Uniform confidence bands: characterization and optimality

Supplemenatry appendix

Joachim Freyberger[†] Yoshiyasu Rai[‡]

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S.1 Useful lemmas

In the lemmas below we use the following notation. For $S \subset \mathbb{R}^K$, let $conv(\cdot)$ denote the convex hull of S. For $S_1 \subset \mathbb{R}^K$ and $S_2 \subset \mathbb{R}^K$ and $\lambda \in (0,1)$ let $\lambda S_1 \oplus (1-\lambda)S_2 = \{s \in R^K : s = \lambda s_1 + (1-\lambda)s_2$ for some $s_1 \in S_1$ and $s_2 \in S_2\}$.

Lemma S1. *Let $CI_1 = \{\beta : \beta - \hat{\beta} \in S_1\}$ and $CI_2 = \{\beta : \beta - \hat{\beta} \in S_2\}$, where S_1 is closed, S_2 is convex, and both sets are nonrandom. Suppose $CI_1 \subseteq CI_2$ and that*

$$P(\beta_0 \in CI_1) = P(\beta_0 \in CI_2) = 1 - \alpha$$

for some $\alpha \in (0,1)$. Then $CI_1 = CI_2$.

Proof. Suppose there is $\bar{s} \in S_2$, but $\bar{s} \notin S_1$. Since S_1 is closed, it follows that there exits $\delta > 0$ such that $\inf_{s \in S_1} \|\bar{s} - s\| \geq \delta$. Let $S_3 = conv(S_1 \cup \bar{s}) \subseteq S_2$. Since S_1 has positive Lebesgue measure, also S_3 has positive Lebesgue measure, and since it is convex, it contains a compact ball B of positive measure (see for example Corollary 2.4.9 and Proposition 4.10.11 of Bogachev (1998)). Now for any $\gamma > 0$ define $S_\gamma = \{s : (1-\alpha)\bar{s} + \gamma s_B$ for some $s_B \in B\}$. Notice that $S_\gamma \subset S_3$, but for γ small enough (but positive), $S_\gamma \cap S_1 = \emptyset$ because $\inf_{s \in S_1} \|\bar{s} - s\| \geq \delta$. It follows that $S_3 \backslash S_1$ has positive Lebesgue measure as well and therefore

$$1 - \alpha = P(\beta_0 - \hat{\beta} \in S_2) \geq P(\beta_0 - \hat{\beta} \in S_3) > P(\beta - \hat{\beta} \in S_1) = 1 - \alpha,$$

which is a contradiction. $\square$

[†]Department of Economics, University of Wisconsin - Madison. Email: jfreyberger@ssc.wisc.edu.
[‡]Department of Economics, University of Wisconsin - Madison. Email: yrai@wisc.edu.

Lemma S2. *Let $Z \sim N(0, I_{K_n \times K_n})$, $\Sigma \in \mathbb{R}^{K_n \times K_n}$ be positive definite, $p(x) \in \mathbb{R}^{K_n}$, and $\sigma(x) = \sqrt{p(x)'\Sigma p(x)} \in (0,\infty)$ for all $x \in \mathcal{X}$. Let $c_u(x)$ and $c_l(x)$ be functions such that*

$$P\left(c_l(x) \leq \frac{p(x)'\Sigma^{1/2}Z}{\sigma(x)} \leq c_u(x)\, \forall x \in \mathcal{X}\right) = 1 - \alpha,$$

where $\alpha \in (0, 1/2)$. Let $\varepsilon_n \to 0$ such that $K_n\varepsilon_n \to 0$. Then

$$P\left(c_l(x) - \varepsilon_n \leq \frac{p(x)'\Sigma^{1/2}Z}{\sigma(x)} \leq c_u(x) + \varepsilon_n\, \forall x \in \mathcal{X}\right) \to 1 - \alpha.$$

Proof. Note that $c_l(x) < -\Phi^{-1}(1-\alpha)$ and $c_u(x) > \Phi^{-1}(1-\alpha)$, where Φ denotes the standard normal cdf. Let

$$\gamma_n^1 = \frac{\Phi^{-1}(1-\alpha) + |\varepsilon_n|}{\Phi^{-1}(1-\alpha)} \qquad \text{and} \qquad \gamma_n^2 = \frac{\Phi^{-1}(1-\alpha) - |\varepsilon_n|}{\Phi^{-1}(1-\alpha)}.$$

Since $\Phi^{-1}(1-\alpha) > 0$, γ_n^1 and γ_n^2 are well defined and positive for ε_n close enough to 0. Define

$$S_1 = \left\{z \in \mathbb{R}^{K_n} : c_l(x) \leq \frac{p(x)'\Sigma^{1/2}z}{\sigma(x)} \leq c_u(x)\, \forall x \in \mathcal{X}\right\},$$

$$S_{2,n} = \left\{z \in \mathbb{R}^{K_n} : c_l(x) - \varepsilon_n \leq \frac{p(x)'\Sigma^{1/2}z}{\sigma(x)} \leq c_u(x) + \varepsilon_n\, \forall x \in \mathcal{X}\right\}$$

and

$$S_1^\gamma = \left\{z \in \mathbb{R}^{K_n} : \gamma c_l(x) \leq \frac{p(x)'\Sigma^{1/2}z}{\sigma(x)} \leq \gamma c_u(x)\, \forall x \in \mathcal{X}\right\}$$

for $\gamma > 0$. By definition,

$$P(Z \in S_1^{\gamma_n^2}) \leq P(Z \in S_{2,n}) \leq P(Z \in S_1^{\gamma_n^1}) \tag{S1}$$

holds. By a change of variables,

$$\begin{aligned} P\left(Z \in S_1^\gamma\right) &= \int \mathbf{1}\left(z \in S_1^\gamma\right)\phi(z)dz \\ &= \gamma^{K_n}\int \mathbf{1}\left(\gamma z \in S_1^\gamma\right)\phi(\gamma z)dz \\ &= \gamma^{K_n}\int \mathbf{1}\left(z \in S_1\right)\phi(\gamma z)dz. \end{aligned}$$

Since $\phi(\gamma z) < \phi(z)$ for $\gamma > 1$ and $\phi(\gamma z) > \phi(z)$ for $\gamma < 1$,

$$0 \leq P\left(Z \in S_1^{\gamma_n^1}\right) - P\left(Z \in S_1\right) \leq P\left(Z \in S_1\right)\left((\gamma_n^1)^{K_n} - 1\right)$$

and

$$0 \geq P\left(Z \in S_1^{\gamma_n^2}\right) - P\left(Z \in S_1\right) \geq P\left(Z \in S_1\right)\left((\gamma_n^2)^{K_n} - 1\right).$$

Note that

$$(\gamma_n^1)^{K_n} = \left(1 + \frac{|\varepsilon_n|}{\Phi^{-1}(1-\alpha)}\right)^{K_n} = \left(1 + \frac{K_n|\epsilon_n|}{K_n\Phi^{-1}(1-\alpha)}\right)^{K_n} = \exp\left(\frac{K_n|\varepsilon_n|}{\Phi^{-1}(1-\alpha)}\right) + o(1).$$

Thus, if $K_n|\varepsilon_n| \to 0$, then

$$P(Z \in S_1^{\gamma_n^1}) \to P(Z \in S_1) \quad \text{and} \quad P(Z \in S_1^{\gamma_n^2}) \to P(Z \in S_1).$$

