

last revised: 9 October 2009

## 15 Two-Dimensional Nonlinear Models

While we have by no means exhausted the applications of one-dimensional models, we turn in this chapter to two-dimensional nonlinear models. Generically, these systems can be written as

$$\begin{aligned}u_{t+1} &= g_1(u_t, v_t) \\v_{t+1} &= g_2(u_t, v_t)\end{aligned}$$

where the scalars  $u$  and  $v$  are called the *state variables*. Our application in this chapter is a simple group-process model which addresses the mutual interdependence between “friendliness” and “activity” within a group of people. After specifying the model, we show how to analyze it graphically using a phase diagram. We then generalize stability analysis to the two-dimensional case.

### 15.1 A simple group-process model

Drawing on empirical studies of human groups, Homans (1950) argued that positive sentiment (“friendliness”) among group members encourages a higher level of group participation (“activity”). At the same time, he held that activity encourages friendliness. The two variables are thus linked through a positive feedback loop. Homans was also interested in the effect that outside forces (the “external system”) would have on group dynamics (the “internal system”). Building on Homan’s informal theorizing, Simon (1952) developed a formal model that became an early classic of mathematical sociology. In his original paper, Simon focused most of his attention on a linear model, but also discussed a nonlinear version. Here, we’ll consider a simple specification of the nonlinear model.

Following Simon (1952), we begin with a continuous-time specification of the model. Letting the scalars  $F$  and  $A$  denote friendliness and activity, we assume

$$\begin{aligned}dF/dt &= A - \beta F^2 \\dA/dt &= F + \xi - \gamma A^2\end{aligned}$$

where  $\beta$ ,  $\gamma$ , and  $\xi$  are parameters. Fixing the level of activity, the first differential equation indicates that friendliness is rising when it is “too low” relative to activity, is falling when it is “too high” relative to activity, and is stable when

$$F = \sqrt{A/\beta}$$

The nonlinearity in this equation might reflect the “diminishing returns” of additional activity on the level of friendliness. The parameter  $\xi$  in the second differential

equation may be interpreted as the level of “external activity” imposed on the group. (For example, consider the amount of homework that a professor assigns to a study group.) Fixing the levels of friendliness and external activity, that equation indicates that activity is rising when it is “too low” relative to  $F$  and  $\xi$ , is falling when it is “too high” relative to  $F$  and  $\xi$ , and is stable when

$$A = \sqrt{(F + \xi)/\gamma}$$

We might again invoke a “diminishing returns” argument to motivate the nonlinear effect of additional friendliness or external activity on the level of activity.

Moving to a discrete-time specification, the model becomes

$$\begin{aligned}\Delta F &= (A - \beta F^2) h \\ \Delta A &= (F + \xi - \gamma A^2) h\end{aligned}$$

where the parameter  $h$  reflects period length. Equivalently, we may write

$$\begin{aligned}F_{t+1} &= F_t + (A_t - \beta F_t^2) h \\ A_{t+1} &= A_t + (F_t + \xi - \gamma A_t^2) h\end{aligned}$$

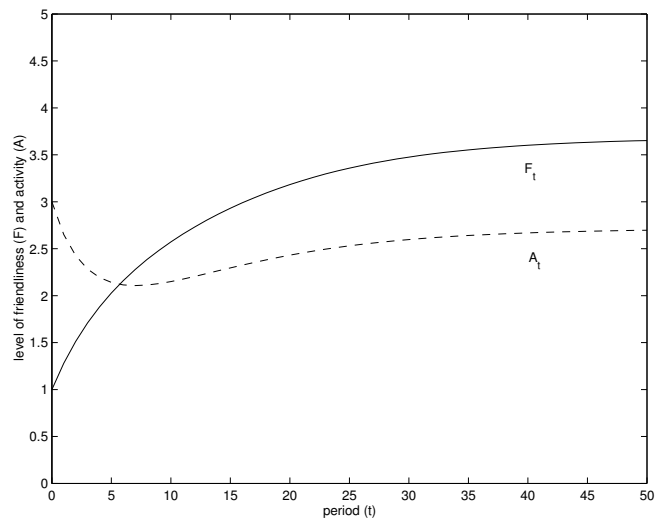
where the  $t$  subscripts index periods. As discussed in Chapter 11, we will set  $h$  small so that our discrete-time model approximates the continuous-time dynamics. While this might seem unnecessary (why not simply set  $h$  to 1?), the importance of period length will be discussed in section 15.7 below.

