect) equilibrium payoff in the associated infinitely repeated game with little or no discounting (where payoff streams are evaluated as average discounted or average values respectively). It is obvious that feasibility and individual rationality are *necessary* conditions for a payoff vector to be an equilibrium payoff. The surprising content of the folk theorems is that these conditions are also (almost) sufficient.

Perhaps the first folk theorem type result is due to Friedman (1971) who showed that any feasible payoff which Pareto dominates a Nash equilibrium payoff of the stage game will be an equilibrium payoff in the associated repeated game with sufficiently patient players. This kind of result is sometimes termed a "Nash threats" folk theorem, a reference to its method of proof. For the more permissive kinds of folk theorems considered here, the seminal results are those of Aumann and Shapley (1976) and Rubinstein (1977, 1979). These authors assume that payoff streams are undiscounted.[2]

Fudenberg and Maskin (1986) establish an analogous result for discounted repeated games as the discount factor goes to 1. Their result uses techniques of proof rather different from those used by Aumann-Shapley and Rubinstein, respectively. See their paper for an insightful discussion of this point, and quite generally for more by way of background. It is a key reference for subsequent work in this area, including our own.

For the two-player case, the result of Fudenberg and Maskin (1986) is a complete if and only if characterization (modulo the requirement of strict rather than weak individual rationality, *which we retain in this note*) and does not employ additional conditions. For three or more players Fudenberg and Maskin introduced a *full dimensionality* condition: The convex hull F, of the set of feasible payoff vectors of the stage game must have dimension n (where n is the number of players), or equivalently a nonempty interior. This condition has been widely adopted in proving folk theorems for related environments such as finitely repeated games (Benoit and Krishna (1985)), and overlapping generations games (Kandori (1992), Smith (1992)).

Full dimensionality is a *sufficient* condition. Fudenberg and Maskin present an example of a three-player stage game in which the conclusion of the folk theorem is false. In this example *all* players receive the same payoffs in all contingencies; the (convex hull of the) set of feasible payoffs is *one*-dimensional. This example violates full dimensionality in a rather extreme way. Less extreme violations may also lead to

[1] This paper combines "The Folk Theorem for Discounted Repeated Games: A New Condition" by Abreu and Dutta and "Folk Theorems: Two-Dimensionality is (Almost) Enough" by Smith, which was the third chapter of his 1991 Ph.D. dissertation at the University of Chicago. The pair of papers independently introduced two equivalent conditions, here replaced by a third equivalent condition which is perhaps the most transparent. Abreu and Dutta covered mixed strategies and established the necessity of their condition. Smith confined attention to pure strategies and extended his analysis to finitely repeated games and overlapping generation games; Smith (1993a) and (1993b) pursue the latter extensions. The present paper follows Abreu and Dutta (1991) closely. We would like to thank David Pearce, Ennio Stacchetti, a co-editor, and two anonymous referees for their comments. Smith is grateful for financial assistance from the Social Sciences and Humanities Research Council of Canada.

[2] Aumann and Shapley (1976) employ the *limit of means* criterion and Rubinstein (1977) considers both the limit of means and the *overtaking criterion*.

difficulties as an example by Benoit and Krishna (1985) shows.[3] In their three-player example *two* of the players receive identical payoffs.

The reason why these examples work is that the folk theorem argument entails nondeviating players punishing a deviant player, and furthermore, since we require perfection, also entails threatening a player involved in punishing a deviant with a *lower* payoff stream for deviating from his/her role in the initial deviant's punishment. This logic obviously breaks down when a pair of players have identical payoffs.[4] The most optimistic conjecture then is that, except in the special case in which a pair of players have identical payoffs, the conclusion of the folk theorem is true. But this guess cannot possibly be exactly right since positive affine transformations of a player's payoffs do not alter the strategic structure of the game. We must at the very least exclude *equivalent* von Neumann-Morgenstern utility functions over outcomes, that is, those which yield identical orderings of lotteries over outcomes. Players i and j have equivalent payoffs if by changing the origin and scale of player i's utility function it may be made identical to player j's; viewed geometrically, equivalent payoffs lie on a straight line with positive slope.

