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Dynamic Deception†

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We characterize the unique equilibrium of a competitive continuous time game between a resource-constrained informed player and a sequence of rivals who partially observe his action intensity. Our game adds noisy monitoring and impatient players to Aumann and Maschler (1966), and also subsumes insider trading models. The intensity bound induces a novel strategic bias and serial mean reversion by uninformed players. We compute the duration of the informed player's informational edge. The uninformed player's value of information is concave if the intensity bound is large enough. Costly obfuscation by the informed player optimally rises in the public deception. (*JEL* D82, D83, G14)

The most valuable commodity I know of is information.
— Gordon Gekko[1]

Deception is an ever topical phenomenon: major political cover-ups such as Watergate and WMDs; massive Ponzi schemes, like Madoff's; entrenched corporate deceptions, such as Enron; pervasive sports subterfuge over performance-enhancing drugs. And deception need not be unethical or illegal—indeed, war and finance heavily rely on it.

The essential model ingredients of dynamic deception are a long-lived player with private information, whose actions are imperfectly observed by either a single uninformed rival or a sequence of them, playing a game with competitive elements. Two distinct literatures have independently explored such dynamic deceptions since the 1980s. In finance, Kyle (1985) studies a large insider with private information about an asset value, facing a sequence of uninformed market players, whose action was a price. In game theory, a large reputation literature models an informed party exploiting valuable private information about its type.

We introduce and fully solve a model that seeks unity between these disparate literatures in finance and game theory. We begin in Section I with a class of potentially infinite-horizon purely competitive games in continuous time. An informed party knows the realization of a binary payoff-relevant state. He allocates a bounded resource between two activities, and his flow payoffs depend on the state and his rival's mixture

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[1] *Wall Street.* (Film). 1987. Dir. Oliver Stone. 20th Century Fox.

between two actions. Specifically, he secures a gain by matching his rival's action, and a smaller gain from aligning his action to the state. In equilibrium, the informed player only profits if his actions reflect his private information. He has a myopic incentive to maximally exploit his informational edge, but his actions are partially obscured from his rival by Gaussian noise (Section II). He faces a dynamic trade-off between immediate payoffs and the capitalized future value of his information edge.

In the unique Markov equilibrium of our dynamic game, strategies and values are sandwiched between those of two one-shot benchmark games in Section III that preserve the action and state space of our dynamic game. As well, static payoffs in each benchmark are proportional to the flow payoffs in our dynamic game. The informed player uniquely knows the state in the asymmetric information benchmark, while neither player is apprised of the state in the symmetric information benchmark. Increasing observational noise conceals early actions for a longer period of time, and shifts our dynamic equilibrium from the symmetric information to the asymmetric information myopic benchmark.

In our dynamic model, whenever the uninformed player is sufficiently convinced in the wrong state, the informed player allocates all resources to the activity favored by the true state. In fact, we show in Section IV that equilibrium play is well-described by a simple scalar *deception parameter*, equal to the interest rate times the observational noise variance divided by the resource bound. When this parameter is large, as with large observational noise, the informed player's strategies in the dynamic and one-shot games coincide for all beliefs—here, the resource constraint binds in one state. Otherwise, the resource constraint does not bind in either state for an interior *confounding interval* of beliefs; here, the uninformed player's strategy coincides with that in the one-shot symmetric information game. In a key contribution, we find that whenever the constraint binds in one state, the uninformed player shades his strategy away from the static best reply toward the best reply for the more likely state. This cross-sectional finding, or *strategic bias*, has a novel time series implication: The uninformed mixture rate mean reverts, drifting up at low beliefs, and down at high beliefs.

By interpreting a unit resource bound as probability allocated between two actions, our model subsumes binary action games. We can thus quantify in Section V a key qualitative finding of the reputation literature—that imperfect monitoring tempts the informed player to chisel away his informational edge, eventually erasing it altogether.[2] In our framework, we deduce an infinite expected time until the truth is *fully revealed*, and then ask how fast the informed player all but monetizes his informational edge. We find that the marginal returns to deception diminish: the expected time until earning fraction ν of his profits is concave in ν.

In Section VI, we use the baseline solution to investigate the incentives of uninformed players to acquire information. Recall that the value of Gaussian information in a static decision-theory context is nonconcave—its marginal value is initially zero, then rises and falls.[3] This, for instance, precludes small purchases of information. But in our dynamic, competitive world, when the public learns more about the state of the world, the informed party exploits his advantage less intensively. This induces a diminishing returns to informational sleuthing that is

[2] See Fudenberg and Levine (1992) and Cripps, Mailath, and Samuelson (2004)

[3] Radner and Stiglitz (1984); Chade and Schlee (2002); and Keppo, Moscarini, and Smith (2008).

absent from decision theory contexts. As a result, when the deception parameter is not too large, no informational nonconcavity emerges for interior beliefs, but instead its marginal value is globally falling, like typical economic goods. Accordingly, this means that engaging in small amounts of commercial or military espionage can possibly be optimal.

We modify the base model in Section VII to investigate a form of deception that is common in applications of our model. *Obfuscation* is the costly veiling of one's action by raising the observational noise—such as diversions to distract the enemy's attention before military invasions, or Enron's refusal to release balance sheets with its earnings statements. Intuitively, greater variance in what is seen about actions leads to a lower variance belief process. We assume the informed player may raise the action observation noise at some increasing, convex cost. We explore when his incentives to veil his actions are greatest. For small deception parameters, obfuscation is single-peaked in his beliefs, and maximal when the uninformed player places equal weight on the states. If marginal obfuscation costs are not too concave, then optimal obfuscation drifts down over time.

Our deception framework generalizes the first work on repeated games of incomplete information, as formulated by Aumann and Maschler (1966) in discrete time. With a unit resource bound, our stage game falls precisely in their class of constant sum games pitting a long-run player who knows the game, against an uninformed player. But that paper assumed perfectly patient players and observed actions. There has since been a large literature, but to our knowledge, ours is the first to relax both constraints.[4] Our generalization allows us to explore a different and yet economically interesting dynamic trade-off: Imperfect monitoring and impatience affords the informed player both an opportunity and an incentive to erode his informational advantage for immediate gain.

For our most relevant link, a special case of our solution converges to the insider trading model of Back and Baruch (2004) as the resource constraint relaxes. Thus, insider trading is a corner example in our larger class of deception games.

Deception *per se* has been explored in static settings. Hendricks and McAfee (2006) model deception as a one-shot zero-sum sender-receiver game in which the attacker's actions are imperfectly signaled. Under some specifications of the noisy observation technology, an equilibrium with a positive weight on "feinting" emerges. Crawford (2003) considers purposeful deception in a one-shot zero sum binary action game with a cheap talk stage and behavioral types hardwired to either tell the truth or lie. If such behavioral types are common, then rational agents can effectively lie in equilibrium.[5]

We employ tools from the nascent literature on continuous time dynamic games with unobserved actions and incomplete information.[6] Our paper differs from reputation models as there are no behavioral types committed to an action. Both informed player's types are rationally motivated. As Hendricks and McAfee point out, this kills the

[4] The survey by Zamir (1992) separately explores the discounted and imperfect monitoring cases.

[5] Models with a long-run informed player, choosing actions noisily observed by sequences of rivals, have a long tradition, e.g., Matthews and Mirman (1983). Sobel (1985) considers a dynamic sender-receiver game. In equilibrium, a sender with myopic incentives to deceive tells the truth with positive probability in order to enhance future opportunities to lie.

[6] See the seminal work by Faingold and Sannikov (2011) for a good summary of this literature.

$\theta = 0$

	a	b
A	$-1-\xi$	$1-\xi$
B	$1+\xi$	$\xi-1$

$\theta = 1$

	a	b
A	$\xi-1$	$1+\xi$
B	$1-\xi$	$-1-\xi$

Figure 1. Flow Payoffs

Notes: Flow payoffs for the informed (row) player; the uninformed (column) player receives the negative of these payoffs. The depicted game is zero sum, but obviously, any positive affine transformation leaves behavior unchanged.

monotonicity that signaling games rely on. And unlike here, one may exploit a reputation and pool with the crazy type without revealing private information.

That the equilibrium value of information may offer surprises to economists echoes the well-known general equilibrium finding of Hirshleifer (1971), also in Schlee (2001). They show how increased information may inhibit risk-sharing, thereby harming everyone. Their framework was static and did not involve the strategic use of market power as ours does.

I. The Model

We study a continuously-repeated constant sum game of incomplete information and partially observed actions played on the time interval $[0, \infty)$. There is one *long-run informed player* ("he"), who alone knows which of two fixed *states* $\theta = 0, 1$ obtains. He plays against a single long-run *uninformed player* ("she"), or a unit density stream of them. Uninformed players have prior belief $q(0) \in (0,1)$ in state $\theta = 1$ at time 0.

The long-run player is risk neutral. His payoffs earn a bank interest rate $i > 0$. The game exogenously ends with chance $\phi\, dt \geq 0$ in any interval $[t, t + dt]$, and so ends after time $t > 0$ with chance $e^{-\phi t}$. The player's net discount factor is then $r = \phi + i > 0$.

The uninformed player (players) has two actions a and b, and chooses a with chance $p(t) \in [0,1]$ at time $t \geq 0$. [7] The informed player chooses activities A and B with respective intensities $\alpha(t) \in [0,M]$ and $\beta(t) \in [0,M]$ at time t.[8] We assume constant sum flow payoffs, such as in Figure 1. Specifically, Player 1 receives the sum of α times the $(p, 1-p)$ weighted average of the listed A row payoffs plus β times the $(p, 1-p)$ weighted average of the B row payoffs, which yields state contingent flow payoffs:

$$(1) \qquad (\alpha - \beta)(1 - 2p - \xi) \quad \text{in state } \theta = 0$$

$$\text{and} \quad (\alpha - \beta)(1 - 2p + \xi) \quad \text{in state } \theta = 1.$$

Thus, payoffs only depend on the actions of the informed player via the *intensity difference* $\Delta(t) \equiv \alpha(t) - \beta(t) \in [-M, M]$ and have two components. First, the

[7] Continuous time mixed strategies is an intuitive extrapolation from discrete time. See Bolton and Harris (1999) for a more formal justification.

[8] We will suppress the dependence on t from now on whenever clear, for expositional ease.

informed player faces a strategic trade-off: He gains or loses 1 depending on how closely his action matches his rival's action. Second, the *information edge* $\xi > 0$ parameterizes the informed player's adverse selection advantage for matching his action to the State.

A critical benchmark for us is the limit of our model as the *intensity bound* explodes ($M \to \infty$). This limit is only well defined when the information edge $\xi \leq 1$; otherwise, the informed player has a dominant strategy to match the state, earning payoff:

$$\begin{cases} \Delta(1 - 2p - \xi) \geq \Delta(1 - \xi) > 0 & \text{in state } \theta = 0, \text{ if } \Delta < 0 \\ \Delta(1 - 2p + \xi) \geq \Delta(\xi - 1) > 0 & \text{in state } \theta = 1, \text{ if } \Delta > 0 . \end{cases}$$

The uninformed player can only defend herself against action surprise provided the private information edge obeys $0 < \xi \leq 1$— henceforth assumed.

We now turn to the players' information sets. To capture *misinformation*, we assume that a Gaussian noise process[9] obscures observations of the informed player's actions $\alpha(t), \beta(t)$. Specifically, we assume that all players commonly observe the scalar signal process $Y(t)$, which only depends on actions through the intensity difference $\Delta(t)$, obscured by a Weiner noise process $W(t)$:[10]

$$dY = \Delta \; dt + \sigma \, dW. \tag{2}$$

Here, $\sigma W(t)$ is a driftless Brownian motion with variance σ^2. Such a process is standard in market maker models in finance—for instance, Back and Baruch (2004) assume that only the net order flow (buys minus sells) is observed, and there is an additional exogenous stochastic net order flow process. Our assumption is more restrictive than is sometimes assumed in the nascent continuous time game literature. Sannikov (2007) would allow that public signals depend on $(\alpha(t), \beta(t))$. Intuitively, our assumption makes sense when what is observed reflects a symmetric garbling of the intensities, with α action flows misinterpreted as β action flows just as often as the opposite.

In the case of a single long-lived uninformed player, we assume that she does not observe her past payoffs. This makes sense if payoffs are only revealed when the game eventually ends. Our war-time example will have this property.

Example 1 (War): Our theory captures competitive settings ranging from resource theft to privately informed trade. For example, consider a wartime struggle in which one of the belligerents chooses between two locations for a decisive invasion.[11] Assume that this invasion will happen at a random time, determined by the arrival of special but unpredictable weather conditions.[12] If we ignore natural time preference

[9] For simplicity, we suppress explicit mention of the associated filtrations in all expectations.
[10] Unlike Hendricks and McAfee (2006) and Crawford (2003), signals have full-support.
[11] For example, in World War II, two such famous choices were Operations Overlord and Mincemeat, for the respective invasions of France and Italy.
[12] For example, the Battle of the Bulge ended when clear skies allowed the Allies to use their air superiority.

$\theta = 0$

	$\tilde{p} = 1$	$\tilde{p} = 0$
Buy	–1	0
Sell	1	0

$\theta = 1$

	$\tilde{p} = 1$	$\tilde{p} = 0$
Buy	0	1
Sell	0	–1

FIGURE 2. UNIT SHARE TRADING GAME

Note: Here $\xi = 1$ and storage payoffs are scaled by $1/2$.

(so that $i = 0$), then the arrival rate of these conditions is $r = \phi > 0$. Preparations for the invasion are done in secrecy, and are only observed with noise by the defender.

Assume the attacker can allocate a fixed flow $M > 0$ of resources per unit time toward preparing either of two possible invasion locations A and B. The defender has a fixed resource flow to allocate to his defenses, which we normalize to 1. Defensive resources reduce the attacker's payoff at the associated location one for one. Assume the attacker has a comparative advantage at one of the locations, namely, the one whose terrain is better suited to his forces. Attacking resources yields an additive bonus of ξ at the preferred location and a penalty of $-\xi$ at the suboptimal location.