Together with (S1) we can conclude $P(Z \in S_{2,n}) \to 1 - \alpha$. □

Lemma S3. *Let $\lambda \in (0,1)$. Let ν denote the Lebesgue measure on $\mathbb{R}^K$, let Φ denote the K-dimensional standard normal measure, and let ϕ denote the standard normal pdf. Let S_1 and S_2 be bounded sets on $\mathbb{R}^K$. Suppose there is a set $S_3 \subseteq \lambda S_1 \oplus (1-\lambda)S_2$ with $\nu(S_3) > 0$ such that for all $s_3 \in S_3$ and some $\delta > 0$*

$$\inf_{(s_1,s_2)\in S_1\times S_2 : \lambda s_1 + (1-\lambda)s_2 = s_3} \|s_1 - s_2\|^2 > \delta.$$

Then

$$\Phi(\lambda S_1 \oplus (1-\lambda)S_2) \geq \Phi(S_1)^\lambda \Phi(S_2)^{1-\lambda} + \inf_{s\in S_1\cup S_2} \phi(s)\left(\exp\left(\frac{\lambda(1-\lambda)\delta}{2}\right) - 1\right)\nu(S_3).$$

Proof. From simple algebra it follows that

$$\ln\phi(\lambda s_1 + (1-\lambda)s_2) - (\lambda \ln\phi(s_1) + (1-\lambda)\ln\phi(s_2)) = \frac{\lambda(1-\lambda)}{2}\|s_1 - s_2\|^2$$

and thus

$$\phi(\lambda s_1 + (1-\lambda)s_2) = \phi(s_1)^\lambda \phi(s_2)^{1-\lambda} \exp\left(\frac{\lambda(1-\lambda)}{2}\|s_1 - s_2\|^2\right).$$

Now define

$$h(x) = \sup_{y\in\mathbb{R}^K} \phi\left(\frac{x-y}{\lambda}\right)^\lambda \mathbf{1}\left(\frac{x-y}{\lambda} \in S_1\right) \phi\left(\frac{y}{1-\lambda}\right)^{1-\lambda} \mathbf{1}\left(\frac{y}{1-\lambda} \in S_2\right).$$

Notice that

$$x = \lambda\frac{x-y}{\lambda} + (1-\lambda)\frac{y}{1-\lambda}$$

and therefore, log-concavity of $\phi(\cdot)$ implies that for all $y \in \mathbb{R}^K$

$$\phi(x) \geq \phi\left(\frac{x-y}{\lambda}\right)^{\lambda} \phi\left(\frac{y}{1-\lambda}\right)^{1-\lambda}.$$

Moreover, $\mathbf{1}(x \in S_1 \oplus (1-\lambda)S_2) \geq \mathbf{1}\left(\frac{x-y}{\lambda} \in S_1\right) \mathbf{1}\left(\frac{y}{1-\lambda} \in S_2\right)$ and thus for all $x \in \mathbb{R}^K$

$$\phi(x)\mathbf{1}(x \in S_1 \oplus (1-\lambda)S_2) \geq h(x).$$

Similarly, if $x \in S_3$, then for any $y \in \mathbb{R}^K$

$$\phi(x) \geq \phi\left(\frac{x-y}{\lambda}\right)^{\lambda} \phi\left(\frac{y}{1-\lambda}\right)^{1-\lambda} \exp\left(\frac{\lambda(1-\lambda)}{2}\left\|\frac{x-y}{\lambda} - \frac{y}{1-\lambda}\right\|^2\right).$$

If $\frac{x-y}{\lambda} \in S_1$ and $\frac{y}{1-\lambda} \in S_2$, then $\left\|\frac{x-y}{\lambda} - \frac{y}{1-\lambda}\right\|^2 \geq \delta$ and $x \in \lambda S_1 \oplus (1-\lambda)S_2$. Therefore, for all $x \in S_3$

$$\phi(x) \geq h(x)\exp\left(\frac{\lambda(1-\lambda)\delta}{2}\right)$$

Moreover, take any $x \in \lambda S_1 \oplus (1-\lambda)S_2$. Then there exist $s_1 \in S_1$ and $s_2 \in S_2$ such that $x = \lambda s_1 + (1-\lambda)s_2$. Let $y = (1-\lambda)s_2$. Then $s_1 = \frac{x-y}{\lambda} \in S_1$. Thus

$$h(x) \geq \phi(s_1)^{\lambda}\mathbf{1}(s_1 \in S_1)\phi(s_2)^{1-\lambda}\mathbf{1}(s_2 \in S_2) \geq \inf_{s \in S_1 \cup S_2} \phi(s) > 0.$$

We now get

$$\begin{aligned}
\phi(x)\mathbf{1}(x \in \lambda S_1 \oplus (1-\lambda)S_2) &= \phi(x)\mathbf{1}(x \in \lambda S_1 \oplus (1-\lambda)S_2)\mathbf{1}(x \notin S_3) + \phi(x)\mathbf{1}(x \in S_3) \\
&\geq h(x)\mathbf{1}(x \notin S_3) + h(x)\exp\left(\frac{\lambda(1-\lambda)\delta}{2}\right)\mathbf{1}(x \in S_3) \\
&= h(x) + h(x)\left(\exp\left(\frac{\lambda(1-\lambda)\delta}{2}\right) - 1\right)\mathbf{1}(x \in S_3) \\
&\geq h(x) + \inf_{s \in S_1 \cup S_2} \phi(s)\left(\exp\left(\frac{\lambda(1-\lambda)\delta}{2}\right) - 1\right)\mathbf{1}(x \in S_3).
\end{aligned}$$

By Theorem 1.8.2 in Bogachev (1998),

$$\int h(x)dx \geq \Phi(S_1)^{\lambda}\Phi(S_2)^{1-\lambda},$$

Taking the integral of the inequality above yields

$$\Phi(\lambda S_1 \oplus (1-\lambda)S_2) \geq \Phi(S_1)^{\lambda}\Phi(S_2)^{1-\lambda} + \inf_{s \in S_1 \cup S_2} \phi(s)\left(\exp\left(\frac{\lambda(1-\lambda)\delta}{2}\right) - 1\right)\nu(S_3).$$

$\square$

Lemma S4. *Let* $\lambda \in (0,1)$*. Let* S_1 *and* S_2 *be convex and compact subsets of* $\mathbb{R}^K$ *such that for some* $C > 0$*,* $\underline{S} \equiv \{s : \|s\| \leq C\} \subset S_1$ *and* $\underline{S} \subset S_2$*. Let* $c_1(x) = \sup_{\gamma \in S_1} p(x)'\gamma$ *and* $c_2(x) = \sup_{\gamma \in S_2} p(x)'\gamma$*, where* $\sup_{x \in \mathcal{X}} \|p(x)\| < \infty$*. If* $\sup_{x \in \mathcal{X}} |c_1(x) - c_2(x)| > \varepsilon$ *for some* $\varepsilon > 0$*, then there exists a set* $S_3 \subseteq \lambda S_1 \oplus (1-\lambda)S_2$ *and a* $\delta > 0$ *such that for all* $s_3 \in S_3$

$$\inf_{(s_1,s_2) \in S_1 \times S_2 : \lambda s_1 + (1-\lambda)s_2 = s_3} \|s_1 - s_2\| > \delta$$

and $\nu(S_3) > \delta$*, where* δ *only depends on* λ*,* ε*,* $\sup_{x \in \mathcal{X}} \|p(x)\|$*,* C*, and* $\sup_{\gamma \in S_1 \cup S_2} \|\gamma\|$*.*