In this form the conjecture is in fact correct; the simple condition that no pair of players have equivalent utility functions is sufficient. We term this requirement *non-equivalent utilities* (NEU). This condition is easy to understand, and, of course, weaker than full dimensionality. Furthermore, it is a "tight" condition in the sense that it is also often *necessary*. While one can imagine contexts in which full dimensionality is violated but NEU not, the primary advantage of NEU is conceptual: the condition is simple, minimal, and clarifies the essential elements of the folk theorem proof.

This paper is organized as follows. Section 2 presents some notation and preliminary results about normal form (stage) games. Section 3 contains the main theorems for infinitely repeated games, and Section 4 concludes.

2. PRELIMINARIES

2.1. *The Stage Game*

We consider a finite n-player game in normal form $G = \langle A_i, \pi_i;\ i = 1, \dots, n\rangle$ where A_i is player i's finite set of actions, and $A = \times_{i=1}^{n} A_i$. Player i's payoff function is π_i: $A \to \mathbb{R}$. The game G satisfies NEU if for i and j, there do not exist scalars c, d where $d > 0$ such that $\pi_i(a) = c + d\pi_j(a)$ for all $a \in A$.

Let M_i be the set of player i's mixed strategies, and let $M = \times_{i=1}^{n} M_i$. Abusing notation, we write $\pi_i(\mu)$ for i's expected payoff under the mixed strategy $\mu = (\mu_1, \dots, \mu_n) \in M$. For any n-element vector $v = (v_1, \dots, v_n)$, the corresponding $(n-1)$-element vector with element v_i missing is denoted v_{-i}. Let $\pi_i^*(\mu_{-i}) = \max_{a_i} \pi_i(a_i, \mu_{-i})$ be player i's best response payoff against the mixed profile μ_{-i}. Denote by $m^i = (m_1^i, \dots, m_n^i) \in M$ a mixed strategy profile which satisfies $m_{-i}^i \in \operatorname{argmin}_{\mu_{-i}} \pi_i^*(\mu_{-i})$ and $m_i^i \in \operatorname{argmax}_{\mu_i} \pi_i(\mu_i, m_{-i}^i)$. In words, m_{-i}^i is an $(n-1)$-profile of mixed strategies which minimax player i, and m_i^i is a best response for i when being minimaxed. We have adopted the normalization $\pi_i(m^i) = 0$ for all i. Let $F = \operatorname{co}\{\pi(\mu) \colon \mu \in M\}$, so that the set of feasible and (strictly) individually rational payoffs is $F^* = \{w \in F \colon w_i > 0 \text{ for all } i\}$.

[3] Their analysis was for the finitely repeated case, but the example works equally well in the infinitely repeated setting.

[4] In the context of *Nash* equilibrium and its refinements, only *single* person deviations need be deterred. Hence, the focus on *pairs* (an original deviant and a single subsequent deviant) of players as opposed to *coalitions* of players.

2.2. *An Equivalence Result*

NEU has two quite powerful and equivalent representations that are developed in the lemmas below. For $j \neq i$, let F_{ij}, denote the projection of F on the i-j coordinate plane and dim F_{ij} the dimension of F_{ij}.

DEFINITION: The set F satisfies the *projection condition* if for all players i, either (a) $\dim F_{ik} = 2$ for all $k \neq i$, or (b) $\dim F_{ij} = 1$ for some $j \neq i$ and $\dim F_{ik} = 2$ for all $k \neq i, j$. Furthermore, in the latter case, F_{ij} is a line with negative slope.