Example 2 (**Insider Trading**)**:** The war example naturally demands interpreting M as a resource flow constraint. We next turn to a class of finance examples. This setting first sees strategies interpreted as mixtures, and later as intensities.

Assume that an asset is worth either 0 or 1. Only the informed player knows the value, and he can buy or sell a share of its stock, or randomly mix between trades. The uninformed player is allowed to mix between setting the share prices $\tilde{p} = 0, 1$ with respective chances $(1 - p, p)$ in the normal form game of Figure 2; in this case, the mixture chance p admits an interpretation as the price. In general, we can rescale $M = 1$, and then interpret α and β as mixtures for the normal form game in Figure 1. Equivalently, they are the market-clearing quantities of shares traded in equilibrium. Alternatively, had the uninformed player simply been a risk neutral market maker forced to pick a single *price* in a one-shot trading game, then by the logic of Back and Baruch (2004), he would have chosen the mixture p.[13]

While the setup of our trading model differs somewhat from Back and Baruch (2004), the equilibrium in our model converges to theirs as $M \to \infty$ for the special case $\xi = 1$. *Within our class of games, insider trading models in finance have the largest information edge consistent with a well-defined unbounded intensity model.*

II. Preliminary Equilibrium Analysis

We explore equilibria where strategies at any time depend on the current public belief. In the end, this Markovian restriction does not bind. Since the observation process is common knowledge, so too are the induced public beliefs. Their evolution depends on: (i) the *actual* mixed strategies of the informed player, (ii) the *expected equilibrium* mixed strategies of the uninformed player, and (iii) random observational noise.

[13] A single price is the benchmark in the rational expectations literature. Clearly, competitive market makers would lose money against an informed insider if they could not set a higher ask than bid.

Let $\delta_\theta(q)$ be the uninformed player's *expectation of the intensity difference* Δ *in state* θ *at public belief* q. Unconditional on θ, she expects the intensity difference:

$$\delta(q) \equiv (1 - q)\delta_0(q) + q\delta_1(q). \tag{3}$$

The Appendix contains a heuristic Bayesian derivation of the belief process.[14]

LEMMA 1 (Public Beliefs)*: For the realized intensity difference* Δ, *beliefs obey:*

$$dq(t) = [q(1 - q)(\delta_1 - \delta_0)(\Delta - \delta)/\sigma^2]\,dt + [q(1 - q)(\delta_1 - \delta_0)/\sigma]\,dW. \tag{4}$$

This process obtains in and out of equilibrium, and has many intuitive properties central to our theory. First, belief volatility is greater for smaller signal noise σ^2. This makes sense, since a stronger inference can be drawn from any given intensity Δ. Next, if the uninformed player expects the same intensity in the two states ($\delta_0 = \delta_1$), then she attributes all fluctuations in the signal process dY to noise, and the public belief never changes. Third, if the informed player chooses the unconditionally expected intensity difference $\Delta = \delta(q)$, then beliefs do not drift; however, they still are volatile, as the uninformed player does not know what the intensity difference Δ is, and so seeks to learn from the signal innovations. Finally, when the realized intensity Δ differs from its expectation $\delta(q)$, the belief drift responds linearly, while belief volatility is unaffected. Unexpected changes in the observation process cannot raise volatility, since they are unobserved.

Assume for now a stream of uninformed players, each around for a moment in time. Since each does not know the state, her strategy cannot directly impact the belief evolution. So if the informed player only conditions his intensity on the current public belief, his rival's choice cannot impact the future. At each time t, the informed player chooses an intensity difference $\delta_\theta(q(t))$ in state θ. Simultaneously,[15] the uninformed player sets the mixture $p(q(t))$, knowing the choices of the uninformed players and the signal process $Y(s)$ for times $s < t$.

Consistent with the static payoffs (1) introduced in Section 2, define the *unit flow payoffs*:

$$u_0(p) = 1 - 2p - \xi \quad \text{and} \quad u_1(p) = 1 - 2p + \xi. \tag{5}$$

The informed player's flow payoff is the *product* of his unit flow payoff u_θ and his intensity difference Δ. If $u_\theta < 0$, then $\Delta < 0$ secures a positive flow payoff.

Uninformed players minimize their myopic expected loss, the *expected dividend*:

$$d(q) \equiv q\delta_1(q)u_1(p(q)) + (1 - q)\delta_0(q)u_0(p(q)). \tag{6}$$

[14] The formal argument is in the proof of Theorem 1 in Back and Baruch (2004). Henceforth, we will often suppress the q argument, whenever it is clear.

[15] The sequential versus simultaneous choices distinction in this continuous setting is immaterial, as the realized observation process sample paths are continuous: Knowing $Y(s)$ on $[0,s)$ tells us $Y(t)$.

In light of these definitions, the uninformed player is indifferent across mixtures p in this linear minimization exactly when she expects zero intensity difference:

$$\delta(q) = 0. \tag{7}$$

The informed player engages in a far-sighted optimization. His *interim* or *conditional value* $V_\theta(q)$ is the present value of his payoffs in state θ given the belief q:

$$V_\theta(q) = E\Big(\int_0^\infty e^{-rt}\Delta(q_t)u_\theta(p(q_t))dt | q_0 = q\Big). \tag{8}$$

This value obeys a Bellman equation—as usual captured by the intuitive asset value equation that "the return equals the expectation of the dividend plus capital gains." Here, the "dividend" is the flow payoff, while the "capital gains" are value losses. Using Ito's Lemma with the belief process (4), we have:

$$rV_\theta = \sup_{\Delta \in [-M, M]} \Delta u_\theta(p) + q(1-q)(\delta_1 - \delta_0)(\Delta - \delta)V_\theta'/\sigma^2 + \frac{1}{2}q^2(1-q)^2(\delta_1 - \delta_0)^2 V_\theta''/\sigma^2. \tag{9}$$

The present discounted value of the uninformed player's losses is $V \equiv qV_1 + (1-q)V_0$, or equivalently, this is the informed player's *unconditional* or *ex ante value.*

The informed player can profit from his informational edge only if his intensity difference varies in this knowledge. In choosing a best response to the mixture p, the informed player balances a myopic gain $u_\theta(p)\Delta$ and a capital loss from eroding his edge. Specifically, the more intensely he exploits his information, the better is the signal to the uninformed players—the change in public beliefs is $\Delta q(1-q)(\delta_1(q) - \delta_0(q))/\sigma^2$. All told, the marginal static gains and dynamic losses with an increment in Δ sum to:

$$u_\theta(p) + q(1-q)(\delta_1(q) - \delta_0(q))V_\theta'(q)/\sigma^2. \tag{10}$$

A *Markov equilibrium* is a 5-tuple $(\delta_0, \delta_1, p, V_0, V_1)$ in which:

(i) the public belief q obeys the law of motion (4);

(ii) the uninformed player's mixture p minimizes (6);

(iii) the informed player's value V_θ is defined by equation (9);

(iv) the intensity differences $\delta_\theta = \Delta_\theta$ ensure that (10) vanishes, or that $\delta_\theta = \pm M$ as the expression (10) is positive or negative, respectively.

PROPOSITION 1: *There exists a unique Markov equilibrium.*

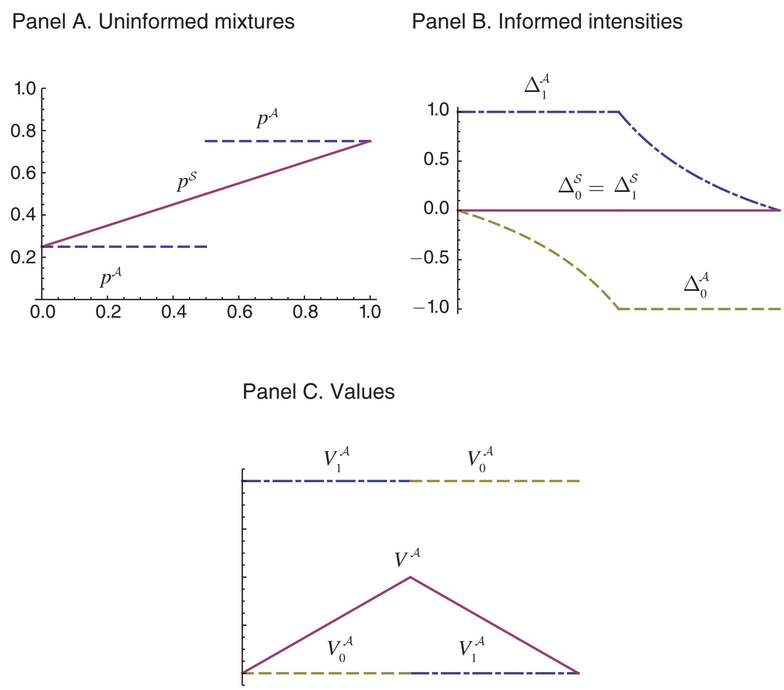

FIGURE 3. BENCHMARK EQUILIBRIA

Notes: Panel A shows the uninformed mixtures in asymmetric and symmetric information games $\mathcal{A}$ and $\mathcal{S}$ (dashed and solid). Panel B shows the informed intensities, pooled at zero in $\mathcal{S}$, and in states 0 and 1 (dashed and dot-dashed) in $\mathcal{A}$. Panel C depicts the values in $\mathcal{A}$: the ex ante value (solid) is concave and the state θ value is positive if and only if the uninformed player thinks state θ is less likely.

Appendix I constructs a Markov equilibrium and includes a precise description of strategies and values. We prove uniqueness by exploiting the zero sum structure of the game.

III. Benchmark Equilibria and Historical Backdrop

Two different benchmark one-shot games provide insight into our dynamic model: the *asymmetric information* game $\mathcal{A}$ that is the stage game of our dynamic model, and the one-shot *symmetric information* game $\mathcal{S}$ with neither player informed of the state. Figure 3 depicts the unique equilibria in each of these benchmark games.

PROPOSITION $\mathcal{A}$: *In asymmetric information game* $\mathcal{A}$, *when* $q < 1/2$, *the price is* $p(q) = (1 - \xi)/2$, *intensity differentials* $\Delta_0(q) = -Mq/(1 - q)$ *and* $\Delta_1(q) = M$, *informed values* $V_0(q) = 0$ *and* $V_1(q) = 2\xi M$, *and ex ante value* $V(q) = 2\xi Mq$.

Game $\mathcal{A}$ yields an extreme informational advantage to player 1, which he maximally exploits whenever the uninformed player places more than 50 percent weight on the wrong state. Having no way to learn from past interactions, the uninformed player must choose a price p in the game that guards against the two types of informed players. Player 1's unconditional intensity is zero, which rationalizes the interior price.

In game $\mathcal{S}$, the equilibrium reflects player 1's ignorance of the state, while player 2 must vary her mixture rate with the common belief q to ensure player 1's indifference.

PROPOSITION $\mathcal{S}$: *In the symmetric information one-shot game, the equilibrium is* $\Delta_0 = \Delta_1 = 0$ *and* $p(q) = 1/2 + (q - 1/2)\xi$, *and the expected value zero.*

Aumann and Maschler (1966) introduced one-shot symmetric information games like $\mathcal{S}$ in their study of repeated games of incomplete information. They showed that when the value of such games is concave—such as our $\mathcal{S}$ with a constant expected value—the informed player cannot profit from his information in the infinitely repeated game with observed actions and the (liminf) time-average payoff criterion. This describes the limit case of our model as the observational noise vanishes ($\sigma \to 0$)—as continuous time obviates the need for $r \to 0$. Absent noise, the informed player can only conceal his information by choosing the same mixture in each state; he is willing to do so because the uninformed player chooses a mixture that prices out the expected informational edge. We enrich Aumann and Maschler (1966) in the special case of a zero value, by making the more standard assumption of impatient players, and also by assuming imperfectly observed actions—as in the recent game theory reputation literature.[16] These changes work in concert: impatient players prefer payoffs sooner rather than later, while the veil on actions enables just such a trade-off. The informed player will then exploit his information and secure a positive expected payoff.

IV. Equilibrium Strategies

Appendix IB gives explicit expressions for equilibrium strategies: Here we focus on their salient properties. We argue that $\mathcal{A}$ and $\mathcal{S}$ are critical benchmark games. To that end, define the scalar *deception parameter* $\psi \equiv r\sigma^2/M^2$ capturing the players' impatience r, the action observation noise σ, and the intensity bound M. Intuitively, the Aumann and Machler game with no discounting most resembles $\psi = 0$; thus, the equilibrium of game $\mathcal{S}$ describes the dynamic game with small $\psi > 0$. Conversely, in game $\mathcal{A}$ the informed player can fully conceal his actions, and so resembles a repeated game with very noisy action observations: Its equilibrium should approximate dynamic play with large $\psi < \infty$.

Figure 4 illustrates the informed intensities for $\psi < 1$ (left) and $\psi \geq 1$ (right). Since the informed player wishes to intensely exploit his informational edge when his opponent is most deceived, the intensity constraint binds at the wrong extreme beliefs. But the intensity constraint cannot bind for both states, except at $q = 1/2$. For the uninformed player strictly mixes in equilibrium, and so must be indifferent among his actions; this requires that his expected intensity vanish (7), i.e., $q = 1/2$. When neither constraint binds, the informed player confounds his opponent by playing each action at a positive intensity. We call this interior belief interval the *confounding* region. It is nonempty when $\psi < 1$, namely, an interval $[q^*(\psi), 1 - q^*(\psi)]$.

[16] See Fudenberg and Levine (1992) and Cripps, Mailath, and Samuelson (2004).