Proof. Without loss of generality assume that for some $\bar{x} \in \mathcal{X}$ it holds that $c_1(\bar{x}) - c_2(\bar{x}) > \varepsilon$. Also notice that $c_1(\bar{x}) = p(\bar{x})'\gamma_1$ for $\gamma_1 \in S_1$ and $c_2(\bar{x}) = p(\bar{x})'\gamma_2$ for $\gamma_2 \in S_2$. Let $c_3 = \lambda c_1(\bar{x}) + (1-\lambda)c_2(\bar{x})$. Let c be the midpoint of $(c_2(\bar{x}), c_3)$, which is

$$c = c_2(\bar{x}) + \frac{1}{2}\lambda(c_1(\bar{x}) - c_2(\bar{x})),$$

and define

$$H_1 = \{\gamma \in \mathbb{R}^K : p(\bar{x})'\gamma \geq c\} \qquad \text{and} \qquad H_2 = \{\gamma \in \mathbb{R}^K : p(\bar{x})'\gamma \leq c_2(\bar{x})\}.$$

Now define $S_3 = \lambda S_1 \oplus (1-\lambda)S_2 \cap H_1$. Notice that $\lambda\gamma_1 + (1-\lambda)\gamma_2 \in H_1$ and thus, S_3 is not empty. We will now prove that S_3 satisfies the properties in the lemma.

Let $A = conv(\underline{S} \cup \lambda\gamma_1 + (1-\lambda)\gamma_2)$. Since $\underline{S} \subseteq \lambda S_1 \oplus (1-\lambda)S_2$, $\lambda\gamma_1 + (1-\lambda)\gamma_2 \in \lambda S_1 \oplus (1-\lambda)S_2$, and $\lambda S_1 \oplus (1-\lambda)S_2$ is convex, it follows that $A \subseteq \lambda S_1 \oplus (1-\lambda)S_2$.

Let $\underline{c} = \inf_{\gamma \in \underline{S}} p(\bar{x})'\gamma$. Then for $\bar{\lambda} = \frac{c-\underline{c}}{c_3-\underline{c}}$ and $\gamma \in (\bar{\lambda}(\lambda\gamma_1 + (1-\lambda)\gamma_2) \oplus (1-\bar{\lambda})\underline{S})$ we have $p(\bar{x})'\gamma \geq c$ and thus,

$$(\bar{\lambda}(\lambda\gamma_1 + (1-\lambda)\gamma_2) \oplus (1-\bar{\lambda})\underline{S}) \subseteq H_1.$$

Moreover,

$$(\bar{\lambda}(\lambda\gamma_1 + (1-\lambda)\gamma_2) \oplus (1-\bar{\lambda})\underline{S}) \subseteq A \subseteq \lambda S_1 \oplus (1-\lambda)S_2.$$

Together this means that

$$(\bar{\lambda}(\lambda\gamma_1 + (1-\lambda)\gamma_2) \oplus (1-\bar{\lambda})\underline{S}) \subseteq S_3.$$

The set $(\bar{\lambda}(\lambda\gamma_1 + (1-\lambda)\gamma_2) \oplus (1-\bar{\lambda})\underline{S})$ is a ball with center at $\bar{\lambda}(\lambda\gamma_1 + (1-\lambda)\gamma_2)$ and radius

$$(1-\bar{\lambda})C = C\frac{c_3 - c}{c_3 - \underline{c}} \geq \frac{C}{2}\frac{\lambda\varepsilon}{c_3 - \underline{c}} > 0.$$

Let $\bar{c} = \sup_{\gamma \in S_1 \cup S_2} p(\bar{x})'\gamma$. Then the radius of the ball, which is contained in S_3, is at least $\frac{C}{2}\frac{\lambda\varepsilon}{\bar{c}-\underline{c}} > 0$ and therefore, $\nu(S_3) > 0$.

Next, define $D = (S_1 \cap H_1)$ which is nonempty. Also notice that $S_2 \subseteq H_2$. Therefore,

$$\inf_{(s_1,s_2)\in D\times S_2} \|s_1 - s_2\| \;\geq\; \inf_{(s_1,s_2)\in H_1\times H_2} \|s_1 - s_2\| \geq \frac{c - c_2(\bar{x})}{\|p(\bar{x})\|} \geq \frac{\lambda\varepsilon}{2\sup_{x\in\mathcal{X}} \|p(x)\|}$$

Now take $s_3 \in S_3$. Then $p(x)'s_3 \geq c$. Write $s_3 = \lambda s_1 + (1-\lambda)s_2$ for $s_1 \in S_1$ and $s_2 \in S_2$. Suppose that $s_1 \notin D$. Then $s_1 \notin H_1$, which implies that $p(\bar{x})'s_1 < c$. But since it also holds that $s_2 \in S_2$, we have $p(\bar{x})'s_3 < c$, which would yield the contradiction that $p(x)'s_2 < c$. It follows that for all $s_3 \in S_3$

$$\inf_{(s_1,s_2)\in S_1\times S_2:\lambda s_1+(1-\lambda)s_2=s_3} \|s_1 - s_2\| \geq \inf_{(s_1,s_2)\in D\times S_2} \|s_1 - s_2\| \geq \frac{\lambda\varepsilon}{2\sup_{x\in\mathcal{X}} \|p(x)\|} > 0.$$

The now conclusion follows with $\delta = \min\left\{\frac{\lambda\varepsilon}{2\sup_{x\in\mathcal{X}}\|p(x)\|}, \nu((1-\bar{\lambda})\underline{S})\right\}$. □

S.2 Proofs

S.2.1 Proofs of main results

Proof of Lemma 1. (1) Suppose $\beta \in CI(\hat{\beta})$. Then for all $x \in \mathcal{X}$ it holds by definition that $g_l(x,\hat{\beta}) \leq p(x)'\beta$ and $p(x)'\beta \leq g_u(x,\hat{\beta})$. Therefore

$$P\left(g_l(x,\hat{\beta}) \leq p(x)'\beta_0 \leq g_u(x,\hat{\beta}) \text{ for all } x \in \mathcal{X}\right) \geq P(\beta_0 \in CI(\hat{\beta})) = 1-\alpha.$$

(2) Let $\mathcal{X}_u = \{x \in \mathcal{X} : w_u(x) > 0\}$ and $\mathcal{X}_l = \{x \in \mathcal{X} : w_l(x) > 0\}$. Then

$$\begin{aligned}
1-\alpha &= P\left(\sup_{x\in\mathcal{X}} \frac{p(x)'(\beta_0-\hat{\beta})}{\sigma(x)} w_u(x) \leq c,\ \inf_{x\in\mathcal{X}} \frac{p(x)'(\beta_0-\hat{\beta})}{\sigma(x)} w_l(x) \geq -c\right)\\
&= P\left(\sup_{x\in\mathcal{X}_u} \frac{p(x)'(\beta_0-\hat{\beta})}{\sigma(x)} w_u(x) \leq c,\ \inf_{x\in\mathcal{X}_l} \frac{p(x)'(\beta_0-\hat{\beta})}{\sigma(x)} w_l(x) \geq -c\right)\\
&= P\left(p(x)'\beta_0 \leq g_u(x,\hat{\beta})\ \forall x \in \mathcal{X}_u,\ p(x)'\beta_0 \geq g_l(x,\hat{\beta})\ \forall x \in \mathcal{X}_l\right)\\
&= P\left(g_l(x,\hat{\beta}) \leq p(x)'\beta_0 \leq g_u(x,\hat{\beta}) \text{ for all } x \in \mathcal{X}\right)
\end{aligned}$$

