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16 Residential Segregation

The racial and ethnic composition of a neighborhood may change over time. The same "ethnic ghetto" might be occupied by a succession of different groups over several generations. Alternatively, a neighborhood may experience more rapid transition from predominantly white to predominantly minority (a process sometimes colloquially called "white flight") or the reverse (as in the "gentrification" of urban neighborhoods). In this chapter, we examine a classic model of residential choice originally developed by Thomas Schelling (1971, 1978). In the model, there are two groups, each characterized by the distribution of "tolerance levels" among its members. Given the numbers of each group living in the neighborhood in the current period, the tolerance distributions determine the numbers of each group for the next period. A key question is whether both groups will be present in the neighborhood in the long run. Formally, the model is a two-dimensional generalization of the one-dimensional threshold models we have already studied in Chapters 12 and 13.

16.1 The model

The model assumes two different racial or ethnic groups. For concreteness, we label the groups "black" and "white." Focusing attention on a single neighborhood, let B_t and W_t denote the number of blacks and whites living in this neighborhood in period t.[1] Because the numbers of each group in period t will determine the numbers of each group in period $t+1$, the model is a two-equation system

$$\begin{aligned} W_{t+1} &= g_1(W_t, B_t) \\ B_{t+1} &= g_2(W_t, B_t) \end{aligned}$$

where the functional forms of g_1 and g_2 will be developed below. To develop the precise specification of the model, we begin with the first equation (governing the number of whites in the neighborhood). The second equation (governing the number of blacks) can then be formulated in an analogous way.

Each white individual i is characterized by a tolerance level $\theta(i)$ indicating the highest black-to-white ratio that she is willing to accept. Following our earlier specification of one-dimensional threshold models, we assume that individuals respond myopically to the current state of the system. Thus, white individual i chooses to live in the neighborhood in period $t+1$ when

$$\theta(i) \geq B_t/W_t$$

[1]For simplicity, we treat B_t and W_t as continuous variables (that need not take integer values). It would thus be more precise (but less conventional) to refer to these variables as population "masses" rather than "numbers."

and chooses to live outside the neighborhood when this inequality is reversed.[2] Given variation in tolerance levels across individuals, let $F_W(x)$ denote the proportion of whites with tolerance levels less than or equal to x. The whites living in the neighborhood in period $t+1$ are those with tolerance levels above B_t/W_t. Thus, the proportion of whites choosing to live in the neighborhood in period $t+1$ is

$$1 - F_W(B_t/W_t)$$

To obtain the *number* of whites living in the neighborhood, we multiply this proportion by the total number of whites who could potentially in the neighborhood. We thus obtain the first equation of our two-equation system,

$$W_{t+1} \;=\; N_W \; [1 - F_W(B_t/W_t)]$$

where N_W is a scalar denoting the total number of whites.

The second equation is developed in similar fashion. Black tolerance levels are specified relative to the white-to-black ratio. Letting $F_B(x)$ denote the proportion of blacks with tolerance levels less than or equal to x, the number of blacks living in the neighbhorhood in period $t+1$ is

$$B_{t+1} \;=\; N_B \; [1 - F_B(W_t/B_t)]$$

where N_B is the total number of blacks who could potentially live in the neighborhood. The two-equation system is thus parameterized by the total numbers of whites and blacks (N_W and N_B) and the distribution of tolerance levels for each group ($F_W(x)$ and $F_B(x)$).

This specification of the system presumes that everyone can move immediately in response to current conditions. However, it may be more realistic to assume that changes in population levels are proportional to period length. Incorporating this idea, our two-equation system can be respecified as

$$\begin{aligned} \Delta W &= (N_W \; [1 - F_W(B/W)] - W) \; h \\ \Delta B &= (N_B \; [1 - F_B(W/B)] - B) \; h \end{aligned}$$

where the parameter h reflects period length. Given this specification, the term $N_W[1 - F_W(B/W)]$ reflects the number of whites who would *prefer* to live in the neighborhood. The difference $(N_W[1 - F_W(B/W)] - W)$ can thus be interpreted as "excess demand" among whites for living in the neighborhood. The second equation can be given an analogous interpretation. While period length obviously affects the (per-period) speed of the dynamics, note that h has no effect on the nullclines $\Delta W = 0$ and $\Delta B = 0$, and hence no effect on the steady states of the system.