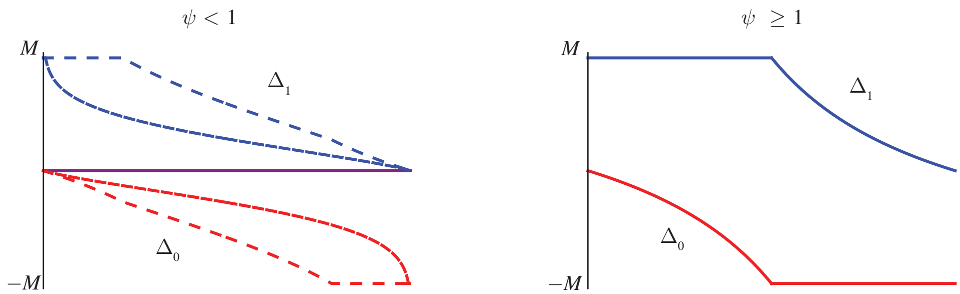

Figure 4. The Informed Intensities

Notes: We graph intensities for two $\psi < 1$ cases on the left, with curves becoming more solid as ψ falls, converging to the game $\mathcal{S}$ intensities $\Delta_\theta^{\mathcal{S}} = 0$. On the right we have the Proposition $\mathcal{A}$ intensities, valid whenever $\psi \geq 1$.

	Confounding region	*Nonconfounding region*
Uninformed	Proposition $\mathcal{S}$ (as in AM66) Equal Back-Baruch with $\xi = 1$	Tend to Proposition $\mathcal{A}$ as $\psi \to \infty$
Informed	Tend to Proposition $\mathcal{S}$ as $r\sigma^2 \to 0$ Tend to Back-Baruch as $M \to \infty$	Proposition $\mathcal{A}$

Figure 5. How Strategies Relate to Benchmarks

Note: This figure summerizes the link between dynamic strategies and static benchmarks contained in Proposition 2.

On its complement, the *constrained* or *information burn* region, one of the constraints always binds.

PROPOSITION 2: *Equilibrium strategies in the bounded dynamic deception game are sandwiched between those in the one-shot games $\mathcal{A}$ and $\mathcal{S}$.*

(*i*) *If $\psi > 1$, the informed strategy coincides with that in Proposition $\mathcal{A}$, and the uninformed strategy jumps up at $q = 1/2$, and tends to those in Proposition $\mathcal{A}$ as $\psi \uparrow \infty$.*

(*ii*) *Let $\psi < 1$. The informed player acts as in Proposition $\mathcal{A}$ outside the confounding region and the uninformed player acts as in Proposition $\mathcal{S}$ inside that region. The informed player's intensities increase in r, σ, tending to those in Proposition $\mathcal{S}$ as $r\sigma^2 \downarrow 0$.*

Figure 5 illustrates this Proposition. The informed player's strategy is the same as in Proposition $\mathcal{A}$ outside of the confounding region, and the uninformed player's strategy the same as in Proposition $\mathcal{S}$ in the confounding region. Our novel strategic findings arise from considering how the informed player behaves in the confounding region, and the uninformed player acts outside it. We will argue that the lessons from insider trading are most salient for the long-run player in the confounding region, while we offer formally new insights about the uninformed player outside this region—adding to the game theory literature on reputation.

PROPOSITION 3: *The informed player's intensities increase in the intensity bound M. In the unbounded limit as $M \to \infty$ and $\xi = 1$, strategies converge to those in the insider trading model of Back and Baruch (2004).*

We can illuminate how our paper subsumes and generalizes Back and Baruch (2004) (see Section I). Fix r and σ, and let $D_0(q), D_1(q)$ be the resulting net purchase rates of their informed trader in states 0, 1. Define the radial expansion of public beliefs $q \mapsto \mathcal{Q}_\psi(q) \equiv c(\psi)q + (1 - c(\psi))/2$, using the constant $c(\psi) \geq 1$ defined in Appendix IB, so that $\mathcal{Q}_\psi(q) \leq c(\psi)q$. Then we can succinctly express the intensity differentials in our bounded model as

$$\Delta_\theta(q) = \frac{\mathcal{Q}_\psi(q)}{c(\psi)q} D_\theta\big(\mathcal{Q}_\psi(q)\big),$$

independent of the information edge $0 \leq \xi \leq 1$. We show in Appendix IE that as $\psi \downarrow 0$, $c(\psi) \downarrow 1$, the confounding interval $[q^*(\psi), 1 - q^*(\psi)]$ monotonically expands to $(0,1)$, and the informed intensities $\Delta_\theta(q)$ monotonically rise $D_\theta(q)$. Since $\psi = r\sigma^2/M^2$, the equilibrium of Back and Baruch (2004) emerges in the limit $M \to \infty$.[17]

In finance, *liquidity* refers to the sensitivity of market prices to order flows: More liquid markets see less price movement in response to trades. Imperfect monitoring of actions in our game theory setting performs an analogous role to noise trade in finance. Having interpreted the uninformed mixture as a price on the two informed actions, we can usefully import this concept of liquidity into our game theoretic deception context. In our model, market liquidity depends on both the responsiveness of the belief drift (4) to trading intensities, and on the slope of the pricing function $p(q)$. For instance, given our linear pricing function $p(q) = 1/2 + \xi(q - 1/2)$ on the confounding region (Proposition 2), the insider trading example with edge $\xi = 1$ is the most liquid.

The finance theory metaphor hints that "prices" $p(q)$ are efficient and a martingale. In fact, neither holds when the intensity constraint binds. The next result complements Propositions 2–3, describing uninformed behavior outside the confounding set.

PROPOSITION 4 (Pricing Bias and Mean Reversion): *The uninformed mixture $p(q)$ is biased toward the more likely state, and this bias increases in the parameter ψ as well as in the information edge ξ. This mixture $p(q)$ mean reverts: It drifts up over time when the public belief q is low, and drifts down when the public belief q is high.*

This result first asserts a strategic bias. As seen in Figure 6, we have $p(q) < 1/2 + \xi(q - 1/2)$ for public beliefs $q < q^*$, with the reverse inequality when $q > 1 - q^*$. For an intuition, consider $q < q^*$. Here, the intensity constraint binds in state $\theta = 1$ but not in state 0. This depresses the expected intensity difference $\delta_1 - \delta_0$. The public belief process (4) is now less sensitive to the actions of the informed

[17] Back and Baruch (2004) establish existence and uniqueness working directly with the unbounded ($M = \infty$) and $\xi = 1$ case, which requires confronting serious technical complications (such as unbounded values) that we avoid by working with the bounded case, and taking limits.

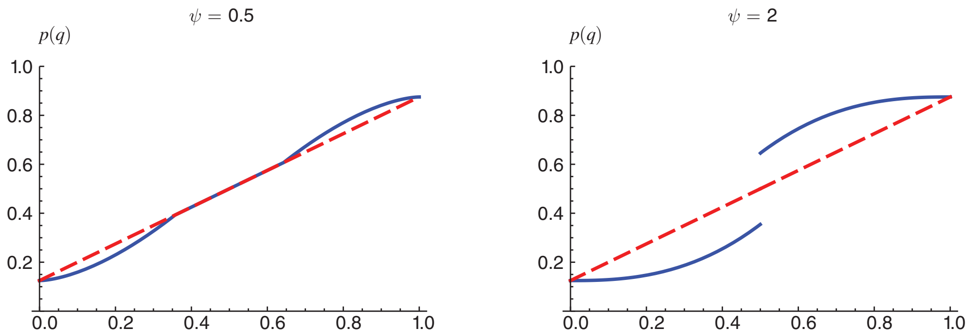

Figure 6. The Uninformed Player's Strategies

Notes: We illustrate Proposition 4, with $\xi = 3/4$, and $\psi = 0.5$ in the left panel and $\psi = 2$ in the right. We contrast our (solid) pricing function $p(q)$ with the (dashed) mixture in Proposition $\mathcal{S}$.

player—equivalently, the "market" is more liquid for any price $p(q)$. The increased market liquidity benefits the informed player in both states. But since he must remain indifferent between his two actions in the low state $\theta = 0$, the price p must fall. Altogether, the intensity constraint causes a downward pricing bias for low beliefs.

The pricing bias in turn leads to predictable price movements. Our proof shows that the function $p(q)$ is strictly locally convex for $q < q^*$, strictly locally concave for $q > 1 - q^*$, and linear in the confounding set. As the belief process (q_t) is a martingale, the price process (p_t) is a submartingale for low beliefs $q < q^*$, a supermartingale for high beliefs $q > 1 - q^*$, and a martingale in the confounding set, as in Figure 6.

V. Time and Money

The arts of deceit and cunning do continually grow weaker and less effectual and serviceable to those that use them.

— John Tillotson, Archbishop of Canterbury (1691–1694)

We now flesh out our equilibrium by examining the time series properties of the unconditional value $V \equiv qV_1 + (1 - q)V_0$. A key insight here is that the informed player exploits his informational edge for static gains. So he incurs a capital loss $(E[dV] < 0)$ because his dividend exceeds his return $d(q) > rV(q)$.[18] How fast does the informed player "burn" through his informational advantage, and thus see his value vanish?

The reputation literature finds that a sufficiently patient informed player will not exploit his informational edge when his actions are perfectly observed; however, he will eventually completely erode his advantage when his actions are ever so slightly obscured by observational noise. Our model unifies this with another dichotomous result in finance: In the static trade model, the no trade theorem precludes exploiting

[18] We can also bound the dividend $d(q) \leq 2rV(q)$. On the confounding region, the right inequality is tight. This allows for a sharp characterization of how fast the unconditional value vanishes a $M \to \infty$. So let $E[-V(q(t))|q(0) = q]$ be *the uninformed player's expected future value after delay* t: this expected value changes at precisely the rate of interest on the confounding region.

an informational advantage, while in a dynamic setting with noise traders, the insider's private information is eventually fully embedded in the price, and the truth is revealed (Back and Baruch 2004). In both cases, *information is fully revealed* given the noise.

Understanding why this occurs is subtle, as both the drift and variance of the public belief vanish as the truth is revealed, so that the learning speed eventually vanishes. Appendix II shows that *complete learning only arises after an infinite expected time*. Thus, we settle for a weaker notion of "nearly complete learning," and then calculate the *precise rate* that learning occurs. When $M = 1$, this also offers a sharp convergence characterization for the repeated game reputation literature—albeit for our specific constant sum game.[19] When $M \to \infty$, this new finding applies for the insider trading model of Back and Baruch (2004).

Focus on the public belief process (q_t), since this is what an outsider observes. Fix $0 < \bar{q} < 1/2$, and let $T(q)$ be the expected time until the belief process (4) starting at $q_0 = q$ leaves the symmetric interval $(\bar{q}, 1 - \bar{q})$. By interpreting $T(q)$ as the value of an asset paying a dividend of 1 per unit time, with zero interest rate (zero return), no drift, and volatility $\varsigma^2(q)$, we can use the standard formula that the return equals the dividend plus the expected capital gain to derive the differential equation for T:

$$(11) \qquad 1 + \frac{1}{2}\varsigma^2(q)T''(q) = 0.$$

The solution T of (11) is a concave function of the beliefs q. Toward an empirically verifiable result, we argue that the expected remaining time is also concave as a function of "money," namely, depending on the fraction $\nu = V(q)/V(1/2)$ of the peak expected value $V(1/2)$. To this end, let $\top_\epsilon(\nu)$ be the market's expected time starting at value $\nu V(1/2)$ until losing a fraction $1 - \epsilon$ of its maximal loss.

PROPOSITION 5 (Diminishing Returns to Time): *For any $0 < \epsilon < \nu$, the expected time $\top_\epsilon(\nu)$ until the value $V(q)$ hits $\epsilon V(1/2)$ is increasing and concave in the initial fraction $\nu \leq 1$ of peak value. When $\psi < 1$, its initial slope in ν is $r\top'_\epsilon(1) = 1$.*

The informed player finds that his deception exercise grows less profitable over time, consistent with the Archbishop's earlier insight. This diminishing return makes sense, as he confronts increasingly adverse "terms of trade" p. The mixture p that he faces drifts up in state $\theta = 1$, and down in state $\theta = 0$, namely $E_1[dp] > 0 > E_0[dp]$, where $dp = p'(q)dq$ for the public belief process (q_t) described in equation (4).

The proof in Appendix II contains an explicit solution for $\top_\varepsilon$ in the $(M \to \infty)$ limit, used below to provide estimates for deception durations. Given the prior $q_0 = 1/2$, the informed player earns r percent of this peak value the first year. His payoff falls over time, by concavity, but he still earns about $\frac{1}{2}\, r$ percent of his peak value through year $0.7/r$. All told, the market expects him to earn 99 percent of this value after a little more than $1.5/r$ years (Figure 7). *With no discovery risk, and*

[19] This is the goal of Cripps, Mailath, and Samuelson (2004), a sequel to Fudenberg and Levine (1992). Unlike our model, these papers assume that one type of Player 1 has a dominant strategy. Both papers conclude that the informed player's true type is revealed in the long run in all equilibria.

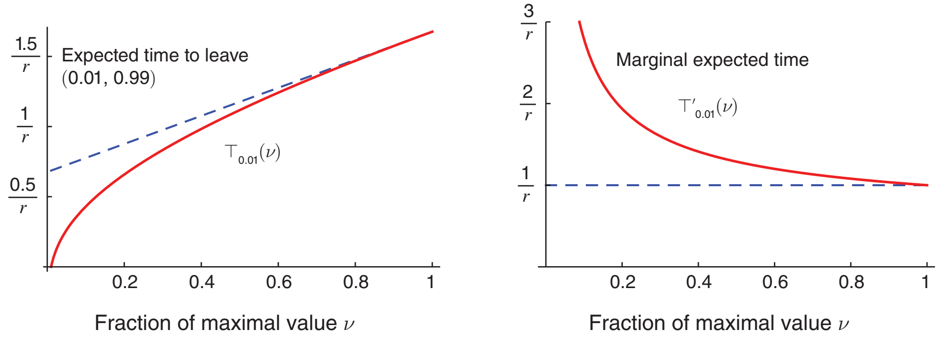

FIGURE 7. DECEPTION DURATION FOR $M \uparrow \infty$

Notes: At left is the expected time until beliefs first leave an interval $(0.01, 0.99)$, drawn as a function of the fraction of the value (solid). Its best linear approximation at $\nu = 1$ (dashed) has slope $1/r$ (time is money). At right is its derivative. For all graphs, we have used the unbounded limit $M \uparrow \infty$ equilibrium for which $\top_\epsilon$ is independent for all parameters σ, r, ξ.

a 10 percent bank interest rate ($\phi = 0$ *and* $r = 10$ *percent*), *the informed player earns 99 percent of his peak value within 15 years.* To wit, deception exercises are long lived without an exogenous risk of discovery.