Moreover, by construction the set $\{\beta : g_l(x,\hat{\beta}) \leq g(x,\beta) \leq g_u(x,\hat{\beta}) \text{ for all } x \in \mathcal{X}\}$ can be written as $\{\beta : \hat{\beta} - \beta \in S\}$ where S is nonrandom. □

Proof of Theorem 1. (1): Let $[g_l(x,\hat{\beta}),\, g_u(x,\hat{\beta})] \in \mathcal{C}$ be an arbitrary taut confidence band and define

$$CI(\hat{\beta}) = \{\beta : g_l(x,\hat{\beta}) \leq p(x)'\beta \leq g_u(x,\hat{\beta}) \text{ for all } x \in \mathcal{X}\}$$

and

$$g_l^*(x,\hat\beta) = \inf_{\beta\in CI(\hat\beta)} p(x)'\beta \text{ and } g_u^*(x,\hat\beta) = \sup_{\beta\in CI(\hat\beta)} p(x)'\beta.$$

By Lemma 1, $P(g_l^*(x,\hat\beta) \le g_0(x) \le g_u^*(x,\hat\beta)\ \forall x\in\mathcal{X}) \ge 1-\alpha$. Moreover, $g_u^*(x,\hat\beta) \le g_u(x,\hat\beta)$, and $g_l^*(x,\hat\beta) \ge g_l(x,\hat\beta)$ for all x. Hence, if $[g_l(x,\hat\beta),\, g_u(x,\hat\beta)]$ is taut, it holds that $[g_l(x,\hat\beta),\, g_u(x,\hat\beta)] = [g_l^*(x,\hat\beta),\, g_u^*(x,\hat\beta)]$. Finally, by definition of the class $\mathcal{C}$, there exists a nonrandom set $S\subset\mathbb{R}^K$ such that

$$g_l(x,\hat\beta) = \inf_{\beta\in CI(\hat\beta)} p(x)'\beta = p(x)'\hat\beta + \inf_{\beta-\hat\beta\in CI(\hat\beta)-\hat\beta} p(x)'(\beta-\hat\beta) = p(x)'\hat\beta \ + \inf_{\gamma\in S} p(x)'\gamma$$

and similarly

$$g_u(x,\hat\beta) = p(x)'\hat\beta \ + \sup_{\gamma\in S} p(x)'\gamma.$$

It now immediately follows that the taut band $[g_l(x,\hat\beta),\, g_u(x,\hat\beta)]\in\mathcal{C}$ can be obtained by a projection on

$$\{\beta\in\mathbb{R}^K : \inf_{\gamma\in S} p(x)'\gamma \le p(x)'(\beta-\hat\beta) \le \sup_{\gamma\in S} p(x)'\gamma \text{ for all } x\in\mathcal{X}\}.$$

Now suppose that $[g_l(x,\hat\beta),\, g_u(x,\hat\beta)]$ is a confidence band obtained by a projection on

$$CI(\hat\beta) = \{\beta\in\mathbb{R}^K : c_l(x) \le p(x)'(\beta-\hat\beta) \le c_u(x) \text{ for all } x\in\mathcal{X}\}.$$

Let $S = \{z\in\mathbb{R}^K : c_l(x)\le p(x)'z \le c_u(x) \text{ for all } x\in\mathcal{X}\}$. Then by the arguments above

$$\begin{aligned}[g_l(x,\hat\beta),\, g_u(x,\hat\beta)] &= [p(x)'\hat\beta + \inf_{\gamma\in S} p(x)'\gamma,\ p(x)'\hat\beta + \sup_{\gamma\in S} p(x)'\gamma] \\ &\subseteq [p(x)'\hat\beta + c_l(x),\ p(x)'\hat\beta + c_u(x)]\end{aligned}$$

Since $[g_l(x,\hat\beta),\, g_u(x,\hat\beta)]$ is obtained by a projection on $CI(\hat\beta)$ it also has to hold that

$$CI(\hat\beta) \subseteq \{\beta\in\mathbb{R}^K : g_l(x,\hat\beta) \le p(x)'\beta \le g_u(x,\hat\beta) \text{ for all } x\in\mathcal{X}\}$$

and thus

$$CI(\hat\beta) = \{\beta\in\mathbb{R}^K : g_l(x,\hat\beta) \le p(x)'\beta \le g_u(x,\hat\beta) \text{ for all } x\in\mathcal{X}\}.$$

It follows that $[g_l(x,\hat\beta),\, g_u(x,\hat\beta)]\in\mathcal{C}$. If $[g_l(x,\hat\beta),\, g_u(x,\hat\beta)]$ was not taut, then there exists $[\tilde g_l(x,\hat\beta),\, \tilde g_u(x,\hat\beta)]$ such that $\widetilde{CI}(\hat\beta)\subseteq CI(\hat\beta)$, where

$$\widetilde{CI}(\hat\beta) = \{\beta\in\mathbb{R}^K : \tilde g_u(x,\hat\beta) \le p(x)'\beta \le \tilde g_u(x,\hat\beta) \text{ for all } x\in\mathcal{X}\}$$

and $P(\beta_0 \in \widetilde{CI}(\hat\beta)) = 1-\alpha$. By Lemma S1, $\widetilde{CI}(\hat\beta) = CI(\hat\beta)$. But since $[g_l(x,\hat\beta),\, g_u(x,\hat\beta)]$ is obtained by a projection on $CI(\hat\beta)$ it holds that $[g_l(x,\hat\beta),\, g_u(x,\hat\beta)] \subseteq [\tilde g_l(x,\hat\beta),\, \tilde g_u(x,\hat\beta)]$ for all $x \in \mathcal{X}$, which is a contradiction.

(2): By the proof of the first part, we can write

$$[g_l(x,\hat\beta),\, g_u(x,\hat\beta)] = \left[p(x)'\hat\beta + \inf_{\gamma\in S} p(x)'\gamma,\; p(x)'\hat\beta + \sup_{\gamma\in S} p(x)'\gamma\right].$$

Let $w_l(x) = -\frac{\sigma(x)}{\inf_{\gamma\in S} p(x)'\gamma}$ if $\inf_{\gamma\in S} p(x)'\gamma > -\infty$ and $w_l(x) = 0$ if $\inf_{\gamma\in S} p(x)'\gamma = -\infty$. Similarly, let $w_u(x) = \frac{\sigma(x)}{\sup_{\gamma\in S} p(x)'\gamma}$ if $\sup_{\gamma\in S} p(x)'\gamma < \infty$ and $w_u(x) = 0$ if $\sup_{\gamma\in S} p(x)'\gamma = \infty$. Then

$$g_l(x,\hat\beta) = \begin{cases} p(x)'\hat\beta - \frac{\sigma(x)}{w_l(x)} & \text{if } w_l(x) > 0 \\ -\infty & \text{if } w_l(x) = 0 \end{cases}$$

and

$$g_u(x,\hat\beta) = \begin{cases} p(x)'\hat\beta + \frac{\sigma(x)}{w_u(x)} & \text{if } w_u(x) > 0 \\ \infty & \text{if } w_u(x) = 0 \end{cases}$$