[2] Note that an individual chooses to live in the neighborhood when she has a *high* tolerance level. This contrasts with our specification of one-dimensional threshold models, in which an individual participated in the collective action when she had a *low* threshold level.

16.2 A numerical example

To develop a numerical example, we need to fix the tolerance distribution for each group, along with the number of blacks and whites that could potentially live in the neighborhood.[3] We'll initially suppose that white tolerance levels are distributed uniformly between 0 and 2. This implies that the least tolerant white would prefer to move if any blacks entered the neighborhood, the most tolerant white would prefer to stay unless there were more than 2 blacks per white, and the remaining whites are "spread evenly" between these extremes. More formally, the distribution of white tolerance levels is characterized by the probability density function

$$f_W(\theta) \;=\; \begin{cases} 1/2 & \text{for } \theta \in [0,2] \\ 0 & \text{otherwise} \end{cases}$$

and the associated cumulative distribution function is given by

$$F_W(x) \;=\; \int_0^x f_W(\theta)\; d\theta \;=\; \begin{cases} (1/2)x & \text{for } x \in [0,2] \\ 1 & \text{for } x > 2 \end{cases}$$

We will further assume that there are $N_W = 100$ whites who could potentially live in the neighborhood. The first equation thus becomes

$$\begin{aligned} \Delta W \;&=\; \begin{cases} (100\;[1-(1/2)(B/W)] - W)\; h & \text{for } B/W \le 2 \\ (100\;[1-1] - W)\; h & \text{for } B/W > 2 \end{cases} \\ &=\; \begin{cases} (100 - 50(B/W) - W)\; h & \text{for } B/W \le 2 \\ -Wh & \text{for } B/W > 2 \end{cases} \end{aligned}$$

We assume that blacks have the same distribution of tolerance levels as whites (so that $F_B(x) = F_W(x)$), but that there are only $N_B = 50$ blacks who could potentially live in the neighborhood. The second equation thus becomes

$$\begin{aligned} \Delta B \;&=\; \begin{cases} (50\;[1-(1/2)(W/B)] - B)\; h & \text{for } W/B \le 2 \\ (50\;[1-1] - B)\; h & \text{for } W/B > 2 \end{cases} \\ &=\; \begin{cases} (50 - 25(W/B) - B)\; h & \text{for } W/B \le 2 \\ -B\; h & \text{for } W/B > 2 \end{cases} \end{aligned}$$

Using these equations, we can begin to analyze the model graphically by deriving the nullclines. From the first equation, we find that $\Delta W = 0$ implies $W = 0$ or

$$B \;=\; W(100 - W)/50$$

Thus, there two W-nullclines. One W-nullcline (at $W = 0$) follows the B axis. Intuitively, if there are no whites in the neighborhood, then the black-to-white ratio

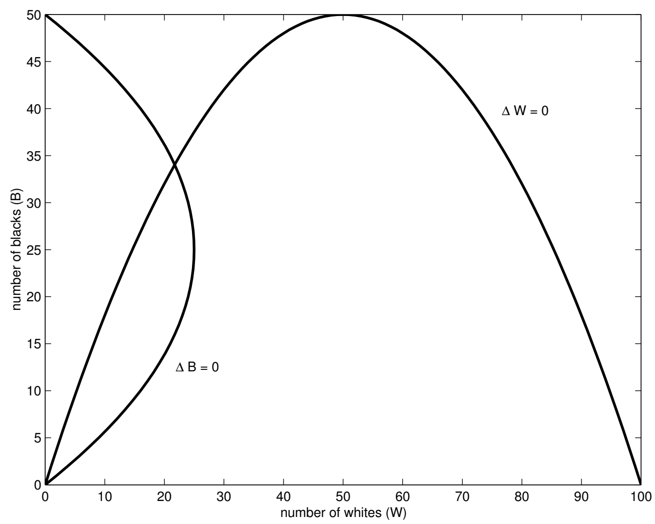
[3] Throughout the remainder of this chapter, our numerical examples closely follow those presented (in a less formal manner) in Schelling, *Micromotives and Macrobehavior*, Chapter 4.