Wikipedia lists a host of Ponzi schemes, with durations ranging from a few months to 14 years. Charles Ponzi's infamous scheme lasted just two years, from 1918–1920. Bernie Madoff's 15-year deception was among the longest—he admitted that he stopped trading and started fabricating earnings in the mid-1990s. Beyond Ponzi schemes, the accounting deception by Enron began in the early 1990s, and lasted until 2001.

Having fully explored the equilibrium of our model, we now change tacks and in the next two sections, modify the model to allow for information acquisition by the uninformed players, and obfuscation by the informed player.

VI. Information or Sleuthing

If afforded an opportunity, uninformed players would engage in actions that reduce their expected losses $V(q)$. For instance, before investing, individuals often undertake market research. Information sleuthing subsumes a host of applications—from trying to learn the fundamental value in our insider trading example, to military or industrial espionage. For instance, Germany long sought to uncover the planned D-Day landing location. Such information acquisition is always valuable, since it undermines the informed player's edge, and always effects a mean-preserving spread of the belief distribution. Thus the value function $-V(q)$ for the uninformed player must be convex.

Yet while information is valuable, it critically differs from other goods: Under standard conditions, its marginal value in static contexts is initially zero, so the value of information cannot possibly be globally concave.[20] Rather, the marginal value of

[20] See Radner and Stiglitz (1984) and Chade and Schlee (2002). As Keppo, Moscarini, and Smith (2008) show, this nonconcavity even holds for two state models with Gaussian information, such as this one.

information is first rising and then falling, so that individuals either acquire a lot of information, or none at all. So is the parallel assertion true in our dynamic setting, namely, that no one ever has an incentive to engage in just a little espionage?

Our model of espionage assumes that the uninformed player can access an information process similar to that afforded in equilibrium by the passage of time: At any instant, she can see the realization of a signal process like (2), except with fixed unit drift and volatility. The belief process evolves according to $dQ = Q(1 - Q)dZ$ in "metaphorical time" on $[0, \tau]$, for a Wiener process Z. If the uninformed player ends with the random posterior belief $Q(\tau)$, then she earns the "terminal reward" $V(Q(\tau))$. Defining $\mathcal{V}(q_0, \tau) \equiv E[V(Q(\tau))|Q(0) = q_0]$, the value of information is the reduction $V(q_0) - \mathcal{V}(q_0, \tau)$ in the uninformed player's expected loss. Not surprisingly, the initial marginal value of information is positive and finite. Also, it peaks at the public belief $q_0 = 1/2$, and vanishes as q_0 tends to 0 or 1. More strongly:

PROPOSITION 6 (Value of Information): *There exists a threshold* $\overline{\psi} \approx 8.16$ *s.t.*

(*i*) *For all deception parameters* $\psi \geq \overline{\psi}$, *the marginal value of information initially rises and then falls.*

(*ii*) *For any deception parameter* $\psi < \overline{\psi}$, *there exists* $\hat{q}$ *in* $(0, 1/2)$, *such that the marginal value of information is everywhere falling for initial public beliefs* q_0 *in* $(\hat{q}, 1 - \hat{q})$, *but rises and then falls for more extreme initial beliefs* q_0. *As* $\psi \downarrow 0$, *we have* $\hat{q} \downarrow \overline{q} \approx 0.077$.

In our dynamic competitive environment, the classic informational nonconcavity disappears on an interior interval of beliefs, as long as the informed player has enough latitude to exploit his informational advantage—namely, a large enough intensity bound M. Inside the arrow-shaped area of Figure 8, information has a globally decreasing marginal value. When beliefs are sufficiently diffuse and the deception parameter ψ low enough—e.g., a high enough intensity bound or patient enough players or easily observed actions—the demand for information behaves like any standard economic good: It vanishes above a choke-off price, and rises continuously from zero as the price falls.[21] This is consistent with the observation that even a little spying or sleuthing is often worthwhile in wartime, industrial competition, or crime fighting, instances with a long conflict horizon (i.e., low r). Otherwise, as in standard decision theory, individuals either acquire a lot of information, or none at all.

We now offer an intuition for why the nonconcavity disappears. The essential insight is that public information acquisition changes the economic environment. As the public grows more informed, the insider exploits his edge less intensely. This provides an extra source of diminishing returns to the public information purchases. The magnitude of this effect depends on the informed player's responsiveness to the public belief. The extra effect is only large enough to secure a concave value of

[21] That information may have counterintuitive properties in a general equilibrium context is the message of Hirshleifer (1971) and Schlee (2001). By contrast, we argue that an equilibrium setting (now in game theory) *rescues an intuitive economic feature of information*.

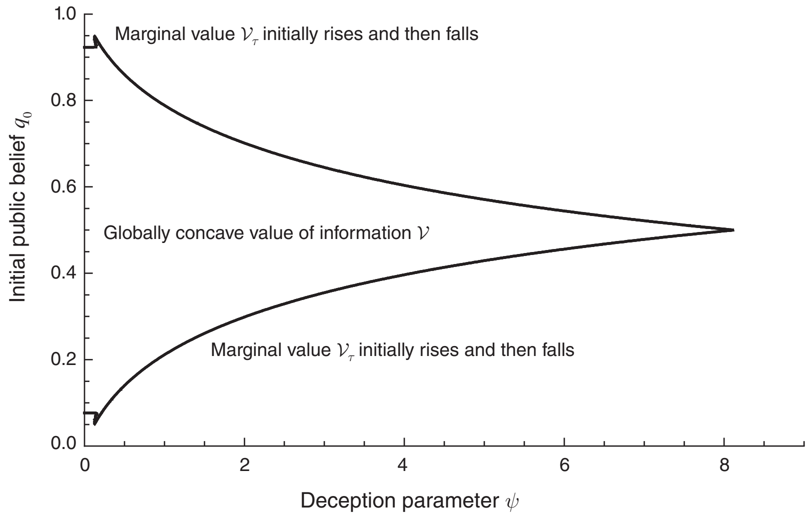

Figure 8

Notes: For low deception parameters ψ and non extreme prior beliefs q_0, the value of information is globally concave in the information quantity τ—as in decision theory. Otherwise, the marginal value of information is initially increasing and then decreasing.

information for sufficiently non-extreme prior beliefs, for instance, when his intensity is sufficiently unconstrained (large M).

Consider our examples in light of Proposition 6. Small levels of market research by individual traders are largely inconsistent with the static theory of information demand. But the financial trading environment is surely one characterized by large intensity bounds and low interest rates, and thus small ψ. In this case, the value of information is concave in our strategic model. On the other hand, intensity resource bounds are surely binding in wartime settings, and impatience plays a critical role. Thus, ψ is larger, and Proposition 6 predicts a nonconcave value, and thus no small espionage in warfare.

VII. Obfuscation

Let's turn to the flip-side of espionage, by exploring actions of the informed player with a concealing flavor. One might think that he surely benefits by increasing the veil on his actions. While this holds unconditionally—i.e., the ex ante value V increases in the noise parameter σ—it curiously fails for the conditional value.

PROPOSITION 7 (More Noise May Hurt the Informed Player)*: The conditional value V_θ falls in observational noise σ for sufficiently accurate public beliefs.*

Fixing the strategy of the uninformed player, the informed player prefers that his actions be better concealed by noise, but the endogenous response of the uninformed player to noise changes may harm the informed player in one state. Assume state $\theta = 0$. The uninformed player's mixture p falls in σ for low public beliefs due to our earlier comparative static in Proposition 4—namely, that the strategic bias increases in $\psi = r\sigma^2/M^2$. This intuitively harms the informed player—e.g., a lower price hurts him, as he wishes to sell in state $\theta = 0$.

Next, let us consider the effect of increasing noise on the unconditional value. *Obfuscation* is the deliberate and costly concealment of actions. In war and competition, this can be secured by simple means such as camouflage or other "smoke and mirrors" activities that distract attention, like the fake electronic chatter preceding D-Day. A gentle modification of our deception model captures this. Assume that the informed player can raise the signal noise.[22] By paying the flow cost $c(\sigma)$, he may secure the noise σ. Assume that $c(\sigma)$ is smooth, strictly increasing, and convex above a baseline level of noise σ. Assume that $c(\sigma)$ is smooth, strictly increasing, and convex above a baseline level of noise $\underline{\sigma} \geq 0$, with initial marginal cost $c'(\underline{\sigma}) = 0$, and $c(\sigma)/\sigma \uparrow \infty$ as $\sigma \uparrow \infty$.

Appendix IVB proves the well-known fact that *volatility can be instantaneously deduced.* This easily precludes any separating equilibrium, where the informed player chooses different volatilities in the two states—for that would immediately reveal the state and yield zero payoff thereafter. By similar logic, it also rules out informative partial pooling equilibria. More strongly, we will next show that no type ever wishes to mix, and that only a pure pooling equilibrium on a unique noise is possible.

Assume that the informed player chooses a common volatility schedule $\sigma(q)$ and state intensity differences δ_0, δ_1. Denote by $w(q)$ the flow dividend (6) for the optimal intensity difference, and $W(q) = qW_1(q) + (1 - q)W_0(q)$ the associated present value. Toward a specific choice of pooling equilibrium, modify optimization (10), using unconditional value and subtracting the flow cost $c(\sigma)$:

$$(12) \quad rW(q) = w(q) + \max_{\sigma \geq \underline{\sigma}} \frac{1}{2} q^2(1 - q)^2(\delta_1(q) - \delta_0(q))^2 W''(q)/\sigma^2 - c(\sigma).$$

This choice of pooling equilibrium is focal, and also embeds an intuitive refinement that with repeated play, the informed player will slowly learn his ex ante optimal noise.[23]

The first derivative of (12) in σ is negative at $\sigma = \underline{\sigma}$, and thus a corner solution is not optimal. But the optimal noise is finite, because the average obfuscation costs $c(\sigma)/\sigma$ explode in σ.[24] The volatility choice is therefore interior, where the marginal cost balances the expected capital gains from more obfuscation:

$$(13) \quad c'(\sigma) = -q^2(1 - q)^2(\delta_1(q) - \delta_0(q))^2 W''(q)/\sigma^3.$$

[22] In our baseline model, the informed player loses from any Blackwell improvement in public information. Here we consider changes in signalling that preserve our Gaussian information structure.

[23] (12) is necessary but not sufficient for our pooling equilibrium. In Appendix IVE, we posit an off-path belief process such that neither type can profitably deviate from the proposed equilibrium.

[24] The only possibility is that intensity differences and obfuscation jointly explode. But then flow payoffs must explode and therefore the difference to the obfuscation edge explodes. This entails dominating flow costs under our cost function assumption.

Differentiating the right side of (12) a second time, we see that it is globally concave. Now substitute the FOC (13) into the Bellman equation (12) to discover:

$$w(q) = rW(q) + \frac{1}{2}\sigma c'(\sigma) + c(\sigma). \tag{14}$$

We next show that for large intensity bounds $M < \infty$, obfuscation is greatest when public beliefs are most diffuse, and that it attenuates over time.

PROPOSITION 8: *Inside any belief interval $I \subset (0,1)$, for some $\overline{M} < \infty$, obfuscation $\sigma(q)$ is quasiconcave, peaking at $q = 1/2$ for any bound $M > \overline{M}$. If $4c''(\sigma) + \sigma c'''(\sigma) > 0$, then $\sigma(q)$ is concave, and obfuscation drifts down.*

The cost condition for concavity essentially requires that marginal costs be not too concave. This holds for all geometric costs, e.g., and at the knife-edge, $c''(\sigma) \propto 1/\sigma^4$.

Loosely, when the uninformed player is most misled, the informed player has the greatest incentive to guard his dividend gains behind a cloud of smoke. The Appendix derives this result for an exploding intensity bound $M \to \infty$. To be precise, this result holds inside any confounding interval of beliefs—as in our original model for deception parameters $\psi < 1$. While we cannot solve this revised model in closed form, intuitively the confounding interval should still increase to $(0,1)$ as players grow sufficiently patient ($r \downarrow 0$), or as the baseline action noise $\underline{\sigma}$ vanishes. In either case, the conclusion of Proposition 8 holds.

This result is consistent with the timing of stealth activities in warfare. For instance, stealth jets have recently been the first used in US military conflicts, and invasions invariably are initiated under cover of darkness.