Moreover, let $\mathcal{X}_u = \{x \in \mathcal{X} : w_u(x) > 0\}$ and $\mathcal{X}_l = \{x \in \mathcal{X} : w_u(x) > 0\}$. Then

$$\begin{aligned}
1-\alpha &= P\left(p(x)'\hat\beta + \inf_{\gamma\in S} p(x)'\gamma \le p(x)'\beta_0 \le p(x)'\hat\beta + \sup_{\gamma\in S} p(x)'\gamma \text{ for all } x \in \mathcal{X}\right) \\
&= P\left(p(x)'\beta_0 \le p(x)'\hat\beta + \inf_{\gamma\in S} p(x)'\gamma \;\forall x \in \mathcal{X}_u, p(x)'\beta_0 \ge p(x)'\hat\beta + \sup_{\gamma\in S} p(x)'\gamma \;\forall x \in \mathcal{X}_l\right) \\
&= P\left(p(x)'\beta_0 \le p(x)'\hat\beta + \frac{\sigma(x)}{w_u(x)} \;\forall x \in \mathcal{X}_u, p(x)'\beta_0 \ge p(x)'\hat\beta - \frac{\sigma(x)}{w_l(x)} \;\forall x \in \mathcal{X}_l\right) \\
&= P\left(\sup_{x\in\mathcal{X}_u} \frac{p(x)'(\beta_0-\hat\beta)}{\sigma(x)} w_u(x) \le 1,\, \inf_{x\in\mathcal{X}_l} \frac{p(x)'(\beta_0-\hat\beta)}{\sigma(x)} w_l(x) \ge -1\right) \\
&= P\left(\sup_{x\in\mathcal{X}} \frac{p(x)'(\beta_0-\hat\beta)}{\sigma(x)} w_u(x) \le 1,\, \inf_{x\in\mathcal{X}} \frac{p(x)'(\beta_0-\hat\beta)}{\sigma(x)} w_l(x) \ge -1\right).
\end{aligned}$$

□

Proof of Corollary 1. First notice that since $\sigma(x) = \sqrt{p(x)'\Sigma p(x)} > 0$ for all $x \in \mathcal{X}$ and since Σ is positive definite, it follows that $p(x)$ is not the zero vector and $\nu(x) = \sqrt{p(x)'\Omega p(x)} > 0$. Next, for any $c_1 > 0$ define $CI_1(\hat\beta, c_1) = \{\beta : (\beta-\hat\beta)\Omega^{-1}(\beta-\hat\beta) \le c_1^2\}$. Then

$$\sup_{\beta\in CI_1(\hat\beta,c_1)} p(x)'\beta = p(x)'\hat\beta + \max_{\gamma:\gamma'\Omega^{-1}\gamma\le c_1^2} p(x)'\gamma.$$

The Lagrangian of the maximization problem is $p(x)'\gamma + \lambda(c_1^2 - \gamma'\Omega^{-1}\gamma)$ with (necessary and sufficient) first order conditions $p(x) = 2\lambda\Omega^{-1}\gamma$ and $\lambda(c_1^2 - \gamma'\Omega^{-1}\gamma) = 0$. Since $p(x) \neq 0$ it follows that $\gamma \neq 0$ and $\lambda \neq 0$ and therefore it is easy to solve for

$$\lambda = \frac{1}{2c_1}\sqrt{p(x)'\Omega p(x)} \qquad \text{and} \qquad \gamma = \frac{c_1 \Omega p(x)}{\sqrt{p(x)'\Omega p(x)}}.$$

Therefore,

$$\sup_{\beta \in CI_1(\hat{\beta}, c_1)} p(x)'\beta = p(x)'\hat{\beta} + c_1\sqrt{p(x)'\Omega p(x)}.$$

Analogously,

$$\inf_{\beta \in CI_1(\hat{\beta}, c_1)} p(x)'\beta = p(x)'\hat{\beta} - c_1\sqrt{p(x)'\Omega p(x)}.$$

Next define

$$CI(\hat{\beta}, c_1) = \{\beta : -c_1\sqrt{p(x)'\Omega p(x)} \leq p(x)'(\beta - \hat{\beta}) \leq c_1\sqrt{p(x)'\Omega p(x)} \text{ for all } x \in \mathcal{X}\}.$$

Since $CI(\hat{\beta}, c_1)$ is obtained by a projection it has to hold that $CI_1(\hat{\beta}, c_1) \subseteq CI(\hat{\beta}, c_1)$. Next notice that there is a unique c such that $P(\beta_0 \in CI(\hat{\beta}, c)) = 1-\alpha$, because $P(\beta_0 \in CI(\hat{\beta}, c_1))$ is strictly increasing in c_1. It follows that

$$P\left(\sup_{x \in \mathcal{X}} \left| \frac{p(x)'(\beta_0 - \hat{\beta})}{\sigma(x)} w(x) \right| \leq c\right) = 1 - \alpha,$$

where $w(x) = \frac{\sigma(x)}{\nu(x)}$. Finally, since $CI_1(\hat{\beta}, c) \subseteq CI(\hat{\beta}, c)$, we have

$$p(x)'\hat{\beta} + c\sqrt{p(x)'\Omega p(x)} = \sup_{\beta \in CI_1(\hat{\beta}, c)} p(x)'\beta \leq \sup_{\beta \in CI(\hat{\beta}, c)} p(x)'\beta$$

and by definition of $CI(\hat{\beta}, c)$ it holds that

$$\sup_{\beta \in CI(\hat{\beta}, c)} p(x)'\beta \leq p(x)'\hat{\beta} + c\sqrt{p(x)'\Omega p(x)}.$$

Therefore, $\sup_{\beta \in CI(\hat{\beta}, c)} p(x)'\beta = p(x)'\hat{\beta} + c\sqrt{p(x)'\Omega p(x)}$ and similarly $\inf_{\beta \in CI(\hat{\beta}, c)} p(x)'\beta = p(x)'\hat{\beta} - c\sqrt{p(x)'\Omega p(x)}$. It follows that the confidence band obtained from the sup t-statistic, $[p(x)'\hat{\beta} - c\nu(x),\ p(x)'\hat{\beta} + c\nu(x)]$, coincides with the taut projection confidence band obtained from a projection on $CI(\hat{\beta}, c)$, and by Theorem 1, the confidence band obtained from projecting on $CI(\hat{\beta}, c)$ is taut. $\square$

Proof of Corollary 2. Suppose $p_1(x) = 1$ and let

$$CI(\hat{\beta}) = \{\beta : -c \le p(x)'(\beta - \hat{\beta}) \le c \text{ for all } x \in \mathcal{X}\}.$$

Then $P(\beta_0 \in CI(\hat{\beta})) = 1 - \alpha$. By Theorem 1 it suffices to show that

$$\sup_{\beta \in CI(\hat{\beta})} p(x)'\beta = p(x)'\hat{\beta} + c \qquad \text{and} \qquad \inf_{\beta \in CI(\hat{\beta})} p(x)'\beta = p(x)'\hat{\beta} - c.$$

By the definition of the supremum,

$$\sup_{\beta \in CI(\hat{\beta})} p(x)'\beta \le p(x)'\hat{\beta} + c.$$

Now define $\tilde{\beta} = \hat{\beta} + \begin{pmatrix} c & 0 & \ldots & 0 \end{pmatrix}'$ and notice that $\tilde{\beta} \in CI(\hat{\beta})$ because $p_1(x) = 1$. Therefore,

$$\sup_{\beta \in CI(\hat{\beta})} p(x)'\beta \ge p(x)'\tilde{\beta} = p(x)'\hat{\beta} + c.$$