To begin exploring the behavior of this model, we initially assume no external activity ( $\xi = 0$ ) and fix the other parameters at  $\beta = 0.2$ ,  $\gamma = 0.5$ , and  $h = 0.1$ . The diagrams on the next page show the time path of  $F$  (solid curve) and  $A$  (dotted curve) for the first 50 periods. The first diagram assumes the initial conditions ( $A_0 = 3, F_0 = 1$ ) while the second diagram assumes the initial conditions ( $A_0 = 2, F_0 = 5$ ). In both cases, the system converges to the same long-run outcome ( $A^* = 2.7144$ ,  $F^* = 3.6841$ ). Thus, one immediate question is whether the system converges to this equilibrium for every possible set of initial conditions. We will address this issue graphically in the next section.

```

>> b = .2; g = .5; h = .1; A = 3; F = 1; y1 = [A F];
for t = 1:50;
    dF = (A - b*F^2)*h; dA = (F - g*A^2)*h; F = F+dF; A = A+dA; y1 = [y1; A F];
end;
plot(0:50,y1(:,1),0:50,y1(:,2))
% time paths of F and A given A0 = 3 and F0 = 1

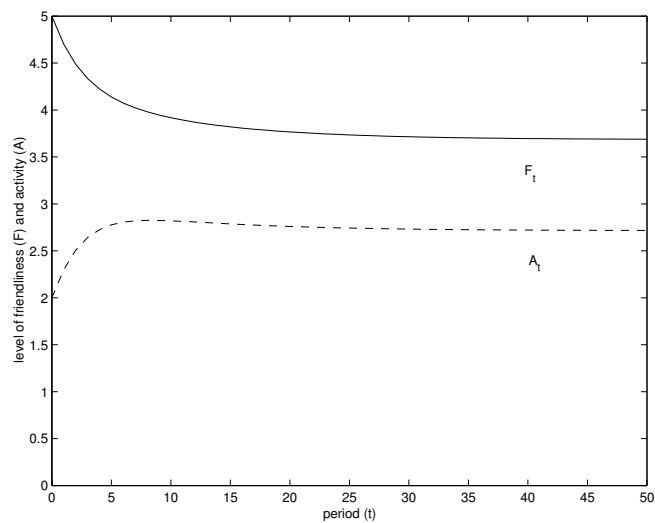
```



```

>> b = .2; g = .5; h = .1; A = 2; F = 5; y2 = [A F];
for t = 1:50;
    dF = (A - b*F^2)*h; dA = (F - g*A^2)*h; F = F+dF; A = A+dA; y2 = [y2; A F];
end;
plot(0:50,y2(:,1),0:50,y2(:,2))
% time paths of F and A given A0 = 2 and F0 = 5

```



## 15.2 The phase diagram

Graphical analysis of two-dimensional models is facilitated by use of a *phase diagram*. In steady state, both of the state variables must be stable. For a generic two-equation system with state variables  $u$  and  $v$ , steady state thus requires

$$\Delta u = g_1(u, v) - u = 0$$

$$\Delta v = g_2(u, v) - v = 0$$

Each of these conditions defines a relationship between the state variables. When plotted (in either  $(u, v)$  or  $(v, u)$  space), these curves are called *nullclines*. In particular, the curve corresponding to the  $\Delta u = 0$  condition may be called the  $u$ -nullcline. This terminology reflects the fact that  $u$  is stable (neither rising nor falling) everywhere along the  $u$ -nullcline. Similarly,  $v$  is stable everywhere along the  $v$ -nullcline determined by the  $\Delta v = 0$  condition. An intersection of the nullclines, corresponding to a pair  $(u^*, v^*)$  such that  $\Delta u^* = 0$  and  $\Delta v^* = 0$ , indicates a steady state of the system.

Returning to our present example, the system is in steady state when both  $F$  and  $A$  are stable. We have already noted that  $\Delta F = 0$  implies

$$F = \sqrt{A/\beta}$$

which is the equation for the  $F$ -nullcline. Rewriting the  $\Delta A = 0$  condition to facilitate plotting in  $(A, F)$  space, we obtain

$$F = \gamma A^2 - \xi$$

which is the equation for the  $A$ -nullcline. Setting the parameter values (so that  $\beta = 0.2$ ,  $\gamma = 0.5$ , and  $\xi = 0$ ), we obtain the phase diagram shown below. Note that the equilibrium identified in our time-path computations corresponds to the intersection of the nullclines at  $(A^* = 2.71, F^* = 3.68)$ . However, as indicated by the phase diagram, there is a second equilibrium at  $(A^* = 0, F^* = 0)$ . Intuitively, a group which initially has no friendliness and no activity would (in the absence of shocks or external activity) remain in that state indefinitely.