is infinite, and no whites wish to enter. The other W-nullcline is given by the preceding (quadratic) equation. Similarly, we see from the second equation that $\Delta B = 0$ implies $B = 0$ or

$$W = B(50 - B)/25$$

Thus, there are also two B-nullclines. One follows the W axis, indicating that, if there are no blacks in the neighborhood, then none wish to enter. The other B-nullcline is again given by a quadratic equation. Plotting the quadratic nullclines, we obtain the phase diagram below.

```
>> W = 0:100; nullW = W.*(100-W)./50; B = 0:50; nullB = B.*(50-B)./25;
plot(W,nullW,nullB,B)   % plotting the nullclines
```



Recognizing that the vertical axis is also a W-nullcline, and that the horizontal axis is also a B-nullcline, this diagram reveals four steady states. Three lie along the edge of the diagram at $(W^* = 100, B^* = 0)$, $(W^* = 0, B^* = 50)$, and $(W^* = 0, B^* = 0)$.[4] The final steady is determined by the intersection of the quadratic nullclines at $(W^* = 21.74, B^* = 34.03)$.

Having identified two "segregated" equilibria (one all-white and the other all-black) and one "integrated" equilibrium (with both whites and blacks), the interesting question is which of these equilibria are stable. Following the procedure discussed in Chapter 15, we could assess stability graphically by determining the sign of ΔW and ΔB for each region of the phase diagram. (This was the approach originally

[4]We have been skirting some technical issues that arise because the ratios B/W and W/B are undefined at the origin. In a more rigorous treatment of this model, Dokumaci and Sandholm (2009) address this issue by assuming that small masses of blacks and whites have infinite tolerances and are thus always present in the neighborhood. However, because the steady state at the origin is unstable (see below) and perhaps less relevant empirically, we will (following Schelling's original presentation) continue simply to gloss over the technicalities.

taken by Schelling (1971, 1978) and remains an excellent exercise for readers.) But here, we'll simply use Matlab to plot the vectorfield. To do so, it will be helpful to write our two-equation system in another form. Note that the cdfs can be written as

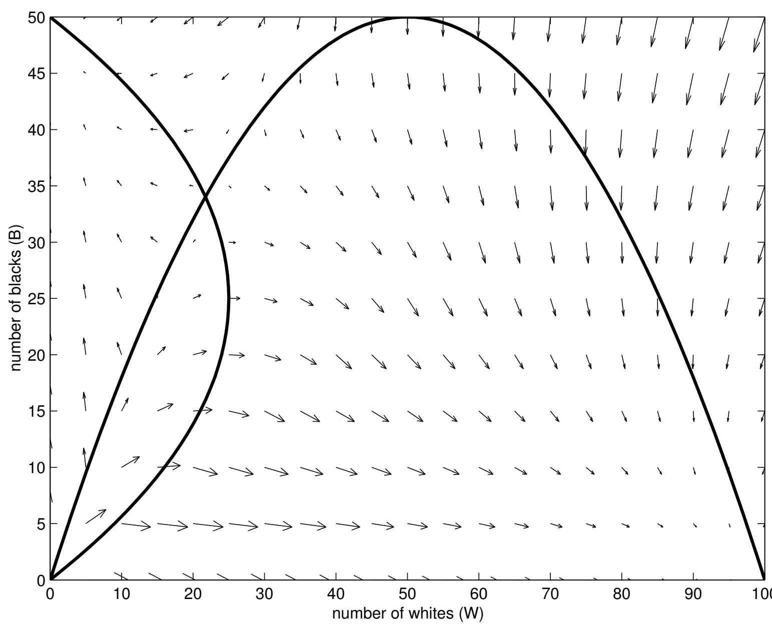
$$F_W(x) \;=\; F_B(x) \;= \min\{(1/2)x, 1\}$$

where $\min\{a, b\}$ denotes the minimum of the scalars a and b. Thus, the two equations can be written as

$$\begin{aligned} \Delta W &= (100\;[1 - \min\{(1/2)(B/W), 1\}] - W)\; h \\ \Delta B &= (50\;[1 - \min\{(1/2)(W/B), 1\}] - B)\; h \end{aligned}$$

Using this specification, we first compute the vectorfield, and then superimpose it on the phase diagram.