LEMMA 1: *Suppose that NEU holds and that no player is indifferent over all action profiles. Then F satisfies the projection condition.*

PROOF: Since no player is indifferent over all possible action profiles, it follows that $\dim F_{ij} \geqslant 1$ for all $i \neq j$. Now suppose the $\dim F_{ij} = \dim F_{ik} = 1$ for some $j \neq k$. NEU applied to the payoffs of players i and j (and similarly players i and k) implies that the payoffs are perfectly negatively correlated. This in turn implies that the payoffs of players j and k are perfectly *positively* correlated. That is, players j and k have *equivalent* payoffs, in violation of NEU. *Q.E.D.*

DEFINITION: The vectors $\{v^1, \ldots, v^n\}$ satisfy *payoff asymmetry* if $v_i^i < v_i^j \; \forall i, j, i \neq j$.

LEMMA 2: *Suppose that F satisfies the projection condition. Then there exist feasible payoff vectors which satisfy payoff asymmetry.*

PROOF: By Lemma 1, for each pair of players j and k, there exist some feasible payoff vectors v^{jk} and v^{kj} such that $v_j^{jk} > v_j^{kj}$ and $v_k^{kj} > v_k^{jk}$. For each player i, order the $n(n-1)$ payoff vectors v^{jk} $(\forall j \neq k)$ in *increasing* size (break ties arbitrarily) from the point of view of player i, and assign these ordered vectors strictly *decreasing* weights θ_h, $h = 1, 2, \ldots, n(n-1)$, summing to one. Let v^i be the resulting convex combination of the payoff vectors v^{jk}. Note that in defining the v^i's we use the same weights θ_h for all i. Then by construction, $v_i^i < v_i^j$ for all $i \neq j$, establishing payoff asymmetry. *Q.E.D.*

Finally, it is straightforward to see that the existence of asymmetric payoffs implies NEU (and rules out universal indifference for any player). In other words, NEU, the projection condition, and the existence of asymmetric payoffs are equivalent assumptions (absent universal indifference).

3. THE MAIN THEOREMS

We will analyze the infinitely repeated game with *discounting* that is associated with the stage game G. We assume perfect monitoring; that is each player can condition his action in period t on the past actions of all players. In addition, we permit *public randomization*. That is, in every period players publicly observe the realization of an exogenous continuous random variable and can condition on its outcome. This assumption can be made without loss of generality in the infinitely repeated game; a result due to Fudenberg and Maskin (1991) shows explicitly how public randomization can be replaced by "time-averaging" (see also, Sorin (1986)). Denote by $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{it}, \ldots)$ a (behavior) strategy for player i and by $\pi_{it}(\alpha)$ his expected payoff in period t given the strategy profile α. Each player i's objective function is his infinite-horizon *average* expected discounted payoff $(1-\delta)\Sigma_0^\infty \delta^t \pi_{it}(\alpha)$ under the (common) discount factor δ. Let $V(\delta)$ denote the set of subgame perfect equilibrium payoffs.

3.1. *Sufficient Conditions for a Folk Theorem*

We will establish here that NEU is *sufficient* for the folk theorem, and later on that for a wide class of games, it is also a *necessary* condition for the folk theorem to hold.

THEOREM 1: *Under NEU any (strictly) individually rational payoff in the stage game is a subgame perfect equilibrium payoff of the infinitely repeated game when players are sufficiently patient. That is* $\forall u \in F^*$,

$$\exists \delta_0 < 1 \quad \text{so that } \delta \in [\delta_0, 1) \Rightarrow u \in V(\delta).$$

PROOF: For expositional simplicity, we will first prove the theorem under the assumption that mixed strategies are observable; the argument is then extended to the unobservable mixed strategy case.

Step 1—Observable Mixed Strategies: If $F^* = \varnothing$, then the theorem is trivially true. Now suppose $F^* \neq \varnothing$. This implies that each player has distinct payoffs. Hence, by NEU and Lemma 2, there exist payoff vectors $v^1, \ldots, v^n$ which satisfy payoff asymmetry. Fix $u \in F^*$. We will show that in fact there exist vectors $x^1, \ldots, x^n$ such that $\forall i$,

(1) $\quad x^i \gg 0 \quad$ strict individual rationality,

(2) $\quad x_i^i < u_i \quad$ target payoff domination,

and $\forall i, j,\ i \neq j$,

(3) $\quad x_i^i < x_i^j \quad$ payoff asymmetry.