Of course, we could also have used algebra to derive the steady states of the model. Having assumed  $\xi = 0$ , we can substitute  $A = \beta F^2$  into  $F = \gamma A^2$  to obtain

$$F = \gamma \beta^2 F^4$$

Because this equation is satisfied when  $F = 0$ , there is a steady state at  $(A^* = 0, F^* = 0)$ . Alternatively, assuming that  $F$  is positive, we obtain

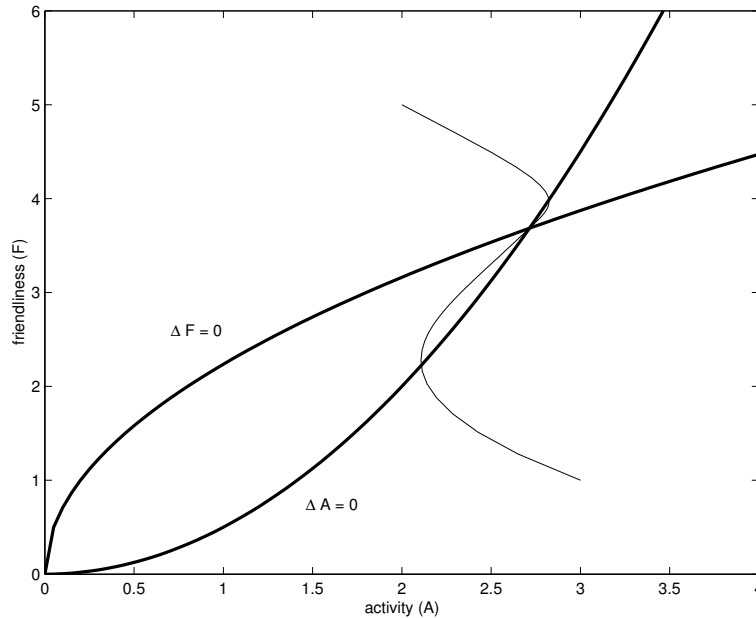
$$F = \gamma^{-1/3} \beta^{-2/3}$$

which, given our parameter assumptions  $\beta = 0.2$  and  $\gamma = 0.5$ , implies  $F^* = 3.68$  and hence a second steady state at  $(A^* = 2.71, F^* = 3.68)$ .

```

>> b = .2; g = .5; A = 0:.05:4; nullF = (A/b).^(1/2); nullA = g * A.^2;
>> plot(A,nullF,A,nullA,y1(:,1),y1(:,2),y2(:,1),y2(:,2))
>> % phase diagram with trajectories

```



In addition to the nullclines, our phase diagram also shows the trajectories corresponding to the two cases in Figure 1. Each *trajectory* is a sequence

$$\{(A_0, F_0), (A_1, F_1), (A_2, F_2), \dots\}$$

In contrast to time-path diagrams, in which each state variable is plotted as a function of time, the time dimension is merely implicit in the plot of a trajectory. Consequently, there is some loss of information in moving from the time-path diagram to a trajectory. While a trajectory shows how the process moves through the two-dimensional state space, it does not show how much time was required for the process to reach equilibrium.<sup>1</sup> However, offsetting this disadvantage, trajectories can be plotted directly on the phase diagram, making it easy to reconcile numerical examples with the graphical analysis of the system.

### 15.3 Graphical analysis of stability

Beyond identifying the steady states of the system, we can also use the phase diagram to help assess the stability of each steady state. The nullclines partition the phase

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<sup>1</sup>Using Matlab to plot trajectories, one can partially overcome this problem by adding markers at each point  $(A_t, F_t)$ . However, if we set period length  $h$  small to approximate continuous-time dynamics, the markers will be spaced closely together. Moreover, as the dynamics become slower in the neighborhood of an equilibrium, the markers begin to overlap, making it impossible to distinguish visually between time steps.

diagram into regions in which each state variable ( $u$  or  $v$ ) is either rising or falling. Once we determine the sign of  $\Delta u$  and  $\Delta v$  in each region (and notate our phase diagram accordingly), we may be able to determine whether the system would return to a particular steady state ( $u^*, v^*$ ) after a small shock.

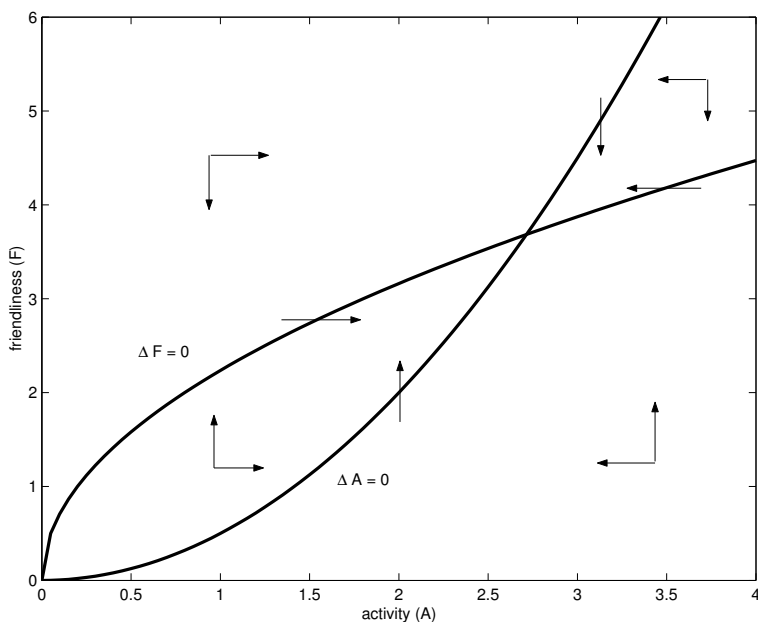
For our present example, note that

$$\Delta F > 0 \quad \text{implies} \quad F < \sqrt{A/\beta}$$

while  $\Delta F < 0$  implies  $F > \sqrt{A/\beta}$ . Graphically,  $F$  is rising at points below the  $F$ -nullcline, and falling at points above the  $F$ -nullcline. Note further that