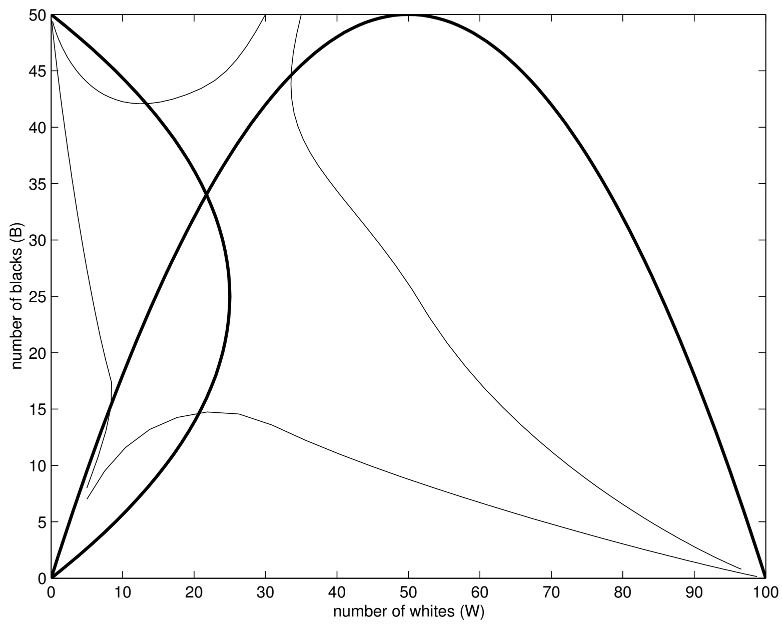
```
>> [W,B] = meshgrid(0:5:100,0:5:50);
dW = (100*(1-min((1/2)*(B./W),1))-W); dB = (50*(1-min((1/2)*(W./B),1))-B);
hold on; quiver(W,B,dW,dB);
W = 0:100; nullW = W.*(100-W)./50; B = 0:50; nullB = B.*(50-B)./25;
plot(W,nullW,nullB,B); %   phase diagram with vectorfield
```



This diagram shows that the two segregated equilibria are stable. Trajectories in the upper left corner of the phase diagram (where B/W is relatively large) will flow toward the all-black equilibrium, while trajectories in the remainder of the diagram will flow toward the all-white equilibrium. In contrast, the interior integrated equilibrium is unstable.

From the phase diagram, we can also see that small differences in initial conditions can potentially lead to very different long-run outcomes. To illustrate further, we plot below the trajectories for several initial conditions.

```
>> W = [5 5 30 35]; B = [8 7 50 50]; h = .1; yW = W; yB = B;
for t = 1:50;
    dW = (100*(1-min((1/2)*(B./W),1))-W)*h; dB = (50*(1-min((1/2)*(W./B),1))-B)*h;
    W = W+dW; B = B+dB; yW = [yW; W]; yB = [yB; B];
end
>> plot(yW(:,1),yB(:,1),yW(:,2),yB(:,2),yW(:,3),yB(:,3),yW(:,4),yB(:,4),
0:100,nullW,nullB,0:50) %   phase diagram with trajectories
```



Note that the initial condition ($W_0 = 5, B_0 = 8$) lies on a trajectory that "veers upwards" toward the all-black equilibrium, with the nearby point ($W_0 = 5, B_0 = 7$) lies on a trajectory that "veers rightwards" toward the all-white equilibrium. Similarly, while the initial conditions ($W_0 = 30, B_0 = 50$) and ($W_0 = 35, B_0 = 50$) are close, they lie on trajectories leading to very different long-run outcomes.

