Slutsky equation

The basic consumer model is $\max_{x:p\cdot x\leq y} U(x)$, which is solved by the Marshallian demand function $X(p,y)$.

The value of this primal problem is the indirect utility function $V(p,y) = U(X(p,y))$.

The dual problem is $\min_x \{p\cdot x \mid U(x) = u\}$, which is solved by the Hicksian demand function $h(p,u)$.

The value of the dual problem is the expenditure or cost function $e(p,u)) = p\cdot h(p,u)$.

First show that the Hicksian demand function is the derivative of the expenditure function.

$$\begin{aligned} e(p,u) = p\cdot h(p,u) \leq & p\cdot h(p+\Delta p,u) \\ e(p+\Delta p,u) = (p+\Delta p)\cdot h(p+\Delta p,u) \leq & (p+\Delta p)\cdot h(p,u) \end{aligned}$$

Add these:

$$p(p,u) + (p+\Delta p)\cdot h(p+\Delta p,u) \leq p(p+\Delta p,u) + (p+\Delta p)\cdot h(p,u)$$

Cancel terms:

$$\Delta p\cdot h(p+\Delta p,u) \leq \Delta p\cdot h(p,u)$$

$$\Delta p\cdot \Delta h \quad \leq \quad 0$$

meaning that the substitution effect is negative.

$$\Delta e = (p+\Delta p)\cdot h(p+\Delta p,u) - p\cdot h(p,u)$$

Replace the first term by something bigger:

$$\Delta e \leq (p+\Delta p)\cdot h(p,u) - p\cdot h(p,u) = \Delta p\cdot h(p,u)$$

and replace the second term by something bigger:

$$\Delta e \geq (p+\Delta p)\cdot h(p+\Delta p,u) - p\cdot h(p+\Delta p,u) = \Delta p\cdot h(p+\Delta p,u)$$

Thus

$$\Delta p\cdot h(p+\Delta p,u) \leq \Delta e \leq \Delta p\cdot h(p,u)$$

For a positive change in a single price p_i this gives

$$h_i(p+\Delta p,u) \leq \frac{\Delta e}{\Delta p_i} \leq h_i(p,u)$$

In the limit, this shows that the derivative of the expenditure function is the Hicksian demand function (and Shephard's Lemma is the exact same result, for cost minimization by the firm).

$$\frac{\partial e}{\partial p_i} = h_i(p,u)$$

Also, the expenditure function lies everywhere below its tangent with respect to p: that is, it is concave in p:

$$e\left(p+\Delta p, u\right) \leq (p+\Delta p) \cdot h(p, u) = e(p, u) + \Delta p \cdot h(p, u)$$

The usual analysis assumes that the consumer starts with no physical endowment, just money. In the case of leisure, this must obviously be modified. But the modification is simple: just change the origin. Suppose the consumer is endowed with a bundle g. Define $\tilde{x} = x - g$, and do everything in terms of $\tilde{x}$. This has the nice effect of covering security levels at the same time: think of these as negative endowments in the sense that the consumer must buy certain quantities before doing anything else (this is the Stone-Geary specification).

Define the utility function $\tilde{U}\left(\tilde{x}\right) = U(\tilde{x} + g)$. This doesn't change anything – it is just an alternative way to describe the original preference ordering over consumption and leisure bundles. The budget constraint is $p \cdot \tilde{x} = \mu$, where μ is outside income.

The Slutsky equation is derived from the identity

$$\tilde{x}\left(p, e\left(p, \tilde{u}\right)\right) = \tilde{h}\left(p, \tilde{u}\right)$$

This gives

$$\frac{\partial \tilde{x}_i\left(p, \tilde{e}\left(p, \tilde{u}\right)\right)}{\partial p_j} + \frac{\partial \tilde{x}_i\left(p, \tilde{e}\left(p, \tilde{u}\right)\right)}{\partial y} \frac{\partial \tilde{e}\left(p, \tilde{u}\right)}{\partial p_j} = \frac{\partial \tilde{h}_i\left(p, \tilde{u}\right)}{\partial p_j}$$

But $\frac{\partial \tilde{x}_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j}$ and $\frac{\partial \tilde{x}_i}{\partial y} = \frac{\partial x_i}{\partial y}$, so

$$\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial y}\left(h_j - g\right) = \frac{\partial h_i}{\partial p_j}$$

In the case of leisure, $g - x = T - l = L$, where L is hours worked, and $\frac{\partial x_i}{\partial p_j} = -\frac{\partial L}{\partial w}$. So the Slutsky equation for labor supply can be written as

$$\frac{\partial L}{\partial w} - \frac{\partial L}{\partial \mu} L = \frac{\partial L^*}{\partial w}$$

where $\frac{\partial L^*}{\partial w}$ is the substitution effect (the change in hours worked with respect to a wage change, with utility held fixed).

Thus the slope of the Marshallian supply curve is

$$\frac{\partial L}{\partial w} = \frac{\partial L^*}{\partial w} + \frac{\partial L}{\partial \mu} L$$

This shows that the income effect is scaled by hours worked. More explicitly,

$$\frac{\partial L\left(w, \mu\right)}{\partial w} = \frac{\partial L^*\left(w, V\left(w, \mu\right)\right)}{\partial w} + \frac{\partial L\left(w, \mu\right)}{\partial \mu} L\left(w, \mu\right)$$

The substitution effect of a real wage increase is positive (less leisure, more labor supplied).

The income effect is negative.

The strength of the income effect is scaled by the length of the workweek.

So an increase in the wage starting from a low level will have a big income effect, because the workweek is long

Thus the model can actually give a coherent account of observed changes in labor supply in response to rising real wages. Hours worked should fall from an initially high level, but this weakens the income effect, and at some point the income effect is weak enough that it is just offset by the substitution effect, and the quantity supplied is constant from then on.

But for people who were not initially in the market, there is no income effect, and the substitution effect implies an increase in hours worked for these people. So this can explain the rising participation rate for women, while the quantity supplied by participants is actually falling.