$$\Delta A > 0 \quad \text{implies} \quad A < \sqrt{F/\gamma}$$

while  $\Delta A < 0$  implies  $A > \sqrt{F/\gamma}$ . Graphically,  $A$  is rising at points to the left of the  $A$ -nullcline, and falling at points to the right of the  $A$ -nullcline. Returning to our phase diagram, we now can add pairs of arrows to indicate the dynamics of each state variable in each of the four regions created by the nullclines.<sup>2</sup> In addition, we have also drawn arrows along each nullcline to indicate the dynamics of the other (non-stable) state variable. Note that all of the arrows are consistent with the trajectories plotted on the phase diagram above.



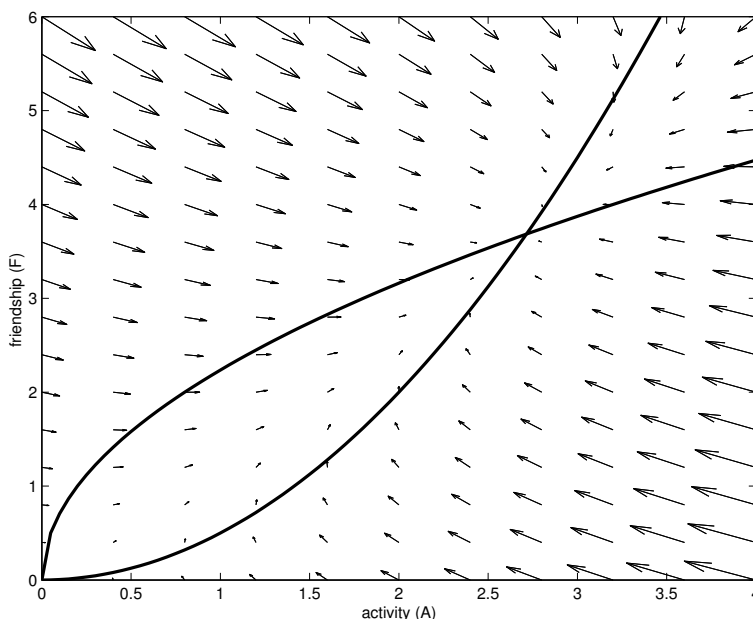
While these arrows merely provide some qualitative information about the dynamics of the system, we can often learn much from them. In particular, for our

<sup>2</sup>All arrows were inserted manually in the Matlab graphics editor. Following a slightly different convention, we might replace each pair of arrows (say one pointing “east” and one pointing “south”) with a single arrow (pointing “southeast”).

present example, this diagram reveals that the upper equilibrium ( $A^* = 2.71, F^* = 3.68$ ) is stable. This result follows essentially from the observation that all arrows are “pointing inward” toward this equilibrium.<sup>3</sup> We can also see that the lower equilibrium ( $A^* = 0, F^* = 0$ ) is unstable. Any small shock to the system would place it on a trajectory headed toward the upper equilibrium.

This sort of qualitative analysis is common and may be sufficient for many purposes. However, using Matlab, it is not difficult to add arrows which indicate more precisely the rate and direction of flow at each point on the phase diagram. We first create a grid in  $(A, F)$  space using the `meshgrid` command, then evaluate  $\Delta A$  and  $\Delta F$  at each point in this grid, then plot the result using the `quiver` command.<sup>4</sup> This type of diagram is called a *vectorfield*. Note that we have also superimposed the nullclines using the `hold on` command.

```
>> [A,F] = meshgrid(0:.4:4,0:.4:6);           % creating grid
>> dF = (A-b*F.^2)*h; dA = (F-g*A.^2)*h;     % evaluating dF and dA at each point
>> hold on; quiver(A,F,dA,dF)                 % plotting the vectorfield
>> A = 0:.05:4; nullF = (A/b).^(1/2); nullA = g*A.^2; plot(A,nullF,A,nullA)
>> % phase diagram with vectorfield
```



<sup>3</sup>Here is a more rigorous argument: Given the arrows on the nullclines, we see that each of the two regions between the nullclines is *trapping region* (i.e., any trajectory which enters the region will never exit). Further, we see that any trajectory entering either of the trapping regions must converge in the long-run to the upper equilibrium. Finally, note that any trajectory which starts in one of the other (non-trapping regions) must either approach the upper equilibrium directly (without entering another region) or else enter one of the trapping regions (resulting in long-run convergence to the upper equilibrium). Thus, all trajectories in the neighborhood of the upper equilibrium will eventually converge to that equilibrium.