Appendix

I. Equilibrium Analysis Results and Proofs

A. *Belief Processes: Derivation of Lemma 1*

Consider the random change in beliefs from $q(t) = q$ over $[t, t+dt]$. Let dY be the signal change observed by the uninformed player over this interval. By Bayes' rule,

$$dq(t) \equiv q(t+dt) - q = \frac{q(1-q)\big(f_1(dY) - f_0(dY)\big)}{qf_1(dY) + (1-q)f_0(dY)}. \tag{15}$$

If the actual intensity difference is Δ, while uninformed players expect state contingent intensities δ_θ, then the probability "density" $f_\theta(dY)$ of increment dY is a normal r.v. with mean $\delta_\theta(q)dt$ and variance $\sigma^2 dt$. We exploit the first order approximation:

$$f_\theta(dY) \propto e^{-\frac{(dY-\delta_\theta dt)^2}{2\sigma^2 dt}} \approx e^{-\frac{1}{2}+\delta_\theta dY/\sigma^2} \propto e^{\delta_\theta dY/\sigma^2} \approx 1 + (\delta_\theta/\sigma^2)dY,$$

where the first approximation follows from $(dY)^2 \cong \sigma^2 dt$. Substituting back into (15):

$$dq(t) \approx \frac{q(1-q)\big(\delta_1(q)-\delta_0(q)\big)dY/\sigma^2}{1+\delta(q)dY/\sigma^2}$$

$$\approx q(1-q)\big(\delta_1(q)-\delta_0(q)\big)dY/\sigma^2[1-\delta(q)dY/\sigma^2].$$

Finally, substitute $dY = \Delta dt + \sigma dW$ and $(dY)^2 \cong \sigma^2 dt$ to complete the derivation:

$$dq(t) \approx q(1-q)\big(\delta_1(q)-\delta_0(q)\big)\big(\Delta-\delta(q)\big)/\sigma^2 dt + q(1-q)\big(\delta_1(q)-\delta_0(q)\big)/\sigma dW.$$

B. *Equilibrium Statement*

Recall $\psi \equiv r\sigma^2/M^2$, and define the cutoff beliefs $q^*(\psi) = 1/2$ for $\psi \geq 1$, and

$$q^*(\psi) \equiv \frac{e^{-\lambda(\psi)^2}\sqrt{\psi}}{2e^{-\lambda(\psi)^2}\sqrt{\psi} - \Phi\big(\lambda(\psi)\big)\sqrt{\pi}} < 1/2 \qquad \forall\, \psi < 1,$$

where $\Phi(s) = 2\Big[\int_0^s e^{-t^2}dt\Big]/\sqrt{\pi}$ and $\lambda(\psi) \equiv (\sqrt{1+8\psi}-3)/(4\sqrt{\psi})$, and the constants:

$$c(\psi) \equiv (2\sqrt{\psi/\pi})\,e^{-\lambda(\psi)^2} - \Phi\big(\lambda(\psi)\big) \qquad \psi \leq 1,$$

as well as $A(\psi) \equiv (1/2)(1+\sqrt{1+8\psi})$ and $\phi(\psi) \equiv 2\min\{\psi, A(\psi)-1\}q^*(\psi)^{1-A(\psi)}$. The cutoff belief partitions equilibrium behavior and values into three regions, as follows:

- Lower information burn region $q \leq q^*(\psi)$:

Uniformed mixture: $p(q) = \frac{1}{2}(1-\xi) + \frac{\xi\phi(\psi)}{2\psi}\, q^{A(\psi)}$

Informed intensities: $\delta_0(q) = -[q/(1-q)]M$ and $\delta_1(q) = M$

Uniformed losses: $rV(q) = 2\xi Mq - \frac{\xi M\phi(\psi)}{A(\psi)(A(\psi)-1)}\, q^{A(\psi)}$

Informed values: $rV_0(q) = \frac{\xi M\phi(\psi)}{A(\psi)}\, q^{A(\psi)}$

$$rV_1(q) = 2\xi M - \frac{\xi M\phi(\psi)}{A(\psi)-1}\, q^{A(\psi)-1} + rV_0(q)$$

- Confounding region $q \in \big(q^*(\psi), 1 - q^*(\psi)\big)$:

Uniformed mixture: $p(q) = 1/2 + \xi(q - 1/2)$

Informed intensities: $\delta_0(q) = -\delta_1(1 - q)$

$$= -\big(\sigma\sqrt{r/\pi}\big)\, e^{-\frac{1}{2}F^*(q)^2}/[(1 - q)c(\psi)]$$

Uniformed losses: $rV(q) = \big(\xi\sigma\sqrt{r}/(c(\psi)\sqrt{\pi})\big)\, e^{-\frac{1}{2}F^*(q)^2}$

Informed values: $rV_0(q) = rV_1(1-q) = rV(q) + \big(\xi\sigma\sqrt{r}/\sqrt{\pi}\big)\, qF^*(q),$

where $F^*(q) \equiv F\big(c(\psi)q + (1 - c(\psi))/2\big)$, given probit function $F(y) \equiv \sqrt{2}\Phi^{-1}(2y - 1)$.

- Upper information burn region $q \geq 1 - q^*(\psi)$: The equilibrium functions here follow from: $V_0(q) = V_1(1 - q)$, $\Delta_0(q) = -\Delta_1(1 - q)$, and $p(q) = 1 - p(1 - q)$.

C. *Equilibrium Construction*

We have $\delta(q) = 0$ on all regions: The uninformed player is indifferent for all beliefs.

Confounding Region.—Verifying the proposed values and informed strategies.

Step 0 (Properties of the probit function): Differentiating the identity $\Phi(F(y)/\sqrt{2}) \equiv 2y - 1$ yields $F'(y) \equiv 2/\Phi'(F(y)/\sqrt{2})$, i.e.,

$$F'(y) \;\equiv\; \sqrt{2\pi}\, e^{F(y)^2/2} \tag{16}$$

$$F''(y) \;\equiv\; F(y)F'(y)\sqrt{2\pi}\, e^{F(y)^2} \;=\; F(y)F'(y)^2. \tag{17}$$

Step 1 ($F^*(q)$ is well defined for $\psi < 1$): This holds if and only if $\frac{1}{2} + c(q - \frac{1}{2}) \in [0, 1]$, or

$$-1 \;\leq\; 2c(\psi)(q \;-\; 1/2) \leq 1 \qquad \forall\; q \;\in\; \big(q^*(\psi), 1 \;-\; q^*(\psi)\big).$$

Since $c(\psi) > 0$, it suffices to check these inequalities at the boundary values $q^*(\psi)$ and $1 - q^*(\psi)$. Both inequalities follow easily from $2c(\psi)(q^*(\psi) - 1/2) \equiv \Phi\big(\lambda(\psi)\big) \geq -1$.

Step 2 (Differential equations for values and strategies): Since $\delta(q) = 0$, the Bellman equation (9) becomes

$$r\sigma^2 V_\theta(q) = \frac{1}{2}q^2(1 - q)^2(\delta_1(q) - \delta_0(q))^2 V''_\theta(q). \tag{18}$$

Combine the two state contingent first order conditions (10) to get

$$\delta_1(q) - \delta_0(q) = \frac{2\xi\sigma^2}{q(1-q)\Lambda'(q)}, \tag{19}$$

where $\Lambda(q) \equiv V_0(q) - V_1(q)$. Define $\rho = r/(\xi^2\sigma^2)$ and substitute (10) into (18) to get

$$\rho V_\theta(q) = 2\frac{V''_\theta(q)}{\Lambda'(q)^2}. \tag{20}$$

Subtracting this at $\theta = 0, 1$ yields a second order ordinary difference equation for Λ:

$$\rho\Lambda(q) = 2\frac{\Lambda''(q)}{\Lambda'(q)^2}. \tag{21}$$

Substitute (19) into the indifference equation (3) and the FOCs (10) to get

$$(1-q)\delta_0(q) = -\frac{2\xi\sigma^2}{\Lambda'(q)} = -q\delta_1(q) \text{ and } p(q) = \frac{1}{2} + \xi\left[\theta - \frac{1}{2} + \frac{V'_\theta(q)}{\Lambda'(q)}\right]. \tag{22}$$

We finish by verifying the state $\theta = 0$ solution to (20) (the $\theta = 1$ case is symmetric), the solution to (21), and the proposed $p(q)$ equation.

Step 3 ($V_0(q) = q\Lambda(q) + 2[\rho\Lambda'(q)]^{-1}$ solves (20)): Differentiating the chosen $V_0, V'_0(q) = q\Lambda'(q) + \Lambda(q) - 2\Lambda''(q)/(\rho\Lambda'(q)^2)$. The last two terms cancel by (21), and so,

$$V'_0(q) = q\Lambda'(q). \tag{23}$$

Differentiate and divide by $\Lambda'(q)^2$, to get $V''_0(q)/\Lambda'(q)^2 = q\Lambda''(q)/\Lambda'(q)^2 + \Lambda'(q)^{-1}$. Substitute for $\Lambda''(q)/\Lambda'(q)^2$ from (21) to get $\frac{V''_0(q)}{\Lambda'(q)^2} = \frac{\rho}{2}\left[q\Lambda(q) + \frac{2}{\rho\Lambda'(q)}\right]$, which verifies that $V_0(q) = q\Lambda(q) + 2[\rho\Lambda'(q)]^{-1}$ solves (20).

Step 4 (The solution to the ODE (21)): One can verify the general solution:

$$\Lambda(q) = \frac{2}{\sqrt{\rho}}\Phi^{-1}(c_1(q - 1/2) + c_1c_2).$$

By symmetry, $V_0(1/2) = V_1(1/2)$ where $\Lambda(1/2) = 0$, and so $c_2 = 0$. To see that our proposed solution lies in this class, set $c_1 = c(\psi)$ and use the identity $F(z) = \sqrt{2}\,\Phi^{-1}(2z - 1)$.

Step 5 (The implied strategies): As $q = V_0'(q)/\Lambda'(q)$ from (23), the mixture $p(q)$ in (22) becomes $p(q) = 1/2 + \xi(q - 1/2)$. The posited informed strategies follow from substituting the Λ' solved for in Step 4 in the left hand equation in (22).

Lower Information Burn Region.—The upper region follows from symmetry.

Step 1 (The Bellman equations are satisfied): Substituting $\delta_1(q) - \delta_0(q) = M(1 - q)^{-1}$ into the $\theta = 0$ state FOC yields:

$$p(q) = \frac{1}{2}(1 - \xi) + \frac{qMV_0'(q)}{2\sigma^2}. \tag{24}$$

Inserting this pricing function, $\delta_1(q) = M$ and $\delta_0(q) = -M[q/(1 - q)]$ into the state contingent Bellman equations (9) yields:

$$\psi V_0(q) = \frac{1}{2} q^2 V_0''(q) \quad \text{and}$$

$$\psi V_1(q) = 2\xi\sigma^2/M - qV_0'(q) + qV_1'(q) + \frac{1}{2} q^2 V_1''(q).$$

Substitution verifies the following general solution of this system:

$$V_0(q) = k_0 q^{A(\psi)} \qquad \text{and} \qquad V_1(q) = \frac{2\xi M}{r} + (k_1 + k_0 q) q^{A(\psi)-1}.$$

Our solution requires $k_0 = \xi M\phi(\psi)/\big(rA(\psi)\big)$ and $k_1 = -\xi M\phi(\psi)\big(r(A(\psi) - 1)\big)$.

Step 2 (Informed best response): Differentiate the given V_0 and substitute the result into (24) to confirm the posited $p(q)$ function. Thus, the State 0 FOC is satisfied, justifying any Δ_0. For $\Delta_1(q) = M$ to be optimal, the $\theta = 1$ FOC must be weakly positive. Substituting $\delta_1(q) - \delta_0(q) = M(1 - q)^{-1}$ into this FOC, substituting out $p(q)$ using (24), and replacing $\Lambda'(q)$ using the listed value functions yields

$$FOC_1(q) = 2\xi + k_1 M\left(\frac{A(\psi) - 1}{\sigma^2}\right) q^{A(\psi)-1}. \tag{25}$$

Since $k_1 < 0$ and $A(\psi) > 1$, $FOC_1(q)$ falls in q. Evaluating this FOC at q^* using the k_1 given in Step 1 yields $\xi[2\psi - 2\min\{\psi, A(\psi) - 1\}]/\psi \geq 0$.

Global Optimality.—Verifying the constants q^*, c, k_0, k_1.

Step 1 (Strategies are feasible): We must have $\Delta_\theta \in [-M, M]$ and $p(q) \in [0,1]$. One can easily verify that strategies are monotonic in q. Feasibility then follows from

$$p(0) = (1 - \xi)/2 \geq 0 \qquad p(1) = (1 + \xi)/2 \leq 1$$

$$|\Delta_0(0)| = \Delta_1(1) = 0 \qquad |\Delta_0(1)| = \Delta_1(0) = M.$$

Step 2 (Value matching and smooth pasting for $\psi \geq 1$): When $\psi \geq 1$, we need only check value matching and smooth pasting in one state, since $q^* = 1/2$, $V_0(1/2) = V_1(1/2)$, and $V_0'(1/2) = -V_1'(1/2)$. Given that $\psi \geq 1$ implies $A(\psi) - 1 \leq \psi$, the given k_0 and k_1 satisfy the value matching and smooth pasting conditions at $q^* = 1/2$.

Step 3: The given $\big(q^*(\psi), c(\psi)\big)$ solve:

$$\text{(26)} \qquad \lim_{q \downarrow q^*} \Lambda(q) = \xi M (A(\psi) - 2)/r \qquad \text{and} \qquad \lim_{q \downarrow q^*} V_0'(q) = 2\xi\sigma^2/M.$$

Substitute the confounding region Λ and V_0' into (26) and use (16) to get

$$\text{(27)} \qquad \Phi^{-1}\big(2c(q - 1/2)\big) = \lambda(\psi) < 0 \qquad \text{and} \qquad 2qc\, e^{\Phi^{-1}(2c(q-1/2))^2} = 2\sqrt{\psi/\pi}.$$

Suppress the arguments of $\lambda(\cdot)$ and invert the first of these two equations to get

$$2c(q - 1/2) = \Phi(\lambda) \Leftrightarrow q = 1/2 + \Phi(\lambda)/(2c) \equiv q_1(c).$$

Using the first equation in (27) to eliminate $\big(\Phi^{-1} 2c(q - 1/2)\big)$ from the second yields

$$qc\, e^{\lambda^2} = \sqrt{\psi/\pi} \Leftrightarrow q = \frac{\sqrt{\psi}}{c\sqrt{\pi}}\, e^{-\lambda^2} \equiv q_2(c).$$

The given solution (q^*, c^*) follows from $q_1(c) = q_2(c)$.

