CONTRACTION MAPPING THEOREM (BANACH FIXED POINT THEOREM)

Let \((X, d)\) be a complete metric space. Suppose \(f : X \to X\) and

\[
d(f(x), f(y)) < cd(x, y)
\]

for all \(x, y \in X\), with \(c < 1\) (that is, \(f\) is Lipschitz, with modulus less than 1). Then \(f\) has a unique fixed point, and it is reached by iterating the function from any starting point.

**Continuity.** Any Lipschitz function is continuous:

suppose \(x_n \to a\). For \(\varepsilon < 0\) choose \(N\) such that \(d(x_n, a) < \varepsilon\) for \(n \geq N\). Then

\[
d(f(x_n), f(a)) < cd(x_n, a)
\]

so \(f(x_n) \to f(a)\)

**Cauchy.** Let \(y_n\) be the sequence defined by iterating (repeatedly applying) the function \(f\), starting from some point \(y_0\).

\[
y_n = f(y_{n-1})
\]

\[
d(y_2, y_1) = d(f(y_1), f(y_0)) < cd(f(y_0), y_0)
\]

\[
d(y_3, y_2) = d(f(y_2), f(y_1)) < cd(y_2, y_1) < c^2d_0
\]

\[
d(y_{n+1}, y_n) < c^nd_0
\]

\[
d(y_{n+k}, y_n) < (c^n + c^{n+1} + \ldots + c^{n+k-1})d_0
\]

\[
= c^n(1 + c + \ldots + c^{k-1})d_0
\]

\[
\leq c^n \sum_{i=0}^{\infty} c^i d_0
\]

\[
= \frac{c^n}{1-c}d_0
\]

so

\[
d(y_m, y_n) < \varepsilon
\]

if

\[
\frac{c^n}{1-c}d_0 \leq \varepsilon
\]

but this holds if \(n\) is large enough, because \(0 < c < 1\).

So this is a Cauchy sequence, and therefore it converges, because \(X\) is a complete metric space.

**Uniqueness.** There can’t be two distinct fixed points, because if \(x\) and \(y\) are both fixed points then

\[
d(f(x), f(y)) < cd(x, y)
\]

and then since these are fixed points

\[
d(x, y) < cd(x, y)
\]

which is impossible unless the distance is zero (meaning \(x = y\).
Iteration. It remains only to show that the limit of the iteration sequence is in fact the fixed point of \( f \) (regardless of where the sequence starts).

\[
y_n \rightarrow y_0
\]

so

\[
f (y_n) \rightarrow f (y_0)
\]

because \( f \) is continuous. But

\[
f (y_n) = y_{n+1}
\]

and

\[
y_{n+1} \rightarrow y_0
\]

so

\[
f (y_0) = y_0
\]