**Constant Returns**

The cost function is linear in output in the case of constant returns:

\[ c(w, q) = qc(w, 1) \]

for all input price vectors \( w \).

To prove this, first note that \( c(w, 1) = w \cdot z^1 \) for some production plan \( z^1 \) such that \( (1, -z^1) \in Y \). Then by constant returns the production plan \( (q, -qz^1) \in Y \), and this plan produces \( q \) units of output at a cost of \( w \cdot (qz^1) = qc(w, 1) \). This proves that \( c(w, q) \leq qc(w, 1) \). Also, \( c(w, q) = w \cdot z^q \) for some production plan \( z^q \) such that \( (q, -z^q) \in Y \). Then by constant returns the production plan \( (1, -\frac{1}{q}z^q) \in Y \), and this plan produces 1 unit of output at a cost of \( \frac{1}{q}c(w, q) \). This proves that \( c(w, 1) \leq \frac{1}{q}c(w, q) \), so \( qc(w, 1) \leq c(w, q) \)

This implies that average and marginal costs are equal.

**CES Cost Function**

**Average Cost**

\[ c = AC = \sum_{\ell=1}^{L} \frac{w_{\ell}z_{\ell}}{q} \]

**Marginal Cost**

\[ c = MC = \frac{w_{\ell}}{MP_{\ell}} \]

**Production Function**

\[ \frac{q^\rho - 1}{\rho} = \sum_{\ell=1}^{L} \theta_{\ell} (z_{\ell})^\rho - 1 \]

**Marginal (and average) Products**

\[ q^{\rho-1}MP_{\ell} = \theta_{\ell} (z_{\ell})^{\rho-1} \]

\[ MP_{\ell} = \theta_{\ell} (AP_{\ell})^{1-\rho} \]

\[ \left( \frac{MP_{\ell}}{\theta_{\ell}} \right)^{\sigma} = AP_{\ell} \]

so from the marginal cost formula above

\[ c^{-\sigma} = \left( \frac{MP_{\ell}}{\theta_{\ell}} \right)^{\sigma} \left( \frac{\theta_{\ell}}{w_{\ell}} \right)^{\sigma} = AP_{\ell} \]

and the average cost formula can be written as

\[ c = \sum_{\ell=1}^{L} \frac{w_{\ell}}{AP_{\ell}} \]

so
\[ c^{1-\sigma} = \sum_{\ell=1}^{L} \frac{w_{\ell}}{AP_{\ell}} AP_{\ell} \left( \frac{\theta_{\ell}}{w_{\ell}} \right)^{\sigma} \]

and

\[ c^{1-\sigma} = \sum_{\ell=1}^{L} \theta_{\ell} \left( \frac{w_{\ell}}{\theta_{\ell}} \right)^{1-\sigma} \]

or

\[ c^{1-\sigma} = \sum_{\ell=1}^{L} (\theta_{\ell})^{\sigma} (w_{\ell})^{1-\sigma} \]