Analogously, one can show that $\inf_{\beta \in CI(\hat{\beta})} p(x)'\beta = p(x)'\hat{\beta} - c.$ $\square$

Proof of Theorem 2. First notice that since $(\bar{c}_l(x), \bar{c}_u(x))$ is optimal (and $(\bar{c}_l^J(x), \bar{c}_u^J(x))$ is feasible in the original problem)

$$h(\bar{c}_l^J(x), \bar{c}_u^J(x)) \ge h(\bar{c}_l(x), \bar{c}_u(x)).$$

Next define $\bar{CI}(\hat{\beta}) = \{\beta \in \mathbb{R}^K : \bar{c}_u(x) \le p(x)'(\beta - \hat{\beta}) \le \bar{c}_u(x) \text{ for all } x \in \mathcal{X}_J\}$. Moreover, let $\tilde{c}_u(x) = \sup_{\beta \in \bar{CI}(\hat{\beta})} p(x)'\beta$ and $\tilde{c}_l(x) = \inf_{\beta \in \bar{CI}(\hat{\beta})} p(x)'\beta$ for all $x \in \mathcal{X} \backslash \mathcal{X}_J$ and $\tilde{c}_u(x) = \bar{c}_u(x)$ and $\tilde{c}_l(x) = \bar{c}_l(x)$ for all $x \in \mathcal{X}_J$. It follows that

$$\begin{aligned}
1 - \alpha &= P(\bar{c}_l(x) \le p(x)'\Sigma^{1/2}Z \le \bar{c}_u(x) \text{ for all } x \in \mathcal{X}) \\
&\le P(\bar{c}_l(x) \le p(x)'\Sigma^{1/2}Z \le \bar{c}_u(x) \text{ for all } x \in \mathcal{X}_J) \\
&= P(\tilde{c}_l(x) \le p(x)'\Sigma^{1/2}Z \le \tilde{c}_u(x) \text{ for all } x \in \mathcal{X}_J).
\end{aligned}$$

Since $(\bar{c}_l^J(x), \bar{c}_u^J(x))$ is optimal, it follows that

$$h_J(\bar{c}_l^J(x), \bar{c}_u^J(x)) \le h_J(\tilde{c}_l(x), \tilde{c}_u(x)) = h_J(\bar{c}_l(x), \bar{c}_u(x)).$$

Therefore

$$\begin{aligned}
h(\bar{c}_l^J(x), \bar{c}_u^J(x)) &= h(\bar{c}_l^J(x), \bar{c}_u^J(x)) + h_J(\bar{c}_l^J(x), \bar{c}_u^J(x)) - h_J(\bar{c}_l^J(x), \bar{c}_u^J(x)) \\
&\le h(\bar{c}_l^J(x), \bar{c}_u^J(x)) - h_J(\bar{c}_l^J(x), \bar{c}_u^J(x)) + h_J(\bar{c}_l(x), \bar{c}_u(x)) \\
&\to h(\bar{c}_l(x), \bar{c}_u(x)).
\end{aligned}$$

$\square$

Proof of Lemma 2. Define

$$\bar{c}_l(x,\Sigma) = \inf_{\gamma\in\mathbb{R}^K:\,\Sigma^{-1/2}\gamma\in S(\Sigma)} p(x)'\gamma \quad \text{and} \quad \bar{c}_u(x,\Sigma) = \sup_{\gamma\in\mathbb{R}^K:\,\Sigma^{-1/2}\gamma\in S(\Sigma)} p(x)'\gamma$$

and let $\varepsilon_n \to 0$ such that

$$P\left(\sup_{x\in\mathcal{X}} \frac{|\bar{c}_l(x,\hat{\Sigma}) - \bar{c}_l(x,\Sigma)|}{\sigma(x)} \le \varepsilon_n\right) \to 1 \quad \text{and} \quad P\left(\sup_{x\in\mathcal{X}} \frac{|\bar{c}_u(x,\hat{\Sigma}) - \bar{c}_u(x,\Sigma)|}{\sigma(x)} \le \varepsilon_n\right) \to 1.$$

Then

$$\begin{aligned}
CR &\equiv P\left(g_l(x,\hat{\beta},\hat{\Sigma}) \le g(x,\beta_0) \le g_u(x,\hat{\beta},\hat{\Sigma})\,\forall x\in\mathcal{X}\right)\\
&= P\left(\frac{\bar{c}_l(x,\hat{\Sigma})}{\sigma(x)} \le \frac{\sqrt{n}p(x)'(\beta_0-\hat{\beta})}{\sigma(x)} \le \frac{\bar{c}_u(x,\hat{\Sigma})}{\sigma(x)}\,\forall x\in\mathcal{X}\right)\\
&\le P\left(\frac{\bar{c}_l(x,\Sigma)}{\sigma(x)} - \varepsilon_n \le \frac{\sqrt{n}p(x)'(\beta_0-\hat{\beta})}{\sigma(x)} \le \frac{\bar{c}_u(x,\Sigma)}{\sigma(x)} + \varepsilon_n\,\forall x\in\mathcal{X}\right) + o(1)\\
&= P\left(\frac{\bar{c}_l(x,\Sigma)}{\sigma(x)} - \varepsilon_n \le \frac{p(x)'\Sigma^{1/2}\sqrt{n}\Sigma^{-1/2}(\beta_0-\hat{\beta})}{\sigma(x)} \le \frac{\bar{c}_u(x,\Sigma)}{\sigma(x)} + \varepsilon_n\,\forall x\in\mathcal{X}\right) + o(1)\\
&= P\left(\frac{\bar{c}_l(x,\Sigma)}{\sigma(x)} - \varepsilon_n \le \frac{p(x)'\Sigma^{1/2}Z}{\sigma(x)} \le \frac{\bar{c}_u(x,\Sigma)}{\sigma(x)} + \varepsilon_n\,\forall x\in\mathcal{X}\right) + o(1)\\
&= P\left(\frac{\bar{c}_l(x,\Sigma)}{\sigma(x)} \le \frac{p(x)'\Sigma^{1/2}Z}{\sigma(x)} \le \frac{\bar{c}_u(x,\Sigma)}{\sigma(x)}\,\forall x\in\mathcal{X}\right) + o(1)\\
&= 1-\alpha+o(1).
\end{aligned}$$

The fifth line follows from the normal approximation assumption, the sixth line follows from Lemma S2, and the last line follows from the definition of the confidence band.