Step 4 (The smooth pasting conditions obtain for $\psi < 1$): Using the proposed V_0, we find $\lim_{q \uparrow q^*} V_0'(q^*) = 2\xi\sigma^2/M$, which establishes state $\theta = 0$ smooth pasting by Step 3. Routine algebra establishes $q^* \lim_{q \uparrow q^*} \Lambda'(q) = 2\xi\sigma^2/M$. Altogether,

$$\lim_{q \uparrow q^*} V_0'(q) = \lim_{q \downarrow q^*} V_0'(q) \qquad \text{and} \qquad \lim_{q \uparrow q^*} V_0'(q) = q^* \lim_{q \uparrow q^*} \Lambda'(q).$$

These two equations combined with $\lim_{q \downarrow q^*} V_0'(q) = q^* \lim_{q \downarrow q^*} \Lambda'(q)$ (implied by 23), yield $\lim_{q \uparrow q^*} \Lambda'(q) = \lim_{q \downarrow q^*} \Lambda'(q)$ and completes the step.

Step 5 (The value matching conditions obtain for $\psi < 1$): Using the proposed value functions on the lower information burn region we may directly evaluate:

$$\lim_{q \uparrow q^*} \Lambda(q) = \xi M(A(\psi) - 2) r^{-1} = \lim_{q \downarrow q^*} \Lambda(q) \qquad \text{(by Step 3).}$$

To verify value matching for the $\theta = 0$ state, substitute $\lim_{q \downarrow q^*} \Lambda(q) = \xi M(A(\psi) - 2) r^{-1}$ (by Step 3) and $\lim_{q \downarrow q^*} \Lambda'(q) = 2\xi\sigma^2/(Mq^*)$ (by Step 3 and 23) into the confounding region value $V_0(q) = q\Lambda(q) + 2[\rho\Lambda'(q)]^{-1}$ to get $\lim_{q \downarrow q^*} V_0(q) = \xi M(A(\psi) - 1) q^* r^{-1}$. Routine algebra establishes the same limit result for $\lim_{q \uparrow q^*} V_0(q)$.

D. *Verification and Uniqueness*

Our proposed values and strategies solve the Bellman equation (9), but need not maximize (8). However, since value functions are bounded and C^2, the variance of beliefs is boundedly positive on $(0,1)$, and the discounted value (8) is bounded for all admissible intensities, we may apply Theorem 5.1 in Fleming and Soner (2006) to conclude that our values correspond to the maximized discounted values (8).

For uniqueness, let (δ, p) be an equilibrium of the game with payoffs $(V, -V)$. Assume a different equilibrium payoff $(V^*, -V^*)$ generated by some strategy (δ^*, p^*). The uninformed player does weakly better at (δ^*, p^*) than at (δ^*, p). Because this is a zero-sum game, the informed player does weakly better at (δ^*, p) than at (δ^*, p^*). Allowing him to re-optimize, he does no worse at (δ, p) than at (δ^*, p). Since $V^* \neq V$ by assumption, we must have $V > V^*$. But our starting point was arbitrary, and the same logic establishes that $V^* > V$. So $V^* = V$ for all equilibria. Since this holds for all initial beliefs q, there is a unique equilibrium value $V(q)$, and it is Markovian, as described by our formulas. Our strategies uniquely follow from the value functions and their derivatives.

E. *Properties of Equilibrium Strategies*

CLAIM 1 (Limits of q^* and c): *We have:* $q^* \to 0$ *and* $c \to 1$ *as* $\psi \to 0$.

PROOF:

Since $\lim_{\psi \downarrow 0} \lambda(\psi) = -\infty$, we have $e^{-\lambda(\psi)^2} \to 0$ and $-\Phi(\lambda(\psi)) \to 1$.

CLAIM 2 (Monotonicity of q^* and c): q^* *and* c *are strictly increasing for* $\psi < 1$.

PROOF:

Differentiating q^* we find:

$$\frac{\partial q^*(\psi)}{\partial \psi} = \left(\frac{3A(\psi) - 2}{\pi\psi(2A(\psi) - 1)} \right) e^{-2\lambda(\psi)^2}$$

$$- \; \Phi(\lambda(\psi)) \left(\frac{5A(\psi) - 4 + 2\psi(2A(\psi) - 4)}{2\psi^{3/2}(2A(\psi) - 1)\sqrt{\pi}} \right) e^{-\lambda(\psi)^2}.$$

The first term is positive by $A > 1$. The second is positive, as $\psi < 1 \Rightarrow \lambda(\psi) < 0 \Rightarrow \Phi(\lambda) < 0$, and $5A(\psi) - 4 + 2\psi(2A(\psi) - 4 > 0$ for $\psi < 1$. Differentiating $c(\psi)$:

$$e^{\lambda(\psi)^2} c'(\psi)\sqrt{\pi\psi} = \frac{2A(\psi)}{2A(\psi) - 1} + \frac{1 - A(\psi)}{\psi} > 0.$$

CLAIM 3 (Informed Strategy): *The absolute intensity $|\Delta_\theta|$ is monotonic in q.*

PROOF:

The result is trivial on the burn regions. On the confounding region, we need:

$$\left(\frac{q^2 c(\psi)\sqrt{\pi}}{\sigma\sqrt{r}}\right)\frac{\partial\Delta_1(q)}{\partial q} = -e^{-\frac{1}{2}F^*(q)^2} - 2c(\psi)q\sqrt{\pi}F^*(q)/\sqrt{2} \equiv -\varphi(q) < 0.$$

Since $\varphi'(q) = 2c(\psi)^2\pi q e^{\frac{1}{2}F^*(q)^2} > 0$, it is sufficient to show $\varphi(q^*) > 0$.

$$\frac{\varphi(q^*)}{2q^*c(\psi)} = \frac{e^{-\lambda(\psi)^2}}{2q^*c(\psi)} + \sqrt{\pi}\lambda(\psi)$$

$$= \sqrt{\pi}\left(\frac{1}{2\sqrt{\psi}} + \lambda(\psi)\right) = \frac{\sqrt{\pi}}{4\sqrt{\psi}}\left(\sqrt{1 + 8\psi} - 1\right) > 0,$$

where we have used $F^*(q^*)/\sqrt{2} = \lambda(\psi)$ and $q^*c(\psi)e^{\lambda(\psi)^2} = \sqrt{\psi/\pi}$ from (27).

CLAIM 4 (Informed Strategy): *The intensities increase in r, σ, and M.*

PROOF:

The result is clear on the information burn region. Consider changes in M on the confounding region by differentiating $\Delta_1(q)$ in ψ holding $r\sigma^2$ constant:

$$\frac{\partial\Delta_1(q)}{\partial\psi} = \frac{-r\sigma^2 c'(\psi)}{qc(\psi)^2}$$

$$\left[e^{-\Phi^{-1}(2c(\psi)(q-1/2))^2}/\sqrt{\pi} + 2c(\psi)(q - 1/2)\Phi^{-1}\big(2c(\psi)(q - 1/2)\big)\right] < 0,$$

following from $c' > 0$ and $z\Phi^{-1}(z) > 0$. Since ψ falls in M, we have Δ_1 rising in M.

Now differentiate $\Delta_1(q)$ in $r\sigma^2$ to get

$$\frac{\partial\Delta_1(q)}{\partial r\sigma^2} = \kappa\left[(1 - 2q)\psi c(\psi)c'(\psi)F^*(q)/\sqrt{2} + e^{-\frac{1}{2}F^*(q)^2}(c(\psi) - \psi c'(\psi))\right]$$

$$\equiv \kappa\omega(q,\psi)$$

for some $\kappa > 0$. Note that the bracketed expression is symmetric about $q = 1/2$ and rising in q for $q < 1/2$, and thus it is sufficient to show $\omega(q^*(\psi),\psi) > 0$ for

$\psi \in [0,1]$ (i.e., when the confounding region is nonempty). Evaluating $\omega(q^*(\psi),\psi)$ using (27) we find:

$$\omega(q^*(\psi),\psi) = \left[1 - 2q^*(\psi)\right]\psi c(\psi)c'(\psi)\lambda(\psi) + \left[c(\psi) - \psi c'(\psi)\right]e^{\lambda(\psi)^2}.$$

Tedious algebra verifies $\omega_\psi(q^*(\psi),\psi) < 0$ and $\lim_{\psi\downarrow 0}\,\omega(q^*(\psi),\psi) = 5/(3\sqrt{\pi}) > 0$. Altogether, $\omega(q,\psi) > 0$ for all q in the confounding region.

CLAIM 5: *Outside the confounding region, $p(q)$ is biased toward the likely state.*

PROOF:

We focus on $q < q^*$. Define the bias:

$$B(q) \equiv \xi q + \frac{1}{2}(1 - \xi) - p(q) = \xi q\left[1 - \frac{\phi\psi}{2\psi}q^{A(\psi)-1}\right] > 0 \quad \forall q > q^*,$$

where the inequality follows from $\phi(\psi) \leq 2\psi q^{1-A(\psi)}$ for $q < q^*$.

CLAIM 6: *The bias in $p(q)$ increases in ψ.*

PROOF:

If $\psi \leq 1$, then $B(q) = \xi(q - q^{A(\psi)}q^*(\psi)^{1-A(\psi)})$ rises in ψ when $q < q^*$:

$$\frac{\partial[q^{A(\psi)}q^*(\psi)^{1-A(\psi)}]}{\partial\psi} = \left(\log\left(\frac{q}{q^*}\right)q^*A'(\psi) + (1 - A(\psi))\frac{\partial q^*(\psi)}{\partial\psi}\right)\frac{q}{q^*} < 0.$$

Now assume instead that $\psi > 1$. Then we have

$$B(q) = \xi\left(q - \left(\frac{A(\psi) - 1}{2\psi}\right)(2q)^{A(\psi)}\right).$$

At $\psi = 1$, this reduces to $\xi q(1 - 2q) > 0$ for $q < q^*$. But $B(q)$ rises in ψ, and so is positive for all ψ:

$$\frac{\partial B(q)}{\partial\psi} = \frac{\partial p(q)}{\partial\psi} = \frac{(2q)^{A(\psi)}}{2\psi^2(2A(\psi) - 1)}\,g(\psi,q),$$

where $g(\psi,q) \equiv (A(\psi) - 1)\left[1 + \psi\log(4) + 2\psi\log q\right] - 4\psi$. Finally, $g(\psi,q) < 0$ for all $q < 1/2$ and $\psi > 1$, because $g(\psi,1/2) < 0$ and $g_q(\psi,q) = 2\psi(A(\psi) - 1)/q > 0$.

CLAIM 7: *If $\psi > 1$, then $p(q)$ jumps up discretely at $q = 1/2$.*

PROOF:

Given $\psi > 1$, the confounding region is empty, and so we may evaluate:

$$\lim_{q\uparrow 1/2} p(q) = \frac{1}{2} + \xi\left(\frac{A(\psi) - 1 - \psi}{2\psi}\right) < \frac{1}{2},$$

where the inequality follows from $A(\psi) < 1 + \psi$ when $\psi > 1$. Symmetric reasoning proves $\lim_{q\downarrow 1/2} p(q) > 1/2$. Altogether, $p(q)$ must jump up at $q = 1/2$ when $\psi > 1$.

CLAIM 8: $E[dp] > 0$ *when* $q < q^*$ *and* $E[dp] < 0$ *when* $q > 1 - q^*$.

PROOF:

On the lower information burn region, $p''(q) = \xi\phi(\psi)q^{A(\psi)-2} > 0$. By Ito's Lemma, we have $E[dp] = \frac{1}{2}q^2(1-q)^2(\delta_1(q) - \delta_0(q))^2 p''(q)/\sigma^2 > 0$.

II. Time and Money Proofs

CLAIM 9 (Beliefs): *Beliefs converge to the truth, but in infinite expected time.*

Step 1 (Defining the "Scale Functions"): We only consider state $\theta = 0$. As in Section 15.3 in KT81, define the scale functions s_0 and S_0 as follows:

$$s_0(x) \equiv e^{-2\int_0^x \frac{\mu_0(q)dq}{\varsigma^2(q)}} = e^{2\int_0^x \frac{dq}{1-q}} = (1-x)^{-2} \quad \text{and} \quad S_0(x) \equiv \int_0^x \frac{dy}{(1-y)^2} = \frac{1}{1-x}.$$

Step 2 (Beliefs converge to the truth): We satisfy Lemma 6.1 of KT81: If $S_0(b) - \lim_{\epsilon\to 0} S_0(\epsilon) < \infty$ for some $0 < b < 1$, then the truth is revealed in the limit.

Step 3 (The truth is not revealed in finite expected time): By Lemma 6.2 in Section 15.6 of KT81, the true state is not revealed in finite time if for some $0 < c < 1$:

$$\sum(c) = \lim_{b\to 0}\int_b^c\left[\int_y^c\left[q^2(1-q)^2\left(\delta_1(q) - \delta_0(q)\right)^2 s_0(q)\right]^{-2} dq\right] s_0(y)\, dy = \infty.$$

Since intensities rise in M, the truth is revealed most quickly when $M = \infty$. Using these unbounded limit intensities, we find

$$\sum(c) = \lim_{b\to 0}\int_b^c\left[\int_y^c (1-q)^2\,\Lambda'(q)^2\,dq\right](1-y)^{-2}\,dy$$

$$\geq (1-c)^2 \lim_{b\to 0}\int_b^c\left[\int_y^c \Lambda'(q)^2\,dq\right] dy.$$

Next approximate $(\delta_1(q) - \delta_0(q))$ as $q \to 0$ using $\Lambda'(q)^2 \propto F'(q)^2 \propto e^{F(q)^2} \sim -[q^2 \log q]^{-1}$, where we use the asymptotic behavior of F derived by

Dominici (2003).[25] Altogether, $\Lambda'(q)^2 > q^{-2}$ for all small enough $q > 0$, say $q < c$. For such q, we have

$$\sum(c) \geq \lim_{b\to 0}\int_b^c\left[\int_y^c q^{-2}\,dq\right]dy = \lim_{b\to 0}(-\log b) = \infty.$$

PROOF OF PROPOSITION 5:

Diminishing returns to time.