To see this[5], let w^i denote a feasible payoff vector which yields player i his lowest payoff in the game: $w_i^i = \min\{v_i: (v_{-i}, v_i) \in F\}$. Now define

$$x^i = \beta_1 w^i + \beta_2 v^i + \beta_3 u$$

where $\beta_1 = \varepsilon(1-\eta)$, $\beta_2 = \eta\varepsilon$, and $\beta_3 = (1-\varepsilon)$ are convexifying weights *which are independent of* i. By the definition of w^i and v^i, it follows that if β_2 is strictly positive (i.e., $\varepsilon, \eta > 0$), then $x_i^i < x_i^j$ for all $i \neq j$ (payoff symmetry). For small enough $\varepsilon > 0$, we must have $x_i^i > 0$ since $u_i > 0$ (strict individual rationality). Finally, for small enough $\eta > 0$, we must have $x_i^i < u_i$, even if $v_i^i > u_i$.[6,7]

Strategies: Let a (respectively, a^i) denote the publicly randomized action vector whose stage game payoff is u (respectively, x^i). Further, let $\underline{v}^i$ be the payoff vector associated with m^i, i.e. minimizing player i, and recall from Section 2.1 that $\underline{v}_i^i \equiv 0$.

The strategy vector that will generate the target payoff u as an equilibrium payoff (for appropriate choice of parameters) can be defined in Markov strategy terminology as

[5] This construction together with Lemma 2 has a nice geometric intuition. Consider u as a point in $F^* \subset \mathbb{R}^n$. Then a simple choice of x^i is the point in F^* with the smallest i coordinate on the ε-sphere $B_\varepsilon(u)$ about u, for small enough $\varepsilon > 0$. Indeed, Lemma 1 implies that the projection of $B_\varepsilon(u) \cap F^*$ onto any two players' coordinate plane is either an ellipse or a line with negative slope. In the first case, x^i and x^k lie at different locations on the ellipse, while in the second they reside at opposite ends of a line segment.

[6] A referee's comments helped clarify the specification of the β_i's above.

[7] We remark in passing that there is a minor error in the construction of the vectors $x^1, \ldots, x^n$ in Fudenberg and Maskin (1986): They implicitly assume (as pointed out to us by Peter Sorensen of MIT) that because $u \in F^*$, it does not lie on the lower boundary of F. This is false. One way to patch up their construction is suggested by a closer reading of footnote 5.

follows:

1. When in "state" u, play a. If the observed (mixed) action vector a' satisfies $a'_i \neq a_i$ and $a'_{-i} = a_{-i}$, go to "state" $\underline{v}^i$. Else, stay in "state" u.
2. When in "state" $\underline{v}^i$, play m^i. If the observed action vector a' satisfies $a'_j \neq m^i_j$ and $a'_{-j} = m^i_{-J}$, go to "state" $\underline{v}^j$. Else, with probability q stay in "state" $\underline{v}^i$, while with probability $(1-q)$ proceed to "state" x^i.
3. When in state x^i, play a^i. If the observed action vector a' satisfies $a'_j \neq a^i_j$ but $a'_{-j} = a^i_{-j}$, go to "state" $\underline{v}^j$. Else, stay in "state" x^i.

In words, the strategy says: Start with a and continue to play this action till the first single-player deviation (say by player i). Then, minimax player i for one period (with probability one) and (in the event of no observed deviation) continue the minimaxing with probability q. With the remaining probability, terminate the minimaxing and play a^i until further deviations. Treat players symmetrically and subject every single player deviation to this (stochastic) punishment schedule.

Choice of Parameter: The only parameter in the above strategy is the probability q. Let b_i be the best feasible payoff for player i. Choose q to satisfy:

$$b_i < \frac{2-q}{1-q} x^i_i. \tag{4}$$

Since $x^i_i > 0$ by equation (1), we can find such $q \in (0,1)$.

Verification of Equilibrium: We show that no *one-shot deviation* by any player from any state is profitable. Hence the strategy proposed is *unimprovable* and consequently a subgame perfect equilibrium.