Analogous arguments yield $CR \ge 1-\alpha+o(1)$ and hence

$$P\left(g_l(x,\hat{\beta},\hat{\Sigma}) \le g(x,\beta_0) \le g_u(x,\hat{\beta},\hat{\Sigma})\,\forall x\in\mathcal{X}\right) \to 1-\alpha.$$

□

Proof of Theorem 3. The result follows from the assumption and the definition of $S^*(\hat{\Sigma})$. □

Proof of Theorem 4. Let $\varepsilon_n \to 0$ such that

$$P\left(\sup_{x\in\mathcal{X}} \frac{|\bar{c}_l(x,\hat{\beta},\hat{\Sigma}) - \bar{c}_l(x,\beta_0,\Sigma)|}{\sigma(x)} \le \varepsilon_n\right) \to 1,$$

$$P\left(\sup_{x\in\mathcal{X}}\frac{|\bar{c}_u(x,\hat{\beta},\hat{\Sigma})-\bar{c}_u(x,\beta_0,\Sigma)|}{\sigma(x)}\leq\varepsilon_n\right)\to 1,$$

and

$$P\left(\sup_{x\in\mathcal{X}}\sup_{\beta:\|\beta-\beta_0\|\|\leq\|\hat{\beta}-\beta_0\|}\|\nabla_\beta g(x,\tilde{\beta})-\nabla_\beta g(x,\beta_0)\|\|\sqrt{n}(\beta_0-\hat{\beta})\|\|\sigma(x)^{-1}|\leq\varepsilon_n\right)\to 1.$$

Then

$$\begin{aligned}
CR &\equiv P\left(\bar{g}_l(x,\hat{\beta},\hat{\Sigma})\leq g(x,\beta_0)\leq\bar{g}_u(x,\hat{\beta},\hat{\Sigma})\,\forall x\in\mathcal{X}\right)\\
&= P\left(g(x,\hat{\beta})+\bar{c}_l(x,\hat{\beta},\hat{\Sigma})/\sqrt{n}\leq g(x,\beta_0)\leq g(x,\hat{\beta})+\bar{c}_u(x,\hat{\beta},\hat{\Sigma})/\sqrt{n}\,\forall x\in\mathcal{X}\right)\\
&= P\left(\frac{\bar{c}_l(x,\hat{\beta},\hat{\Sigma})}{\sigma(x)}\leq\frac{\nabla_\beta g(x,\tilde{\beta})'\sqrt{n}(\beta_0-\hat{\beta})}{\sigma(x)}\leq\frac{\bar{c}_u(x,\hat{\beta},\hat{\Sigma})}{\sigma(x)}\,\forall x\in\mathcal{X}\right)\\
&\leq P\left(\frac{\bar{c}_l(x,\hat{\beta},\hat{\Sigma})}{\sigma(x)}-\varepsilon_n\leq\frac{\nabla_\beta g(x,\beta_0)'\sqrt{n}(\beta_0-\hat{\beta})}{\sigma(x)}\leq\frac{\bar{c}_u(x,\hat{\beta},\hat{\Sigma})}{\sigma(x)}+\varepsilon_n\,\forall x\in\mathcal{X}\right)+o(1)\\
&\leq P\left(\frac{\bar{c}_l(x,\beta_0,\Sigma)}{\sigma(x)}-2\varepsilon_n\leq\frac{\nabla_\beta g(x,\beta_0)'\sqrt{n}(\beta_0-\hat{\beta})}{\sigma(x)}\leq\frac{\bar{c}_u(x,\beta_0,\Sigma)}{\sigma(x)}+2\varepsilon_n\,\forall x\in\mathcal{X}\right)+o(1)\\
&= P\left(\frac{\bar{c}_l(x,\beta_0,\Sigma)}{\sigma(x)}-2\varepsilon_n\leq\frac{\nabla_\beta g(x,\beta_0)'\Sigma^{1/2}Z}{\sigma(x)}\leq\frac{\bar{c}_u(x,\beta_0,\Sigma)}{\sigma(x)}+2\varepsilon_n\,\forall x\in\mathcal{X}\right)+o(1)\\
&= P\left(\frac{\bar{c}_l(x,\beta_0,\Sigma)}{\sigma(x)}\leq\frac{\nabla_\beta g(x,\beta_0)'\Sigma^{1/2}Z}{\sigma(x)}\leq\frac{\bar{c}_u(x,\beta_0,\Sigma)}{\sigma(x)}\,\forall x\in\mathcal{X}\right)+o(1)\\
&= 1-\alpha+o(1).
\end{aligned}$$

The second line follows from the mean value theorem, where $\tilde{\beta}$ is an intermediate value and $\|\tilde{\beta}-\beta_0\|\leq\|\hat{\beta}-\beta_0\|$. For the third line notice that

$$\left|\frac{\left(\nabla_\beta g(x,\tilde{\beta})'-\nabla_\beta g(x,\beta_0)'\right)\sqrt{n}(\beta_0-\hat{\beta})}{\sigma(x)}\right|\leq\|\nabla_\beta g(x,\tilde{\beta})-\nabla_\beta g(x,\beta_0)\|\|\sqrt{n}(\beta_0-\hat{\beta})\|\|\sigma(x)^{-1}|$$

and hence

$$P\left(\sup_{x\in\mathcal{X}}\left|\frac{\left(\nabla_\beta g(x,\tilde{\beta})'-\nabla_\beta g(x,\beta_0)'\right)\sqrt{n}(\beta_0-\hat{\beta})}{\sigma(x)}\right|\leq\varepsilon_n\right)\to 1.$$

The remaining lines follow similar steps to the ones in the proof of Lemma 2.

Analogous arguments yield $CR\geq 1-\alpha+o(1)$ and hence

$$P\left(\bar{g}_l(x,\hat{\beta},\hat{\Sigma})\leq g(x,\beta_0)\leq\bar{g}_u(x,\hat{\beta},\hat{\Sigma})\,\forall x\in\mathcal{X}\right)\to 1-\alpha.$$

□

Proof of Theorem 5. Let $\varepsilon_n \to 0$ such that $K_n \varepsilon_n \to 0$ and for n large enough

$$\sup_{x\in\mathcal{X}} \frac{|p(x)'E(\hat{\beta}) - g_0(x)|}{\sigma(x)} \leq \varepsilon_n$$

for n large enough and

$$P\left(\sup_{x\in\mathcal{X}} \frac{|\bar{c}_l(x,\hat{\Sigma}) - \bar{c}_l(x,\Sigma)|}{\sigma(x)} \leq \varepsilon_n\right) \to 1 \text{ and } P\left(\sup_{x\in\mathcal{X}} \frac{|\bar{c}_u(x,\hat{\Sigma}) - \bar{c}_u(x,\Sigma)|}{\sigma(x)} \leq \varepsilon_n\right) \to 1.$$

We now get, similar as in the proof of Theorem 4,

$$\begin{aligned}
CR &\equiv P\left(g_l(x,\hat{\beta}_{K_n},\hat{\Sigma}) \leq g_0(x) \leq g_l(x,\hat{\beta}_{K_n},\hat{\Sigma})\, \forall x \in \mathcal{X}\right) \\
&= P\left(\frac{\bar{c}_l(x,\hat{\Sigma})}{\sigma(x)} \leq \frac{p(x)'\sqrt{n}(E(\hat{\beta}_{K_n}) - \hat{\beta}_{K_n})}{\sigma(x)} + \frac{\sqrt{n}(g_0(x) - p(x)'E(\hat{\beta}_{K_n}))}{\sigma(x)} \leq \frac{\bar{c}_u(x,\hat{\Sigma})}{\sigma(x)}\, \forall x \in \mathcal{X}\right) \\
&\leq P\left(\frac{\bar{c}_l(x,\hat{\Sigma})}{\sigma(x)} - \varepsilon_n \leq \frac{p(x)'\sqrt{n}(E(\hat{\beta}_{K_n}) - \hat{\beta}_{K_n})}{\sigma(x)} \leq \frac{\bar{c}_u(x,\hat{\Sigma})}{\sigma(x)} + \varepsilon_n\, \forall x \in \mathcal{X}\right) + o(1) \\
&\leq P\left(\frac{\bar{c}_l(x,\Sigma)}{\sigma(x)} - 2\varepsilon_n \leq \frac{p(x)'\sqrt{n}(E(\hat{\beta}_{K_n}) - \hat{\beta}_{K_n})}{\sigma(x)} \leq \frac{\bar{c}_u(x,\Sigma)}{\sigma(x)} + 2\varepsilon_n\, \forall x \in \mathcal{X}\right) + o(1) \\
&= P\left(\frac{\bar{c}_l(x,\Sigma)}{\sigma(x)} - 2\varepsilon_n \leq \frac{p(x)'\Sigma^{1/2}Z}{\sigma(x)} \leq \frac{\bar{c}_u(x,\Sigma)}{\sigma(x)} + 2\varepsilon_n\, \forall x \in \mathcal{X}\right) + o(1) \\
&= 1 - \alpha + o(1).
\end{aligned}$$