Step 1 (Dividend return link): The equilibrium dividend $d(q)$ and return $rV(q)$ satisfy $rV(q) < d(q) \leq 2rV(q)$, with $d(q) = 2rV(q)$ on the confounding region.[26] This can be seen by evaluating the dividend at the equilibrium strategies and comparing to rV.

Step 2 (A critical ratio of derivatives): Transform the ODE defining $V, rV(q) = d(q) + \frac{1}{2}\varsigma^2(q)V''(q)$, using $r\overline{V}(q) \equiv rV(q) - d(q)$, to get $r\overline{V}(q) = \frac{1}{2}\varsigma^2(q)V''(q)$. Dividing by (11):

$$T''(q) = -\frac{V''(q)}{r\overline{V}(q)}. \tag{28}$$

Step 3 (Critical inequality): We claim that

$$\frac{\overline{V}''(q)}{-r\overline{V}(q)} \geq \frac{V''(q)}{rV(q)}. \tag{29}$$

On the confounding region $rV(q) = -r\overline{V}(q)$ by Step 1, and we get an equality. On the burn regions $d(q)$ is linear, implying $\overline{V}''(q) = V''(q)$, and the result follows by Step 1.

Step 4 (When $\psi < 1$, time is money at $q = 1/2$): Define $\nu(q) = V(q)/V(1/2)$ and $\top(\nu(q)) = T(q)$. Totally differentiating the second identity and rearranging yields

$$r\top'(\nu) = r\frac{T'(q)}{V'(q)}V(1/2). \tag{30}$$

Since $T'(1/2) = V'(1/2) = 0$, we evaluate this limit using (28):

$$\begin{aligned} &r\top'(1) \\ &\quad = r\lim_{q\to 1/2}\frac{T'(q)}{V'(q)}V(1/2) = r\lim_{q\to 1/2}\frac{T''(q)}{V''(q)}V(1/2) \\ &\quad = r\lim_{q\to 1/2}\frac{1}{-r\overline{V}(q)}V(1/2) = 1. \end{aligned}$$

[25] We say that "$A(q)$ behaves like $B(q)$" for $q \approx 0$ or $q \approx 1$ if their ratio tends to 1 near 0 or 1. We write this as $A \sim B$. When we say $A \propto B$, then we mean that their ratio tends to a positive constant.

[26] Using the Feynman-Kac equation:

$$\mathcal{V}_t(q,t) = \frac{1}{2}q^2(1-q)^2\mathcal{V}_{qq}(q,t) = \frac{1}{2}q^2(1-q)^2\frac{\partial^2}{\partial q^2}E[V(Q(t))|Q(0) = q]$$

and $d(q) = 2rV(q)$, one can show that the V obeys $\mathcal{V}_t(q,t) \to e^{-rt}\mathcal{V}_t(q,0)$ as $M \to \infty$.

The final inequality follows $rV(q) = -r\overline{V}(q)$ on the confounding region (Step 1).

Step 5 ($\top$ is concave): Assume WLOG that $q > 1/2$. Differentiate (30) to get

$$\top''(\nu) = V(1/2)\frac{T''(q) - T'(q)V''(q)/V'(q)}{V'(q)^2}, \tag{31}$$

and so, $\top'' \leq 0$ if and only if $T''(q)/V''(q) > T'(q)/V'(q)$. Using $T'(1/2) = 0$ we have

$$T'(q) = \int_{\frac{1}{2}}^{q} T''(s)ds = \int_{\frac{1}{2}}^{q} \frac{\overline{V}''(s)}{-r\overline{V}(s)}\, ds \geq \int_{\frac{1}{2}}^{q} \frac{V''(s)}{rV(s)} ds > \int_{\frac{1}{2}}^{q} \frac{V''(s)}{rV(q)} ds = \frac{V'(q)}{rV(q)},$$

where the weak inequality follows from (29), the strict inequality owes to $V'' < 0$ and $0 < V(q) < V(s)$ for $1/2 \leq s \leq q$, and the final equality follows from $V'(1/2) = 0$. Since $V'(q) < 0$ for $q > 1/2$ we conclude using (28) and Step 1:

$$\frac{T'(q)}{V'(q)} < \frac{1}{rV(q)} < \frac{1}{-r\overline{V}(q)} = \frac{T''(q)}{V''(q)}.$$

Step 6 (Limit solution as $M \to \infty$): In the limit, $q^*(\psi) \to 0$. Thus, we can use our confounding region solution for V from AB and (30), to convert (31) to an ODE in $\top'$ and $\top''$, and then verify the precise formula:

$$r\top(\nu) = \log(\nu)\sum_{n=0}^{\infty}\frac{b_n[\log(\nu)]^n}{n!} + \text{constant} \quad \text{and} \quad r\top'(\nu) = \frac{\sqrt{\pi}\Phi_i\left(\sqrt{-\log(\nu)}\right)}{2\sqrt{-\log(\nu)}},$$

where $\Phi_i(z) \equiv \Phi(z\sqrt{-1})/\sqrt{-1}$, $b_{n+1}/b_n = (n+1)^2/[(n+3/2)(n+2)]$, and $b_0 = 1$.

III. Information or Sleuthing: Proof of Proposition 6

Step 1 (Initial marginal value): Recall $\mathcal{V}(q,t) \equiv E[V(\,Q(t)\,)\,|\,Q(0) = q]$. By the Feynman-Kac equation ((5.4) on p. 214 of Karlin and Taylor 1981—henceforth, KT81):

$$\mathcal{V}_\tau(q,\tau) \equiv \frac{1}{2}q^2(1-q)^2\mathcal{V}_{qq}(q,\tau) \tag{32}$$

with boundary condition $\mathcal{V}(q,0) = -V(q)$. Thus $\mathcal{V}_\tau(q,0+) = -\frac{1}{2}q^2(1-q)^2V''(q) > 0$, since the boundary condition holds for all q, implying $\mathcal{V}_{qq}(q,0) = V''(q)$.

Step 2 (Re-expressing the marginal value of information): Let the time increment $d\tau$ lead to a stochastic belief change dQ. Then the marginal value obeys

$$\mathcal{V}_\tau d\tau = E_q[V(Q(\tau + d\tau))] - E_q[V(Q)] = E_q[V'(Q)(dQ) + V''(Q)(dQ)^2/2]. \tag{33}$$

Since $E_q[dQ] = 0$ and $(dQ)^2 = Q^2(1 - Q)^2 d\tau$, defining $\Upsilon(q) \equiv q^2(1 - q)^2 V''(q)$ yields

$$\mathcal{V}_\tau(q) = kE_q[\Upsilon(Q)]. \tag{34}$$

Step 3 (The marginal value is quasi-concave and eventually vanishes): By (34), the marginal value of information equals $\mathcal{V}_\tau(q,\tau) = \int_0^1 \Upsilon(x) g(x,\tau,q)\,dx$, where $g(x,\tau,q)$ is the density over public beliefs $q(\tau) = x$ at time τ given prior $q(0) = q$. As computed in Keppo, Moscarini, and Smith (2008):

$$g(x,\tau,q) =$$

$$\sqrt{\frac{q(1-q)}{x^3(1-x)^3 2\pi\tau}}\,\exp\left(-\frac{1}{8}\tau - \frac{\left(\log[x/(1-x)] - \log[q/(1-q)]\right)^2}{2\tau}\right).$$

By Theorem 1.2.1 in Karlin (1968), the density g is *totally positive* (TP) of any order jointly in (q,τ). By the "variation diminishing property" (Theorem 5.3.1 in Karlin), $\mathcal{V}_\tau$ is quasi-concave in τ if $\Upsilon(q)$ is quasi-concave in q. This holds if $\Upsilon'(q) \gtrless 0$ as $q \lessgtr 1/2$.

First assume $\psi > 1$. Here, the confounding region is empty, and so when $q < 1/2$:

$$\Upsilon(q) = \frac{\xi M \phi(\psi)}{2r}(1 - q)^2 q^{\frac{1}{2}(1+A(\psi))} \tag{35}$$

$$\Rightarrow \Upsilon'(q) = \left(\frac{\xi M \phi(\psi)}{4r}\right)(1 - q)(1 + (1 - q)A(\psi) - 5q)\, q^{\frac{1}{2}(A(\psi)-1)}.$$

Thus, $\Upsilon'(q) > 0$ if $h(q) \equiv 1 + (1 - q)A(\psi) - 5q > 0$. This follows from $h'(q) < 0$ and $h(1/2) = (A(\psi) - 3)/2 \geq 0$. The analysis for $q > 1/2$ is symmetric. When $\psi \leq 1$, we verify that $\Upsilon'(q) \gtrless 0$ as $q \lessgtr 1/2$ by graphing the level set $\Upsilon'(q) = 0$ for $(q,\psi) \in [0,1]^2$.[27]

Since beliefs converge to the truth in the limit, and $\Upsilon(0) = \Upsilon(1) = 0$, we have

$$\lim_{\tau \to \infty} \mathcal{V}_\tau(q,\tau) = q\Upsilon(1) + (1 - q)\Upsilon(0) = 0.$$

Step 4 (The initial slope of the marginal value of information): By Steps 1 and 2, $\mathcal{V}_\tau(q,0+) = -q^2(1 - q)^2 V''(q)/2 = \Upsilon(q)$, and thus $\mathcal{V}_{\tau qq}(q,0+) = \Upsilon''(q)$. Differentiating (32), we find

$$\mathcal{V}_{\tau\tau}(q,\tau) = \frac{1}{2} q^2(1 - q)^2 \mathcal{V}_{qq\tau}(q,\tau) \quad \Rightarrow \quad \mathcal{V}_{\tau\tau}(q,0+) = \frac{1}{2} q^2(1 - q)^2 \Upsilon''(q).$$

[27] As Υ' is continuous in both regions, numerical evaluation on a fine enough mesh is reliable. A somewhat lengthy analytic proof is available upon request.

Thus, the initial slope of $\mathcal{V}_\tau$ shares the sign of $\Upsilon''(q)$. When $\psi > 1$, (35) holds for all $q < 1/2$. Twice differentiating, we have

$$(36)\quad \Upsilon''(q) = \left(\frac{\xi M \phi(\psi)}{r}\right) q^{\frac{1}{2}(A(\psi)-3)} \left[\psi - q(1 + A(\psi) + 2\psi) + q^2(2 + A(\psi) + \psi)\right]$$

positive for small q. The bracketed term has derivative $-2(3 + A(\psi))h(q)$, which we have shown is negative when $\psi > 1$, and so Υ'' satisfies single crossing in q. Further, letting $\psi^* \approx 8.123$ solve $\Upsilon''(1/2) = 0$, we have $\Upsilon''(1/2) > 0$ for all $\psi > \psi^*$, and thus by single crossing $\Upsilon'' > 0$ when $\psi > \psi^*$. Again by single crossing, there exists a unique $\hat{q}(\psi)$ such that $\Upsilon''(\hat{q}(\psi)) = 0$ for $\psi < \psi^*$. Solving for $\hat{q}$, we find $\hat{q}(\psi) = 1/2$ for $\psi > \psi^*$, and

$$\hat{q}(\psi) = \frac{1 + 2\psi + A(\psi) - \sqrt{2(1 + A(\psi) + 2\psi)}}{2(2 + A(\psi) + \psi)} \qquad \psi \in (1, \psi^*).$$

When $\psi \leq 1$, (36) remains positive for small q and $\Upsilon''(q) > 0$ by (36). Combining the graph of $(\hat{q}(\psi), 1 - \hat{q}(\psi))$ for $\psi > 1/2$ with the level set $\Upsilon''(q_0) = 0$ for $(q_0, \psi) \in [0,1]^2$, we can trace out the region of the parameter space with $\Upsilon''(q_0) \gtrless 0$ in Figure 8.[28]

Step 5 (Putting it together): The marginal value of information is quasi-concave, and so cannot rise after falling. When $\mathcal{V}_{\tau\tau}(q, 0+) < 0$, the marginal value falls for all τ. Since the marginal value eventually tends to zero, when $\mathcal{V}_{\tau\tau}(q, 0+) > 0$, the marginal value rises initially, but eventually crosses a threshold in τ and falls ever after.

IV. Obfuscation Proofs

A. *How Values Change in Noise*

CLAIM 10: *The unconditional value V increases in σ.*

PROOF:

For $q \in (q^*, 1 - q^*)$, the value is $rV = \xi q \Delta_1(q)$, which increases in σ by Claim 4. When $q < q^*$, V rises in σ if $\phi(\psi) q^{A(\psi)}/2\psi$ falls in ψ. When $\psi \leq 1$, $\phi(\psi) q^{A(\psi)}/2\psi = q^{A(\psi)} q^*(\psi)^{1-A(\psi)}$. Differentiating this expression in ψ:

$$\kappa\left(4q^*(\psi)\log(q/q^*(\psi)) + [2(A(\psi) - 1) - 8\psi]\frac{\partial q^*(\psi)}{\partial \psi}\right),$$

for some $\kappa > 0$. The parenthetical expression on the right is negative for $\psi \leq 1$.

[28] The jagged portion of the level set arises because Υ'' is discontinuous at the boundary q^* of the information burn and confounding regions, since Υ'' depends on fourth derivatives of V.

If $\psi > 1$, then $\phi(\psi)q^{A(\psi)}/2\psi = A(\psi)^{-1}(2q)^{A(\psi)}$; and so, for some $\hat{\kappa} > 0$:

$$\frac{\partial \phi(\psi)q^{A(\psi)}/2\psi}{\partial \psi} = \hat{\kappa}(A(\psi)\ \log(2q) - 1) < 0.$$

CLAIM 11: *There exists $\hat{q} \in (0, q^*)$ such that $\partial V_0(q)/\partial \sigma \lessgtr 0$ as $q \lessgtr \hat{q}$.*

PROOF (Upper Information Burn Region):

We have shown that rV rises in ψ. This requires that $\phi(\psi)q^{A(\psi)}/(A(\psi) - 1)$ fall in ψ, which in turn implies

$$rV_1(q) = \xi M\left(2q - \frac{\phi(\psi)}{A(\psi) - 1}\, q^{A(\psi)}(A(\psi)^{-1} + q)\right) \qquad q < q^*$$

increases in ψ. The same holds for $V_0(q)$ for $q > 1 - q^*$ by symmetry $(V_0(1 - q) = V_1(q))$.