State $\underline{v}^i$: Player i's "lifetime" (discounted average) payoff in state $\underline{v}^i$, denoted $L_i(\underline{v}^i)$, satisfies

$$L_i(\underline{v}^i) = \delta\left(qL_i(\underline{v}^i) + (1-q)x^i_i\right)$$

so that

$$L_i(\underline{v}^i) = \frac{\delta(1-q)}{1-\delta q} x^i_i > 0. \tag{5}$$

Note that $L^i_i(\underline{v}^i) \to x^i_i$, as $\delta \uparrow 1$. Player i will not deviate in state $\underline{v}^i$ since the maximal payoff to one-shot deviation is $\delta L_i(\underline{v}^i)$. Player $j \neq i$ will not deviate for high δ, since his maximal payoffs are bounded by $(1-\delta)b_j + \delta L_j(\underline{v}^j)$, which is less than x^i_j, by equation (3).

State x^i: From the definitions it is clear that the difference in the lifetime payoffs to one-shot deviation and conformity is bounded above by

$$\left[(1-\delta)b_i + \delta L_i(\underline{v}^i)\right] - x^i_i = (1-\delta)\left[b_i - \left(\frac{1+\delta-\delta q}{1-\delta q}\right)x^i_i\right] \tag{6}$$

where we have substituted from (5). An immediate implication of inequality (4) defining q is that (6) is strictly negative for all δ close to 1. Since $x^i_j > x^j_j$, $j \neq i$, it is immediate that players $j \neq i$ do not have a profitable one-shot deviation either.

State u: Since, by target payoff domination (2), $u_i > x^i_i$, the arguments above also imply that $(1-\delta)b_i + \delta L_i(\underline{v}^i) < u_i$, for high δ, and hence the action at state u is unimprovable as well.

In sum, for high δ, the posited strategy is unimprovable after all histories, and hence is a subgame perfect equilibrium.

Step 2—Unobservable Mixed Strategies: If the minimax strategy m^i requires that some punishers play nontrivial mixed strategies, then it is necessary to induce minimaxing players $j \neq i$ to play pure strategies in the support of their mixed strategies m^i_j with the appropriate probabilities. (Deviations outside the support of m^i_j are easily deterred by directly punishing player j.) As noted by Fudenberg and Maskin (1986), the only way to do so is to make them indifferent over the pure strategies in the support.

If it is the case that $\dim(F_{ik}) = 1$ for some $k \neq i$, then by Lemma 1, $\dim(F_{ij}) = 2$, for $j \neq i, k$. Furthermore, F_{ik} is a straight line with negative slope. So m^i induces a constant-sum game between i and k, and player k's mixed strategy m^i_k is best response to m^i_{-k}. We thus need only worry about deviations by players $j \neq i, k$.

Such players will be made indifferent across the pure strategies that constitute m^i_j by modifying the strategy when it "escapes" the minimax state $\underline{v}^i$; the modification is to make subsequent play appropriately sensitive to player j's *observed* action choice in state $\underline{v}^i$.

Let c^{ij} be a stage game payoff (with associated action vector a^{ij}) such that $c^{ij}_j \neq x^i_j$ but $c^{ij}_i = x^i_i$. Further, $c^{ij}_j > x^j_j$. (Note that one such vector is specified for every player $j \neq i$ for whom $\dim F_{ij} = 2$; indeed for that reason such a vector exists.) For simplicity, from now on "$j \neq i$" will refer to all players $j \neq i$ such that $\dim F_{ij} = 2$.

Modified Strategies: The play in states u and x^i remain unchanged. Part 2 of the old definition is replaced by 2′ and we have a new part 4:

2′. When in "state" $\underline{v}^i$, play m^i. If the observed action vector is a', then with probability $p^{ij}(a'_j)$, play goes to state c^{ij}, $j \neq i$; with probability q it stays in state $\underline{v}^i$, and with remaining probability, $1 - q - \sum_{j \neq i} p^{ij}(a'_j)$, play proceeds to state x^i. (Notice that the probability $p^{ij}(a'_j)$ *only* depends on player j's action. Furthermore, if there is a player k such that $\dim F_{ik} = 1$, he plays m^i_k.)