<sup>4</sup>Readers are encouraged to consult the respective Matlab help files for more information.

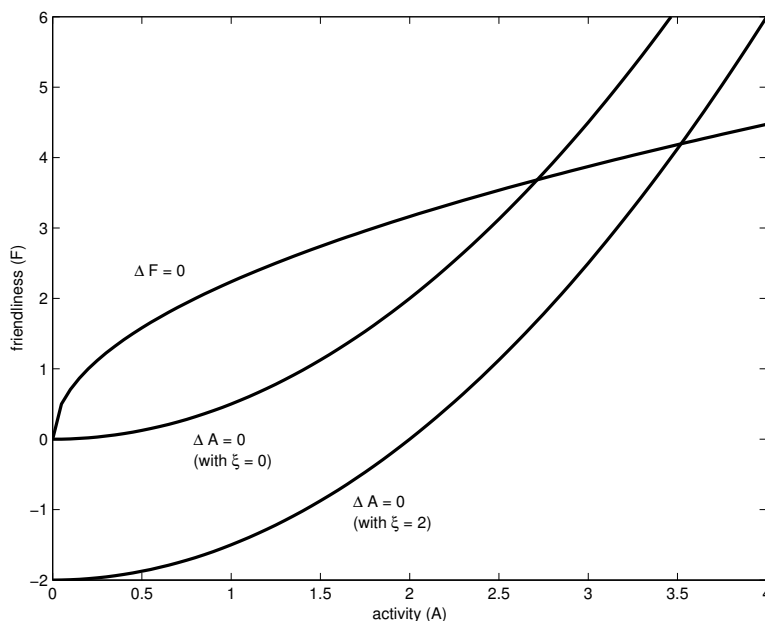
## 15.4 Other uses of the phase diagram

The phase diagram is also useful for conducting other sorts of analyses. To illustrate, we'll now consider the effect of the “external system” on the equilibrium of the “internal system.” Recalling the equation for  $\Delta F$ , it is apparent that the level of external activity (reflected by the parameter  $\xi$ ) has no effect on the  $F$ -nullcline. In contrast, given the equation for the  $A$ -nullcline,

$$F = \gamma A^2 - \xi$$

we see that an increase in  $\xi$  implies a lower level of  $F$  for any level of  $A$ . Graphically, the  $A$ -nullcline shifts downwards as  $\xi$  rises. (Equivalently, rewriting the equation for the  $A$ -nullcline so that  $A$  is a function of  $F$ , we would see that an increase in  $\xi$  implies a higher level of  $A$  for any of level  $F$ , resulting in a rightward shift of the  $A$ -nullcline.) The phase diagram below illustrates the shift in the  $A$ -nullcline as  $\xi$  rises from 0 to 2.

```
>> A = 0:.05:4; nullF = (A/b).^(1/2); nullA1 = g*A.^2; nullA2 = g*A.^2 - 2;
>> plot(A,nullF,A,nullA1,A,nullA2) % phase diagram with shift in A-nullcline
```



From this diagram, we see that an increase in external activity ( $\xi$ ) causes the equilibrium levels of both activity ( $A^*$ ) and friendliness ( $F^*$ ) to rise. Graphically, by shifting the  $A$ -nullcline, an increase in  $\xi$  causes the equilibrium to “move along” the  $N$ -nullcline. Of course, we didn’t need the phase diagram to obtain this result. Given that  $F^*$  is determined implicitly by the equation

$$F^* + \xi = \gamma \beta^2 F^{*4}$$



we could have solved this equation numerically to obtain the new equilibrium ( $A^* = 3.52, F^* = 4.20$ ). Nevertheless, the qualitative graphical analysis may be adequate for many purposes and is often convenient when the nullclines have complicated functional forms.

## 15.5 Stability analysis

In Chapter 11, we explained how to conduct stability analysis for one-dimensional nonlinear models. Here, we extend that method to the two-dimensional case. As before, we suppose that the system is initially in steady state ( $A^*, F^*$ ), but experiences a small, one-time shock in period  $t$ . We assume that this shock could affect either (or both) of the state variables. Thus, we may write

$$\begin{aligned} F_t &= F^* + f_t \\ A_t &= A^* + a_t \end{aligned}$$

where the lower-case variables  $f_t$  and  $a_t$  denote deviations from the steady state. Even with no subsequent shocks, the system will not return immediately to equilibrium. We may thus write

$$\begin{aligned} F_{t+1} &= F^* + f_{t+1} \\ A_{t+1} &= A^* + a_{t+1} \end{aligned}$$

where  $f_{t+1}$  and  $a_{t+1}$  are the deviations from equilibrium in period  $t + 1$ . Given the parallel to the one-dimensional case, you might anticipate that the steady state ( $A^*, F^*$ ) is stable if the derivations become smaller over time. This is essentially correct. However, because there are now two deviations (which could potentially change at different rates and even move in opposite directions), the formal equilibrium condition requires some further development.