Proof Step 1 (Lower information burn region): First consider $\psi > 1$. In this case, $V_0(q)$ equals a positive constant times $q^{A(\psi)}2^{A(\psi)-1}(A(\psi) - 1)/A(\psi)$, and so,

$$\frac{\partial V_0(q)}{\partial \psi} = \kappa^*(1 + \psi\log(4) + 2\psi\ \log(q)),$$

for some $\kappa^* > 0$. Altogether, $\partial V_0(q)/\partial \sigma \lessgtr 0$ as $q \lessgtr (1/2)e^{-\frac{1}{2\psi}} < 1/2 = q^*$.

If instead $\psi \leq 1$, then when $q < q^*$ we have sign $(\partial V_0(q)/\partial \sigma) =$ sign $(h(\psi, q)$, where:

$$h(\psi, q) \equiv q^*(\psi)[2A(\psi) + 4\psi] - 4\psi^2[2A(\psi) - 1]\,\frac{\partial q^*(\psi)}{\partial \psi}.$$

Easily, $h_q(\psi, q) > 0$, $\lim_{q\to 0} h(\psi, q) = -\infty$ and $h(\psi, q^*(\psi)) > 0$, implying the result.

Proof Step 2 (Confounding region): We claim $\partial V_0(q)/\partial \sigma > 0$ on the confounding region. By Step 2, $\partial V_0(q)/\partial \sigma > 0$ at q^* on the information burn region. Since values are continuous in q for all σ, so are the derivatives in σ. Altogether, $\partial V_0(q)/\partial \sigma > 0$ at q^* on the information burn region. We complete the proof by showing $\partial^2 V_0/\partial \sigma \partial q > 0$. Recall our equilibrium value difference, and equation (23):

$$r\big(V_0(q) - V_1(q)\big)/(\xi M\sqrt{2}) = F^*(q) \quad \text{and} \quad V_0'(q) = q\big(V_0'(q) - V_1'(q)\big).$$

Thus $\partial^2 V_0/\partial \sigma \partial q > 0$ if and only if $\partial^2 F^*/\partial \psi \partial q > 0$. Since $F^*(q) \lessgtr 0$ as $q \lessgtr 1/2$, we find

$$\frac{\partial F^*}{\partial \psi} = c'(\psi)(q - 1/2)(\sqrt{2\pi})\,e^{\frac{1}{2}F^*(q)^2} \quad \Rightarrow$$

$$\frac{\partial^2 F^*}{\partial \psi \partial q} = c'(\psi)(e^{\frac{1}{2}F^*(q)^2} + c(\psi)(2q - 1)e^{F^*(q)^2}F^*(q)\sqrt{\pi/2}) > 0.$$

B. *Obfuscation is Perfectly Revealing if it Depends on State*

Let the uninformed player observe a signal process like (2) with an unknown volatility σ over any finite interval $[a,b]$ of time dt. She may partition this interval into n equal segments of width $(b - a)/n$, and let $X_i^{(n)}$ denote n times the ith squared Wiener increment. As a squared normal $N(0,\sigma^2)$ random variable, each $X_i^{(n)}$ is an independent draw from a χ-squared distribution with mean σ^2 and variance $2\sigma^2$. By a version of the strong law of large numbers for a discounted finite variance, the average of the n terms $X_i^{(n)}$ almost surely converges to σ^2 as letting $n \to \infty$.

C. *Beliefs Become Confounding for Large Intensity Bounds*

If not, then for some $\varepsilon > 0$ and subsequence $M_k \uparrow \infty$, the intensity constraint $\Delta_1 \leq M_k$ binds on an interval $(0,\varepsilon)$, for $k = 1, 2, \ldots$. By (5), the flow payoff $\Delta_1 u_1(p)$ explodes on $(0,\varepsilon)$, unless $p \to (1-\xi)/2$. But then $u_0(p) \to -2\xi$. Then, using $\Delta_0 = -M$, informed player earns a flow payoff $2M\xi$ in $(0,\varepsilon)$. This clearly explodes in M.

D. *Identities Valid for Confounding Beliefs*

Twice differentiate $W(q) = qW_1(q) + (1 - q)W_0(q)$:

$$W''(q) = qW_1''(q) + (1 - q)W_0''(q) - 2\Gamma'(q), \tag{37}$$

where $\Gamma = W_0 - W_1$. Combine the intensity FOCs to get

$$\delta_1(q) - \delta_0(q) = \frac{2\xi\sigma^2}{q(1 - q)\Gamma'(q)}. \tag{38}$$

Substitute this into (41) and use $f = 0$ in equilibrium to get

$$\rho W_\theta(q) + (\xi\sigma)^{-2}c(\sigma) = 2\frac{W_\theta''(q)}{\Gamma'(q)^2}. \tag{39}$$

Subtracting this at $\theta = 0, 1$ yields

$$\rho\Gamma(q) = 2\frac{\Gamma''(q)}{\Gamma'(q)^2}. \tag{40}$$

E. *Conditional Incentives For Large M*

Step 1 (Feasible beliefs): We assume belief process:

$$dq(t) = \left[s(q)(\Delta - \delta(q))/\sigma^2 + \mu(\sigma,q)\right]dt + \upsilon(\sigma,q)dW,$$

where $s(q) = q(1 - q)(\delta_1 - \delta_0)$. Beliefs are *feasible* if (μ,υ) is bounded, does not depend on the state, $\mu(\sigma(q),q) = 0$, and $\upsilon(\sigma(q),q) = s(q)/\sigma(q)$ (i.e.,(4) holds in equilibrium).

Step 2 (Conditional maximization): Given Step 1, we have

$$rW_\theta = \sup_{\Delta, \sigma \geq \underline{\sigma}} \Delta u_\theta(p) + \left[s(q)(\Delta - \delta(q))/\sigma^2 + \mu(\sigma, q)\right] W'_\theta + \frac{1}{2}\upsilon(\sigma, q)^2 W''_\theta - c(\sigma).$$

Using $\delta(q) = 0$ from (7) and the FOC in Δ, we focus on the maximization in σ alone:

$$rW_\theta = \sup_{\sigma \geq \underline{\sigma}} \mu(\sigma, q) W'_\theta + \frac{1}{2}\upsilon(\sigma, q)^2 W''_\theta - c(\sigma). \tag{41}$$

Step 3 ($W'_0(q)$ and $W'_1(q)$ cannot share the same sign): Suppose they do. Then by the state contingent FOCs in Δ, u_0 and u_1 must share the same sign, which we WLOG assume positive. However, if this is true, then the informed player can set $\Delta_0 = \Delta_1 = M$, beliefs would not update, and values would explode in M.

Step 4 (Feasible and incentive compatible beliefs): We now construct feasible off-path beliefs such that the solution to (12) also solves (41). First set $\upsilon(\hat{\sigma}, q) = 0$ for all $\hat{\sigma} \neq \sigma(q)$, so that the off-path incentive compatibility constraints become

$$\mu(\hat{\sigma}, q) W'_\theta(q) \leq rW_\theta(q) + c(\hat{\sigma}) \qquad \forall \hat{\sigma} \neq \sigma(q). \tag{42}$$

Assume that $W'_1(q) < 0 < W'_0(q)$ (WLOG by Step 3), then let $\mu(\cdot, q)$ satisfy(42) with equality in State $\theta = 0$. Since $rW_\theta + c > 0$ this yields $\mu(\cdot, q) > 0$. But, then since $W'_1 < 0$, we must have(42) strictly satisfied in State $\theta = 1$.

F. *Proof of Proposition* 8

Step 1 ($w - rW$ is concave): The difference $rW(q) - w(q)$ is convex in q if the objective function in (12) is convex as a function of q. Using(38), we can simplify the terms in q in (12) as

$$\frac{1}{2}q^2(1 - q)^2(\delta_1(q) - \delta_0(q))^2 W''(q)/\sigma^2 = 2\xi^2\sigma^2 \frac{W''(q)}{\Gamma'(q)^2}.$$

Thus, equation (39) yields

$$\frac{2W''(q)}{\Gamma'(q)^2} = 2q\frac{W''_1(q)}{\Gamma'(q)^2} + 2(1 - q)\frac{W''_0(q)}{\Gamma'(q)^2} - 4/\Gamma'(q)$$
$$= \rho W(q) - 4/\Gamma'(q) + (\xi\sigma)^{-2} c(\sigma).$$

Twice differentiate the RHS/ρ in q and use (37), (39), and (40) to get

$$\left(W'(q) + 4\frac{\Gamma''(q)}{\rho\Gamma'(q)^2}\right)' = (W'(q) + 2\Gamma(q))' = W''(q) + 2\Gamma'(q)$$
$$= qW''_1(q) + (1 - q)W''_0(q) > 0.$$

Step 2 (Quasiconcavity): Differentiating (14) we find

$$\sigma'(q) = \frac{2[w'(q) - rW'(q)]}{\sigma(q)c''(\sigma(q)) + 3c'(\sigma(q))} \equiv \frac{2[w'(q) - rW'(q)]}{z(\sigma(q))}. \tag{43}$$

Easily, $w(q) - rW(q) \geq 0$ is symmetric about $q = 1/2$, and satisfies $w(0) - rW(0) = w(1) - rW(1) = 0$. So if $w(q) - rW(q)$ is concave, then $\sigma'(q) \gtreqless 0$ for $q \lesseqgtr 1/2$.

Step 3 (Concavity): Differentiating and rearranging (43) yields

$$\sigma''(q)z(\sigma(q)) + \sigma'(q)^2 z'(\sigma(q)) = 2[w''(q) - rW''(q)].$$

By Step 2, the RHS is negative, thus $z' = 4c'' + \sigma c''' > 0 \Rightarrow \sigma''(q) < 0$.

REFERENCES

Aumann, Robert J., and Michael B. Maschler. 1966. "Game Theoretic Aspects of Gradual Disarmament." In *Report to the US Arms Control and Disarmament Agency*, Chapter 5. Washington, DC: US Arms Control and Disarmament Agency.

Back, Kerry, and Shmuel Baruch. 2004. "Information in Securities Markets: Kyle Meets Glosten and Milgrom." *Econometrica* 72 (2): 433–65.

Birch, Thomas, and John Tillotson. 1820. *Works of John Tillotson, Late Archbishop of Canterbury.* Vol. 4. London: J.F. Dove.

Bolton, Patrick, and Christopher Harris. 1999. "Strategic Experimentation." *Econometrica* 67 (2): 349–74.

Chade, Hector, and Edward E. Schlee. 2002. "Another Look at the Radner-Stiglitz Nonconcavity in the Value of Information." *Journal of Economic Theory* 107 (2): 421–52.

Crawford, Vincent P. 2003. "Lying for Strategic Advantage: Rational and Boundedly Rational Misrepresentation of Intentions." *American Economic Review* 93 (1): 133–49.

Cripps, Martin W., George J. Mailath, and Larry Samuelson. 2004. "Imperfect Monitoring and Impermanent Reputations." *Econometrica* 72 (2): 407–32.

Dominici, Diego. 2003. "The Inverse of the Cumulative Standard Normal Probability Function." Unpublished.

Faingold, Eduardo, and Yuliy Sannikov. 2011. "Reputation in Continuous-Time Games." *Econometrica* 79 (3): 773–876.

Fleming, Wendell, and Halil Soner. 2006. *Controlled Markov Processes and Viscosity Solutions*. New York: Springer Science and Business Media, Inc.

Fudenberg, Drew, and David K. Levine. 1992. "Maintaining a Reputation When Strategies Are Imperfectly Observed." *Review of Economic Studies* 59 (3): 561–79.

Hendricks, Kenneth, and R. Preston McAfee. 2006. "Feints." *Journal of Economics and Management Strategy* 15 (2): 431–56.

Hirshleifer, Jack. 1971. "The Private and Social Value of Information and the Reward to Inventive Activity." *American Economic Review* 61 (4): 561–74.

Karlin, Samuel. 1968. *Total Positivity*. Stanford, CA: Stanford University Press.

Karlin, Samuel, and Howard Taylor. 1981. *A Second Course in Stochastic Processes*. San Diego, CA: Academic Press.

Keppo, Jussi, Giuseppe Moscarini, and Lones Smith. 2008. "The Demand for Information: More Heat Than Light." *Journal of Economic Theory* 138 (1): 21–50.

Kyle, Albert S. 1985. "Continuous Auctions and Insider Trading." *Econometrica* 53 (6): 1315–35.

Matthews, Steven A., and Leonard J. Mirman. 1983. "Equilibrium Limit Pricing: The Effects of Private Information and Stochastic Demand." *Econometrica* 51 (4): 981–96.

Radner, Roy, and Joseph E. Stiglitz. 1984. "A Nonconcavity in the Value of Information." In *Bayesian Models in Economic Theory*, edited by Marcel Boyer and Richard Kihlstrom, 33–52. Amsterdam: Elsevier Science, North-Holland.

Sannikov, Yuliy. 2007. "Games with Imperfectly Observable Actions in Continuous Time." *Econometrica* 75 (5): 1285–329.

Schlee, Edward E. 2001. "The Value of Information in Efficient Risk-Sharing Arrangements." *American Economic Review* 91 (3): 509–24.

Sobel, Joel. 1985. "A Theory of Credibility." *Review of Economic Studies* 52 (4): 557–73.

Zamir, Shmuel. 1992. "Repeated Games of Incomplete Information: Zero-Sum." In *Handbook of Game Theory*. Vol. 1, edited by Robert J. Aumann and Sergiu Hart, 109–54. Amsterdam: Elsevier Science, North-Holland.