16.3 Connection to one-dimensional threshold models

We have already indicated that Schelling's model can be viewed as a two-dimensional generalization of the one-dimensional threshold models we encounted in previous chapters. To better understand this connection, suppose that the number of blacks in the neighborhood is fixed at B while the number of whites follows the dynamics specified previously. Our two-dimensional model is thus reduced to the one-equation system given by

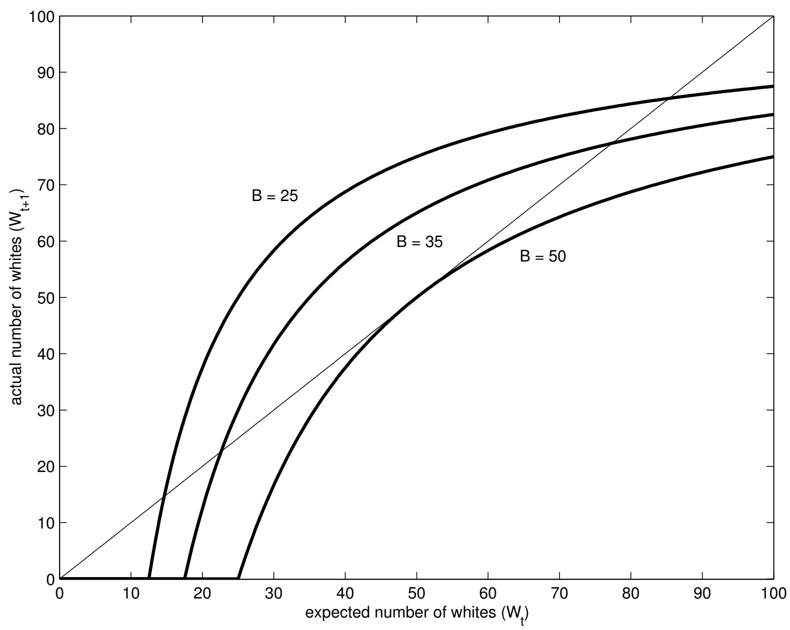
$$\Delta W \;=\; (N_W\;[1 - F_W(B/W)] - W)\;h$$

where B is now viewed as a (fixed) parameter. Setting $h = 1$, and given the parameter values from our preceding example, this equation can be rewritten as

$$W_{t+1} \;=\; 100\,[1 - \min\{(1/2)(B/W_t), 1\}]$$

We plot this function for several values of B to obtain the threshold diagram below.

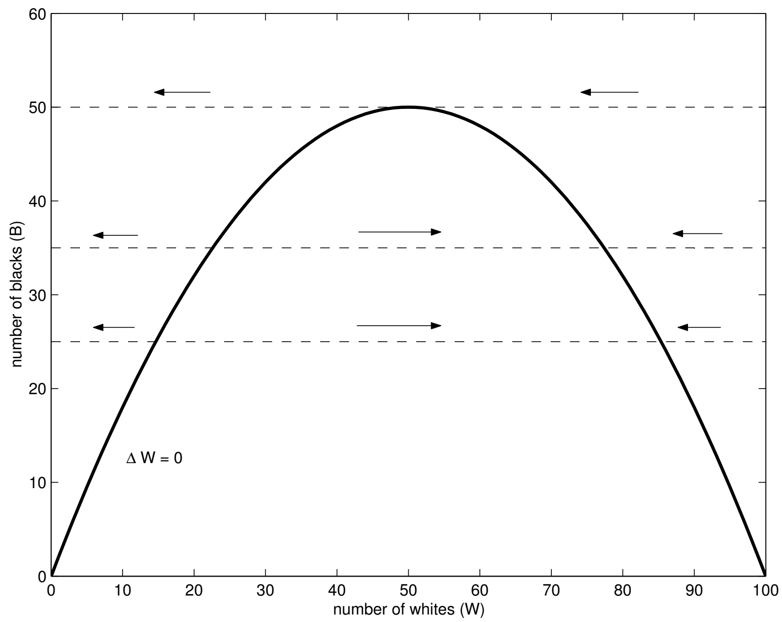
```
>> W = 0:.5:100; plot(W, 100*(1-min((1/2)*(25./W),1)), W, 100*(1-min((1/2)*(35./W),1)),
W, 100*(1-min((1/2)*(50./W),1)),W,W) %   threshold curves
```