We wish to determine how the deviations in period  $t + 1$  depend on the deviations in period  $t$ . To begin, we rewrite the  $F_{t+1}$  equation to obtain

$$\begin{aligned} F_{t+1} &= F_t + (A_t - \beta F_t^2) h \\ F^* + f_{t+1} &= F^* + f_t + (A^* + a_t - \beta(F^* + f_t)^2) h \\ F^* + f_{t+1} &= F^* + f_t + (A^* + a_t - \beta(F^{*2} + 2F^* f_t + f_t^2)) h \\ f_{t+1} &= f_t + (a_t - \beta(2F^* f_t + f_t^2)) h \\ f_{t+1} &\approx f_t + (a_t - 2\beta F^* f_t) h \end{aligned}$$

Note the parallel to our derivations in Chapter 11. The next-to-last line follows because  $F^*$  is an equilibrium; the final line follows because  $f_t^2$  is very small. Similarly, manipulation of the  $A_{t+1}$  equation yields

$$A_{t+1} = A_t + (F_t + \xi - \gamma A_t^2) h$$

$$\begin{aligned}
A^* + a_{t+1} &= A^* + a_t + (F^* + f_t + \xi - \gamma(A^* + a_t)^2) h \\
A^* + a_{t+1} &= A^* + a_t + (F^* + f_t + \xi - \gamma(A^{*2} + 2A^*a_t + a_t^2)) h \\
a_{t+1} &= a_t + (f_t - \gamma(2A^*a_t + a_t^2)) h \\
a_{t+1} &\approx a_t + (f_t - 2\gamma A^* a_t) h
\end{aligned}$$

We have thus obtained a two-equation system relating the deviations in period  $t + 1$  to the deviations in period  $t$ . This system can be written in matrix form as

$$\begin{bmatrix} f_{t+1} \\ a_{t+1} \end{bmatrix} = \begin{bmatrix} 1 - 2\beta h F^* & h \\ h & 1 - 2\gamma h A^* \end{bmatrix} \begin{bmatrix} f_t \\ a_t \end{bmatrix}$$

It is important to note that this system is linear. Indeed, it is customary to say that we have “linearized” the system at the fixed point  $(A^*, F^*)$ . For our example, linearity results from ignoring the small nonlinear (squared deviation) terms.

Generically, the equation relating the deviations in period  $t + 1$  to the deviations in period  $t$  can be written as

$$\mathbf{x}_{t+1} = J\mathbf{x}_t$$

where  $\mathbf{x}_t$  is a column vector reflecting deviations in period  $t$ , and  $J$  is called the *Jacobian* matrix. As indicated in our example, the Jacobian matrix not only depends on the parameters of the model, but must also be evaluated at a particular steady state. Drawing on our knowledge of linear systems from the first half of the book, the deviations in period  $t + s$  will be given by

$$\mathbf{x}_{t+s} = J^s \mathbf{x}_t$$

Moreover, we know that the stable-growth equilibrium is determined by the equation

$$\lambda \mathbf{x} = J\mathbf{x}$$

where  $\lambda$  is the largest (in absolute value) eigenvalue of the  $J$  matrix, and  $\mathbf{x}$  is the associated eigenvector. Recognizing that  $\lambda$  represents the long-run growth factor, the deviations become smaller over time when

$$|\lambda| < 1$$

and become larger over time when

$$|\lambda| > 1$$

Thus, stability of a steady state depends on the leading eigenvalue of the Jacobian matrix. The steady state is stable when this eigenvalue has absolute value less than 1, and is unstable when this eigenvalue has absolute value greater than 1.

To illustrate this approach to stability analysis, we return to our present example, fixing the parameter values  $\beta = 0.2$ ,  $\gamma = 0.5$ ,  $\xi = 0$ , and  $h = 0.1$ . We have already seen that these parameter values imply an upper equilibrium ( $A^* = 2.71$ ,  $F^* = 3.68$ ) and a lower equilibrium ( $A^* = 0$ ,  $F^* = 0$ ). Evaluating the Jacobian matrix at the upper equilibrium, we can use Matlab to compute the eigenvalues of this matrix.

```
>> b = .2; g = .5; e = 0; h = .1;
>> Astar = 2.71; Fstar = 3.68;
>> J = [1 - 2*b*h*Fstar, h; h, 1-2*g*h*Astar]
```

```
J =
    0.8528    0.1000
    0.1000    0.7290
```

```
>> [eigvec, eigval] = eig(J)
```

```
eigvec =
    0.4867   -0.8736
   -0.8736   -0.4867
```

```
eigval =
    0.6733     0
     0     0.9085
```