4. When in "state" c^{ij}, play a^{ij}. If the observed action vector a' satisfies $a'_l \neq a_l$ but $a'_{-l} = a_{-l}$, then go to state $\underline{v}^l$. Else, go back to state c^{ij}.

Choice of Parameters: The probabilities $p^{ij}(a_j)$ satisfy, for all a_j, a'_j in the support of m^i_j,

$$(1-\delta)\left[\pi_j\left(a_j, m^i_{-j}\right) - \pi_j\left(a'_j, m^i_{-j}\right)\right] = \delta\left[p^{ij}(a'_j) - p^{ij}(a_j)\right]\left[c^{ij}_j - x^i_j\right] \tag{7}$$

where $\pi_j(a_j, m^i_{-j})$ is the expected payoff of player j when he plays a_j and the other players' mixed action choice is m^i_{-j}. Since $c^{ij}_j \neq x^i_j$, (7) evidently has a solution whenever δ is high. The solution is not unique; if, say, $c^{ij}_j > x^i_j$, one solution is to set $p^{ij}(a^*_j) = 0$ for $a^*_j \in \operatorname{argmax}_{a_j} \pi_j(a_j, m^i_{-j})$ and then define

$$p^{ij}(a_j) = \left(\frac{1-\delta}{\delta}\right)\left(\frac{\pi_j\left(a^*_j, m^i_{-j}\right) - \pi_j\left(a_j, m^i_{-j}\right)}{c^{ij}_j - x^i_j}\right).$$

Verification of Equilibrium—State $\underline{v}^i$: Since $c^{ij}_i = x^i_i$, for $j \neq i$, it is easy to see from (5) that player i's lifetime payoffs are completely unchanged. Likewise, he has no profitable deviation. The same is true for any player $k \neq i$ for whom $\dim F_{ik} = 1$ and who consequently plays a best response in playing m^i_k.

For players $j \neq i$, the lifetime payoff to any action a_j is

$$\begin{aligned}(1-\delta)\pi_j\left(a_j, m^i_{-j}\right) + \delta\Bigg\{ & qL_j(\underline{v}^i) + \sum_{k \neq i,j}\sum_{a_k} p^{ik}(a_k) m^i_k(a_k) c^{ik}_j + p^{ij}(a_j) c^{ij}_j \\ & + \left[1 - q - \sum_{k \neq i,j}\sum_{a_k} p^{ik}(a_k) m^i_k(a_k) - p^{ij}(a_j)\right] x^i_j\Bigg\}.\end{aligned} \tag{8}$$

Everything in (8), except the first term and the two others involving $p^{ij}(a_j)$, is independent of the choice a_j. Hence, the difference in lifetime payoffs, from the choices a_j and a'_j, is zero iff equation (7) is satisfied. In that event, player j is indifferent between his action choices. In particular, a best response is to play the minimaxing strategy m^i_j.

Verification of Equilibrium—State c^{ij}: The argument that no one-shot deviation is profitable in state c^{ij} is identical to the arguments that have established that no one-shot deviation is profitable from state x^i. (Note that these last arguments do not change at all since no mixed strategies are played in state x^i.) *Q.E.D.*

3.2. *Necessary Conditions for the Folk Theorem*

We turn now to the *necessity* of NEU. Let $f_i = \min\{v_i | v \in F \text{ and } v_j \geqslant 0 \text{ for all } j\}$ be player i's worst payoff in the set of weakly individually rational payoff *vectors*. We will refer to f_i as player i's *minimal attainable payoff*.[8] The *necessity* of payoff asymmetry is shown for games in which no two (maximizing) players can be simultaneously held at or below their minimal attainable payoff. In stage games where every player's minimal attainable payoff is indeed his minimax payoff ($f_i = 0$), the condition stated below (which uses the term mini*miz*ing) is equivalent to the restriction that no pair of players can be simultaneously minimaxed.[9]

Say that a subset of players $S \subseteq \{1, \ldots, n\}$ can be simultaneously minimized if there exists a strategy profile μ such that $\pi_i^*(\mu_i) \leqslant f_i$ for all $i \in S$.