Adopting the terminology used in Chapter 12, this diagram depicts three threshold curves. Each curve shows the relationship between the expected number of whites (W_t) and the actual number of whites (W_{t+1}) for some (fixed) value of B. Recall that steady states are determined by intersections of the threshold curve and the 45-degree line, and are stable when the threshold curve is rising and crosses the 45-degree line from above. For instance, focusing on the threshold curve for $B = 25$, we see that there are two stable equilibria (at $W^* = 0$ and $W^* = 84.36$), and an intermediate unstable equilibrium (at $W^* = 14.64$). Further recall that that the number of whites is rising when the threshold curve is above the 45-degree line (and falling when the threshold curve is below this line). Thus, given $B = 25$, the number of whites is rising for intermediate values (i.e., for $W_t \in (14.64, 85.36)$) but falling for lower or higher values (i.e., $W_t \in (0, 14.64)$ or (84.36, 100)).

Comparing the threshold curves, we see that the threshold curve shifts downwards as B rises. Consequently, as B rises from 25 to 35, the stable upper equilibrium falls (from 84.36 to 77.39) while the intermediate unstable equilibrium rises (from 14.64 to 22.61). We can further see that a catastrophe occurs at $B = 50$. For $B > 50$, the threshold curve lies everywhere below the 45-degree line. Consequently, the number of whites is falling (for any positive W_t), and there is a unique, stable equilibrium at $W^* = 0$.

How can we reconcile the threshold curves above with the phase diagrams in the preceding section? Loosely, each threshold curve corresponds to a one-dimensional "slice" through the W-nullcline. These "slices" are indicated by the horizontal dotted lines in the diagram below.



Following along the dotted line at $B = 25$, the intersections with the W-nullcline indicate the steady states at $W^* = 14.64$ and $W^* = 84.36$. The third steady state at $W^* = 0$ becomes apparent when we recall that the B axis is also a W-nullcline. Similarly, following along the dotted line at $B = 35$, we find the steady states at $W^* = 0$, $W^* = 22.61$, and $W^* = 77.39$. Note that our threshold diagram analysis also helps motivate the sign of ΔW in the regions above and below the W-nullcline. Again fixing $B = 25$, we know from the threshold diagram that the number of whites is rising for $W_t \in (14.64, 85.36)$, corresponding to points below the nullcline. Conversely, fixing $B = 25$, we know that the number of whites is falling for $W_t \in$ (0, 14.64) or (84.36, 100), corresponding to points above the nullcline.

Having considered the ΔW equation when B is fixed, readers might undertake a similar analysis of the ΔB equation when W is fixed. The threshold curves (indicating the relationship between B_t and B_{t+1}) would again correspond to one-dimensional "slices" through the phase diagram, though they would now be represented by vertical lines (for fixed values of W) rather than horizontal lines (for fixed values of B). Combining the results of these analyses, you can obtain the sign of ΔW and ΔB for each region of the phase diagram (and can check your results against the vectorfield plotted above).

16.4 Additional examples

For the example we have just considered, the integrated equilibrium is unstable, and the neighborhood is always occupied by a single group in the long run. Empirically, this result is quite relevant. Many Americans do, in fact, live in neighborhoods that are highly segregated by race (see, e.g., Massey and Denton 1998). However, as a theoretical matter, the existence and stability of integrated equilibria depend crucially on the parameters of the model. In this section, we develop some additional examples to illustrate the variety of equilibrium outcomes.

Intuition might suggest that long-run integration becomes possible when members of both groups are more tolerant. To explore this possibility, we now suppose that tolerance levels in both groups are now distributed uniformly between 0 and 5. Thus, for each group, the most tolerant member stays in the neighborhood unless her group is outnumbered by a 5-to-1 ratio. More formally, we assume that the cdfs are

$$F_W(x) \;=\; F_B(x) \;=\; \begin{cases} (1/5)x & \text{for } x \in [0,5] \\ 1 & \text{for } x > 5 \end{cases} \;=\; \min\{(1/5)x, 1\}$$

