Answer all 5 questions
Time allowed: 2 hours

1. Show that if $f$ is differentiable on an interval with $f'(x) \neq 1$, then $f$ can have at most one fixed point.

2. Show that the following sequence is bounded

$$x_n = \left(1 + \frac{1}{n}\right)^n, n = 1, 2, \ldots$$

3. Show that if the function $f : \mathbb{R} \to \mathbb{R}_{++}$ is continuous on an interval $[a, b]$, then the reciprocal of this function $(\frac{1}{f})$ is bounded on this same interval.

4. Suppose

$$A = \{f : \mathbb{R} \to \mathbb{R}, f \text{ concave}, f(1) = 1, f(3) = 5, f(4) = 6\}$$

Solve the following equations

$$\sup \{f(2) \mid f \in A\} = u$$
$$\inf \{f(2) \mid f \in A\} = v$$

5. A consumer has an income of $300 per week, which is spent entirely on two goods, food ($f$), measured in pounds, and gas ($g$), measured in gallons. The consumer’s utility function is

$$u(f, g) = \sqrt{\frac{f}{100}} + \log(g)$$

(a) If the price of food is $1 per pound, and the price of gas is $4 per gallon, what is the optimal (i.e. utility-maximizing) consumption plan?

(b) If the price of gas rises to $10 per gallon, does the consumer buy more food?