Because the leading eigenvalue of the Jacobian matrix is  $\lambda = 0.9085$ , we have verified that the upper equilibrium is stable. We can assess the stability of the lower equilibrium in a similar fashion.

```
>> Astar = 0; Fstar = 0;
>> J = [1 - 2*b*h*Fstar, h; h, 1-2*g*h*Astar]
```

```
J =
    1.0000    0.1000
    0.1000    1.0000
```

```
>> [eigvec, eigval] = eig(J)
```

```
eigvec =
   -0.7071    0.7071
    0.7071    0.7071
```

```
eigval =
    0.9000     0
     0     1.1000
```

In this case, the leading eigenvalue of the Jacobian matrix is  $\lambda = 1.1000$ , indicating instability of the lower equilibrium.

## 15.6 Another derivation of the Jacobian matrix

Readers who know calculus will be happy to learn that there is a more direct way to obtain the Jacobian matrix. The generic two-dimensional system

$$\begin{aligned}u_{t+1} &= g_1(u_t, v_t) \\v_{t+1} &= g_2(u_t, v_t)\end{aligned}$$

has the Jacobian matrix given by

$$J = \left[ \begin{array}{cc} \partial g_1 / \partial u & \partial g_1 / \partial v \\ \partial g_2 / \partial u & \partial g_2 / \partial v \end{array} \right]_{(u^*, v^*)}$$

where the subscript indicates that the partial derivatives are evaluated at the steady state  $(u^*, v^*)$ . For our present example, where

$$\begin{aligned}g_1(F, A) &= F + (A - \beta F^2) h \\g_2(F, A) &= A + (F + \xi - \gamma A^2) h\end{aligned}$$

you can verify that

$$\begin{aligned}\partial g_1 / \partial F &= 1 - 2\beta h F \\ \partial g_1 / \partial A &= h \\ \partial g_2 / \partial F &= h \\ \partial g_2 / \partial A &= 1 - 2\gamma h A\end{aligned}$$

Evaluating these partial derivatives at the steady state  $(A^*, F^*)$ , we thus obtain the Jacobian matrix derived earlier through our “non-calculus” approach.

Although we are focusing on two-dimensional models, it is useful to know that this calculus approach generalizes in a straightforward way for higher dimensional models (with  $n > 2$ ). (In contrast, because it becomes impossible to draw phase diagrams for  $n > 3$ , our graphical approach to stability analysis is possible only for low-dimensional models.) You might also note that our one-dimensional analysis was merely a special case of our present analysis. For the generic one-dimensional system

$$x_{t+1} = g(x_t)$$

the Jacobian “matrix” is simply

$$J = [\partial g / \partial x]_{x^*} = g'(x^*)$$

and  $g'(x^*)$  also represents the sole eigenvalue of the Jacobian matrix. Thus, even in the one-dimensional case, stability depends on the absolute value of the (only) eigenvalue of the Jacobian matrix.

## 15.7 The importance of period length

Throughout this chapter, we have set period length small ( $h = 0.1$ ) so that our discrete-time model approximates continuous-time dynamics. From our stability analysis, we can see that the dynamics change qualitatively when  $h$  is higher. For instance, setting  $h = 0.62$  (and leaving the other parameters unchanged), the following computations reveal that the upper equilibrium is no longer stable.

```
>> b = .2; g = .5; e = 0; h = .62;
>> Astar = 2.71; Fstar = 3.68; J = [1 - 2*b*h*Fstar, h; h, 1-2*g*h*Astar]

J =
    0.0874    0.6200
    0.6200   -0.6802

>> [eigvec, eigval] = eig(J)

eigvec =

   -0.4867   -0.8736
    0.8736   -0.4867

eigval =

   -1.0256    0
    0         0.4327
```

For this case, the behavior of the system is illustrated by the time-path and phase diagrams on the next page. Further computations show that the system eventually converges to a 2-cycle in which  $(A, F)$  alternates between  $(2.22, 3.89)$  and  $(3.11, 3.39)$ . The reader can verify that longer period lengths (say  $h = 1$ ) result in even “wilder” behavior (with the trajectory leaving the positive quadrant, and the dynamics never converging to an equilibrium).<sup>5</sup>