Further assuming the same number of whites and blacks who could potentially live in the neighborhood (with $N_W = N_B = 100$), our two equations become

$$\Delta W \;=\; \begin{cases} (100 - 20(B/W) - W)\; h & \text{for } B/W \leq 5 \\ -Wh & \text{for } B/W > 5 \end{cases}$$

$$\Delta B \;=\; \begin{cases} (100 - 20(W/B) - B)\; h & \text{for } W/B \leq 5 \\ -Bh & \text{for } W/B > 5 \end{cases}$$

To obtain the nullclines, note that $\Delta W = 0$ if $W = 0$ or

$$B \;=\; (100 - W)W/20$$

Similarly, $\Delta B = 0$ if $B = 0$ or

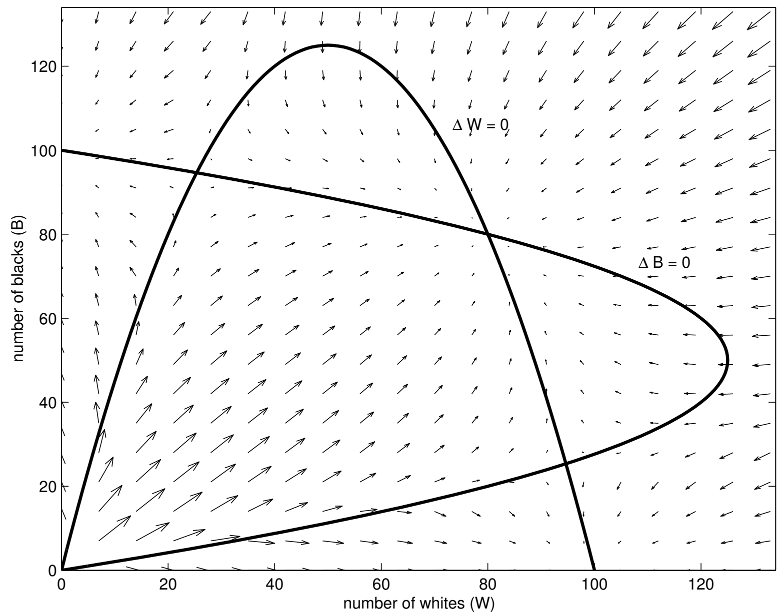
$$W \;=\; (100 - B)B/20$$

To compute the vectorfield, note that the two equations can be rewritten as

$$\begin{aligned} \Delta W &= (100\;[1 - \min\{(1/5)(B/W), 1\}] - W)\; h \\ \Delta B &= (100\;[1 - \min\{(1/5)(W/B), 1\}] - B)\; h \end{aligned}$$

Plotting the nullclines and vectorfield, we obtain the phase diagram below.

```
>> [W,B] = meshgrid(0:7:134,0:7:134);
dW = (100*(1-min((1/5)*(B./W),1))-W); dB = (100*(1-min((1/5)*(W./B),1))-B);
hold on; quiver(W,B,dW,dB);
W = 0:100; nullW = W.*(100-W)/20; B = 0:100; nullB = B.*(100-B)/20;
plot(W,nullW,nullB,B) %   phase diagram
```