DEFINITION: G satisfies *no simultaneous minimizing* [NSM] if no two players of G can be simultaneously minimized.

Under this assumption we obtain a *complete* characterization.

THEOREM 2: *Suppose NSM obtains. The NEU is necessary for the conclusion of the folk theorem.*

PROOF: To establish necessity, we exhibit feasible payoff vectors $x^1, x^2, \ldots, x^n$ such that for all $i \neq j$, $x^i_i < x^j_i$, so that NEU is satisfied by our equivalence result.

Since the conclusion of the folk theorem is valid, $V(\delta) \neq \varnothing$ for sufficiently high δ. Denote by $x^i(\delta)$ an equilibrium payoff vector which yields player i his lowest subgame perfect equilibrium payoff. By adapting the argument of Abreu, Pearce, and Stacchetti (1990), it can easily be shown that $x^i(\delta)$ exists;[10] denote by α^i an equilibrium strategy profile that generates $x^i(\delta)$. By the folk theorem hypothesis, $x^i_i(\delta) \to f_i$, as $\delta \to 1$. By playing his myopic best response in period one and conforming thereafter, i can get at least $(1-\delta)\pi_i^*(\gamma^i_{-i}(\delta)) + \delta x^i_i(\delta)$, where $\gamma^i(\delta)$ is the first period strategy vector in the play of α^i. Hence, $x^i_i(\delta) \geqslant (1-\delta)\pi_i^*(\gamma^i_{-i}(\delta)) + \delta x^i_i(\delta)$, or equivalently $x^i_i(\delta) \geqslant$

[8] Note how f_i differs from w^i_i.

[9] A game in which the minimally attainable payoff is not the minimax payoff for every player is the two-player game specified by the following payoffs: co$[(0,-1),(1,0),(2,1)]$ with $(0,-1)$ a payoff at which player 1 is minimaxed and $(1,0)$ a payoff where player 2 is minimaxed.

[10] The available results on the existence of the worst equilibrium are for the case where mixed strategies are observable; hence we cannot directly appear to any of them. The result is however certainly true and may be proved by, for instance, adapting the self-generation techniques of Abreu, Pearce, and Stachetti (1990) to the present context.

$\pi_i^*(\gamma_{-i}^i(\delta))$. It then follows that along any sequence $\delta'_m \uparrow 1$, for which $\lim \gamma_{-i}^i(\delta'_m)$ exists, $\pi_i^*(\lim \gamma_{-i}^i(\delta'_m)) \leqslant f_i$.

By definition, $x_j^j(\delta) \leqslant x_j^i(\delta)$ for all i, j. Since $x_j^i(\delta) \geqslant (1-\delta)\pi_j^*(\gamma_{-j}^i(\delta)) + \delta x_j^i(\delta)$, if $x_j^i(\delta) = x_j^j(\delta)$ then $x_j^i(\delta) \geqslant \pi_j^*(\gamma_{-j}^i(\delta))$. We claim now that there is $\underline{\delta} < 1$ such that $x_j^j(\underline{\delta}) < x_j^i(\underline{\delta})$ for all $j \neq i$. A contradiction to this claim implies the existence of a sequence $\delta_m \to 1$ and fixed indices $j \neq i$ such that

$$x_j^j(\delta_m) \geqslant \pi_j^*\left(\gamma_{-j}^i(\delta_m)\right) \tag{9}$$

for all m. Assume WLOG that $\lim \gamma_{-i}^i(\delta_m)$ and $\lim \gamma_{-j}^i(\delta_m)$ are well-defined (if necessary, by taking $\{\delta'_m\}$ to be a subsequence of $\{\delta_m\}$). Then the left-hand side of the inequality (9) goes to f_j (by the folk theorem hypothesis), while the right-hand side is strictly greater than f_j, since $\pi_i^*(\lim \gamma_{-i}^i(\delta)) \leqslant f_i$ and simultaneous minimizing is impossible (NSM). This yields the desired contradiction. Finally, $x^i(\underline{\delta})$ is a weighted average of stage payoffs, and so $x^i(\underline{\delta}) \in F$. Now take $x^i = x^i(\underline{\delta})$. *Q.E.D.*