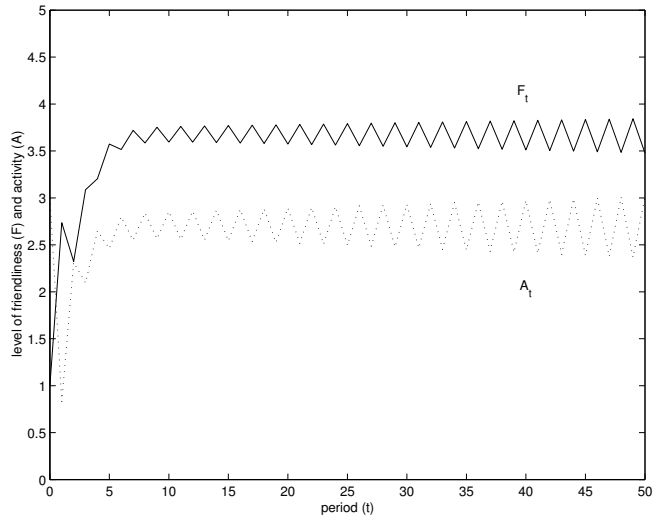
The point of this example is to instill some caution about the use of phase diagrams in the analysis of discrete-time systems. In section 15.3, we drew arrows to indicate the direction of change for each state variable in each region of the phase diagram. These arrows correctly indicate the direction of change for both discrete-time and continuous-time systems. However, our graphical analysis of stability is valid for discrete-time models only if period length is short enough to approximate continuous-time dynamics. For continuous-time systems, the trajectory flows smoothly through the state space. In contrast, for discrete-time systems, we have just seen how the trajectory may “jump” over nullclines and even entire regions of the phase diagram.<sup>6</sup>

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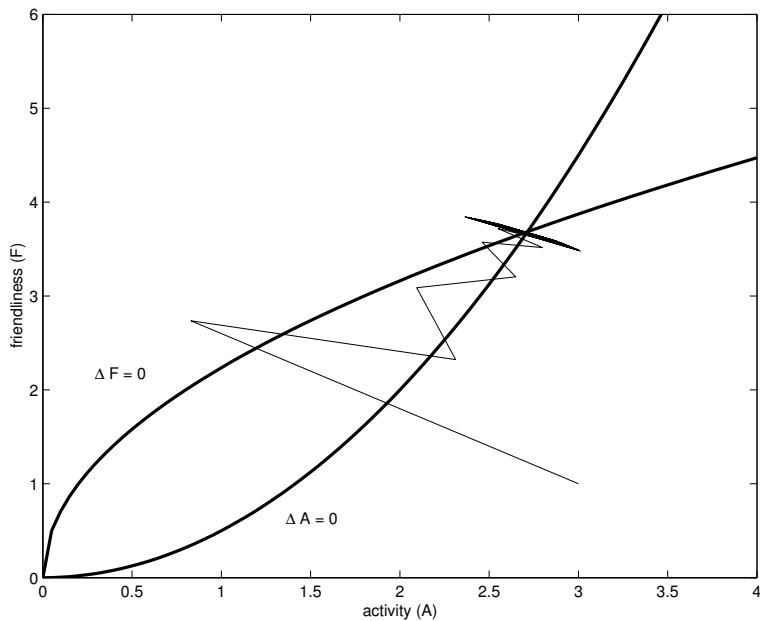
<sup>5</sup>These results might well prompt some reflection on our specification of the model. Should the model permit negative levels of friendliness or activity? If so, how should the dynamics be specified in those regions of the state space?

<sup>6</sup>For one-dimensional models, we could use the cobweb diagram to help visualize this sort of behavior. (Recall our graphical analysis of the 2-cycle in Chapter 11.) Unfortunately, for two-

```
>> A = 3; F = 1; y = [A F]; for t = 1:50; dF = (A - b*F^2)*h; dA = (F - g*A^2)*h;
F = F+dF; A = A+dA; y = [y; A F]; end; plot(0:50,y(:,1),0:50,y(:,2));
% time paths for h = .62
```



```
>> A = 0:.05:4; nullF = (A/b).^(1/2); nullA = g * A.^2;
plot(A,nullF,A,nullA,y(:,1),y(:,2)); % trajectory for h = .62
```



Nevertheless, for many applications, we are in fact interested in the behavior of the system when period length is short. For instance, in the context of our present example, it seems reasonable to assume that the group dynamics would be continually (or at least frequently) modified in light of the current levels of friendliness and activity. Thus, while we will continue to specify models in discrete time (to facilitate computation and reduce the need for calculus), we will also continue to make (judicious) use of continuous-time methods.

## 15.8 Recommended readings

George Homan's book, *The Human Group* (1950 [1992]), is a sociological classic which helped inspire not only group-process theory but also social network analysis. Herbert Simon's formalization was based primarily on Chapter 5. Simon's paper originally appeared in the *American Sociological Review* (1952) and was subsequently reprinted in his book, *Models of Man* (1957). See Thomas Fararo, *The Meaning of General Theoretical Sociology* (1992), for another treatment of the Simon-Homans model and introduction to sociological applications of dynamical systems. See Drazin (1992) or Strogatz (1994) for more rigorous development of stability conditions for both discrete-time and continuous-time systems.

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dimensional models, the analog of the cobweb diagram would need to be four-dimensional (given the two inputs and two outputs of the generator function).