This diagram indicates stable steady states at

$$(W^* = 0, B^* = 100), (W^* = 100, B^* = 0), \text{and } (W^* = 80, B^* = 80)$$

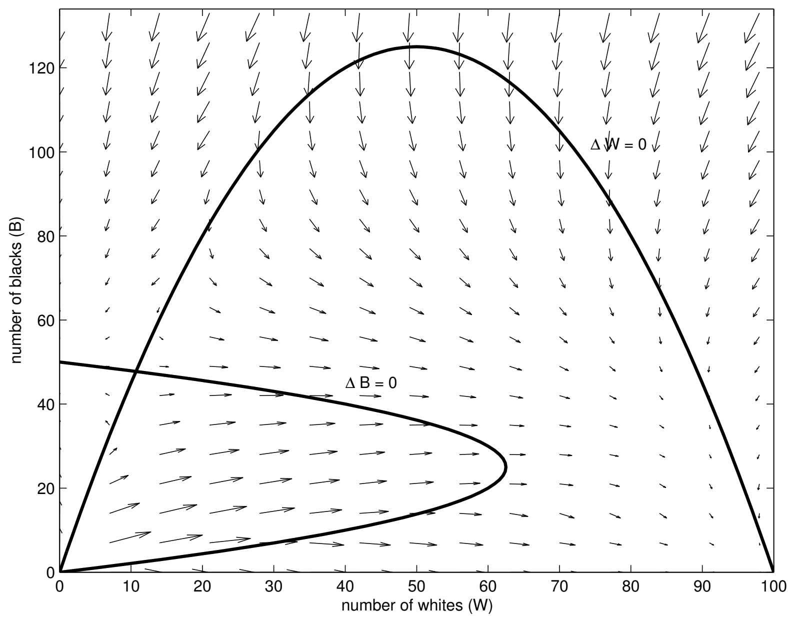
and unstable steady states at

$$(W^* = 25.36, B^* = 94.64), (W^* = 94.64, B^* = 25.36), \text{and } (W^* = 0, B^* = 0)$$

We thus find that increased tolerance does make possible a stable, integrated equilibrium. However, the long-run outcome depends on initial conditions. If the initial B/W ratio is very low or very high, the neighborhood will eventually be occupied by a single group.

Moreover, further examples reveal that increased tolerance is not sufficient for the existence of a stable, integrated equilibrium. To illustrate, we retain the same tolerance distributions (so that tolerance is again distributed uniformly between 0 and 5 for both groups) but now assume twice as many whites as blacks (with $N_W = 100$ and $N_B = 50$). Revising the equations for the nullclines and dynamics given above, we again plot the nullclines and vectorfield.

```
>> [W,B] = meshgrid(0:7:134,0:7:134);
dW = (100*(1-min((1/5)*(B./W),1))-W); dB = (50*(1-min((1/5)*(W./B),1))-B);
hold on; quiver(W,B,dW,dB);
W = 0:100; nullW = W.*(100-W)/20; B = 0:100; nullB = B.*(50-B)/10;
plot(W,nullW,nullB,B) %   phase diagram
```



Qualitatively, this example more closely resembles our first example from section 16.2. The segregated equilibria at $(W^* = 0, B^* = 50)$ and $(W^* = 100, B^* = 0)$ are stable, while the integrated equilibrium at $(W^* = 10.70, B^* = 47.76)$ is unstable. Perhaps ironically, the possibility of a stable intergrated equilibrium is undermined by the presence of "too many" tolerant whites. If the neighborhood is initially occupied by a small number of whites and blacks (say $W_0 = B_0 = 10$), then the numbers of both groups will initially rise. However, as the number of white continues to rise, the number of blacks will eventually begin to fall. (Graphically, this is the point at which the trajectory crosses the B-nullcline.) Following the suggestion by Schelling (1978, pp 162-4), we might attempt to promote integration by limiting the number of whites living in the neighborhood.

16.5 Further reading

Schelling's 1971 article in the *Journal of Mathematical Sociology* presented both the "bounded neighborhood" model discussed in this chapter along with a second "self-forming neighborhood" model in which individuals occupy positions on a "checker-board" and move to new positions if there are too many neighbors of the other racial type. Both of these classic models were also discussed in Chapter 4 of Schelling's 1978 book *Micromotives and Macrobehavior*. Further exposition of the bounded neighborhood model is provided by Granovetter and Soong, *Sociological Methodology*, 1988. For a more rigorous treatment, see Emin Dokumaci and William H. Sandholm, "Schelling Redux: An Evolutionary Dynamic Model of Residential Segregation," University of Wisconsin, Department of Economics, unpublished working paper. Becker and Murphy (*Social Economics*, 2000, Chap 5) extend the model to include housing prices. For empirical work on residential segregation, see Massey and Denton, *American Apartheid*, 1998.

Schelling's model could potentially be applied to many topics beyond residential segregation. In one interesting (informal) application, Lieberson (*Am J Soc* 2000) discussed the process by which first names (the analog to neighborhoods) which are initially used primarily by boys (the analog to one racial group) can eventually become used primarily by girls (the analog to the other racial group).