REMARKS:

1. When NSM is not satisfied, a weaker version of NEU is a necessary condition for the folk theorem. From the proof of Theorem 2 it readily follows that for the folk theorem to hold, the game must satisfy "weak NEU": Namely, any subset of players who cannot all be simultaneously minimized cannot all have equivalent utilities. In other words, it must be possible to simultaneously minimize any subset of players who have equivalent payoffs. Weak NEU can also be shown to be sufficient when mixed strategies are observable (and hence for this specification, it provides an exact characterization of the folk theorem). However, this condition may not be sufficient when mixed strategies are unobservable.[11]

2. One class of games in which NSM is violated is two-player games. Another is the class of those symmetric games in which *all* players can be simultaneously minimaxed. By the preceding remark, all players may have equivalent utilities in such games, and the conclusion of the folk theorem may still go through.[12,13] Indeed, for these games it essentially follows from the two-player analysis of Fudenberg and Maskin (1986) that the folk theorem holds without any conditions on the set of feasible payoffs.

4. CONCLUDING REMARKS

We have established a folk theorem by assuming that players have *nonequivalent utilities*. This condition is weaker than the *full dimensionality* condition introduced by Fudenberg and Maskin (1986). Our condition is appealing in that it is simple and easily interpreted (while full dimensionality is a natural geometric concept, it lacks an immediate strategic interpretation) and also minimal in the sense of being almost necessary. It

[11] Suppose that players in some subset S can be simultaneously minimized by the strategy profile μ. With weak NEU and observable play in μ, the punishment regime for $i \in S$ entails playing μ initially, with deviations by $j \in S$ punished by restarting μ. Indeed, the sufficiency of weak NEU for the pure strategy case follows from Wen (1993). (See the conclusion and footnote 14.) However, with $|S| \geqslant 2$, players in S need not be best responding to μ; so if μ involves mixed play by i, then much as in the proof of Theorem 1, his continuation payoff will have to depend on his observable payoffs in this initial phase. It may thus be impossible to provide incentives to keep him from deviating from prescribed mixed play.

[12] For symmetric games, $f_i = f_j = 0$. Hence NSM reduces precisely to no simultaneous minimaxing.

[13] In any symmetric game it is always possible to simultaneously minimax *two* players, but not necessarily all players.

focuses on the deterrence of *individual* deviations as required by Nash equilibrium theory; full dimensionality permits the greater but unnecessary luxury of providing individually calibrated punishments to all players *simultaneously*.

Dutta (1991) uses some of the ideas presented here in proving a folk theorem for the more general class of stochastic games. Smith (1993a) and Smith (1993b) pursue extensions to finitely repeated games and overlapping generation games. An interesting question is whether our results can be extended to other environments, such as imperfect monitoring, in which full dimensionality has been invoked (see Fudenberg, Levine, and Maskin (1989)) to prove folk theorems.

Wen (1993) extends our results by considering repeated games which do not satisfy the NEU condition. He first shows that all equilibrium payoffs of finitely or infinitely repeated games must dominate his newly defined *effective minimax* payoff.[14] He then proceeds to prove that when players are sufficiently patient and/or long lived, any feasible payoff of the stage game can be supported in a subgame perfect equilibrium if and only if it dominates the effective minimax payoff.

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Manuscript received February, 1991; final revision received November, 1993.

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[14] The minimax payoff of player i is $\min_{a_{-i}} \max_{a_i} \pi_i(a_i, a_{-i})$, whereas the effective minimax payoff is $\min_a \max_{j \in I(i)} \max_{a_j} \pi_i(a_j, a_{-j})$, where $I(i)$ are all players with equivalent utilities to i.

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