Analogous arguments yield $CR \geq 1 - \alpha + o(1)$ and hence

$$P\left(\bar{g}_l(x,\hat{\beta}_{K_n},\hat{\Sigma}) \leq g(x,\beta_0) \leq \bar{g}_u(x,\hat{\beta}_{K_n},\hat{\Sigma})\, \forall x \in \mathcal{X}\right) \to 1 - \alpha.$$

□

S.2.2 Proofs of lemmas from Section A

Proof of Lemma A1. Let C_s be a constant and define

$$S_u = \{\gamma \in \mathbb{R}^K : |\gamma_k| \leq C_s \text{ for all } k = 1,\ldots,K\},$$

$$S_J = \{\gamma \in \mathbb{R}^K : \bar{c}_l^J(x) \leq p(x)'\Sigma^{1/2}\gamma \leq \bar{c}_u^J(x) \text{ for all } x \in \mathcal{X}_J\}$$

and

$$S = \{\gamma \in \mathbb{R}^K : \bar{c}_l(x) \leq p(x)'\Sigma^{1/2}\gamma \leq \bar{c}_u(x) \text{ for all } x \in \mathcal{X}\}.$$

We first show that if $S \subseteq S_u$ and $S_J \subseteq S_u$ for some $C_s < \infty$, then

$$\left|\int (\bar{c}_u(x) - \bar{c}_l(x))w_X(x)dx - \int (\bar{c}_u^J(x) - \bar{c}_l^J(x))w_X(x)dx\right| \to 0.$$

To do so, let $\varepsilon > 0$. Also notice that, by definition, $\bar{c}_u(x) = \sup_{\gamma \in S} p(x)'\gamma$. Therefore, for all $x, x' \in \mathcal{X}$ it holds that

$$\begin{aligned} |\bar{c}_u(x) - \bar{c}_u(x')| &= |\sup_{\gamma \in S} p(x)'\gamma - \sup_{\gamma \in S} p(x')'\gamma| \\ &\leq \sup_{\gamma \in S} |p(x)'\gamma - p(x')'\gamma| \\ &\leq \sup_{\gamma \in S} |\nabla p(\tilde{x})'\gamma(x - x')| \\ &\leq \sup_{\gamma \in S_u} \|\gamma\| \sup_{\tilde{x} \in \mathcal{X}} \|\nabla p(\tilde{x})\| |x - x'|, \end{aligned}$$

where the last line follows from the assumption that $S \subseteq S_u$. Hence,

$$\sup_{x, x' \in \mathcal{X}, x \neq x'} \frac{|\bar{c}_u(x) - \bar{c}_u(x')|}{|x - x'|} \leq \sup_{\gamma \in S_u} \|\gamma\| \sup_{x \in \mathcal{X}} \|\nabla p(x)\|.$$

Moreover,

$$\bar{c}_u(x) \leq \|p(x)\| \sup_{\gamma \in S_u} \|\gamma\|.$$

It follows that for all $x, x' \in \mathcal{X}$

$$\begin{aligned} \frac{|\bar{c}_u(x)w(x) - \bar{c}_u(x')w(x')|}{|x - x'|} &\leq \frac{|(\bar{c}_u(x) - c(x'))w(x)| + |(w(x) - w(x'))\bar{c}_u(x')|}{|x - x'|} \\ &\leq C \sup_{\gamma \in S_u} \|\gamma\| \sup_{x \in \mathcal{X}} \|\nabla p(x)\| + C\|p(x')\| \sup_{\gamma \in S_u} \|\gamma\| \\ &\leq 2C^2 \sup_{\gamma \in S_u} \|\gamma\|. \end{aligned}$$

Next write

$$\begin{aligned} &\left| \int \bar{c}_u(x) w_X(x) dx - \sum_{j=1}^{J-1} \bar{c}_u(x_j) w_X(x_j)(x_{j+1} - x_j) \right| \\ &= \left| \sum_{j=1}^{J-1} \int_{x_j}^{x_{j+1}} \left(\bar{c}_u(x) w_X(x) - \bar{c}_u(x_j) w_X(x_j) \right) dx \right| \\ &\leq \left| \sum_{j=1}^{J-1} \int_{x_j}^{x_{j+1}} 2C^2 \sup_{\gamma \in S_u} \|\gamma\| |x_{j+1} - x_j| dx \right| \\ &= 2C^2 \sup_{\gamma \in S_u} \|\gamma\| \sum_{j=1}^{J-1} (x_{j+1} - x_j)^2. \end{aligned}$$

It follows that

$$\left| \int \bar{c}_u(x) w_X(x) dx - \sum_{j=1}^{J-1} \bar{c}_u(x_j) w_X(x_j)(x_{j+1} - x_j) \right| \to 0$$

Identical arguments imply that

$$\left| \int \bar{c}_u^J(x) w_X(x) dx - \sum_{j=1}^{J-1} \bar{c}_u^J(x_j) w_X(x_j)(x_{j+1} - x_j) \right| \to 0$$

and we get analogous results for $\bar{c}_l(x)$ and $\bar{c}_l^J(x)$. Theorem 2 now implies that

$$\left| \int (\bar{c}_u(x) - \bar{c}_l(x)) w_X(x) dx - \int (\bar{c}_u^J(x) - \bar{c}_l^J(x)) w_X(x) dx \right| \to 0.$$

Next notice that

$$p(x)'\Sigma^{1/2} Z = \tilde{p}(x)' B' \Sigma^{1/2} Z = \tilde{p}(x)' \tilde{Z},$$

where $\tilde{Z} = B'\Sigma^{1/2} Z \sim N(0, B'\Sigma B)$. It follows that we can write the constraint

$$P(c_l(x) \leq p(x)'\Sigma^{1/2} Z \leq c_u(x) \text{ for all } x \in \mathcal{X}) = 1 - \alpha$$

as

$$P(c_l(x) \leq \tilde{p}(x)'\tilde{\Sigma}^{1/2} W \leq c_u(x) \text{ for all } x \in \mathcal{X}) = 1 - \alpha,$$

where $W \sim N(0, I_{L \times L})$ and $\tilde{\Sigma}^{1/2} = (B'\Sigma B)^{1/2}$. Moreover, the confidence band is a projection of $p(x)'\beta$ on

$$\{\beta \in \mathbb{R}^K : \bar{c}_l(x) \leq p(x)'(\beta - \hat{\beta}) \leq \bar{c}_u(x) \text{ for all } x \in \mathcal{X}\},$$

or equivalently a projection of $\tilde{p}(x)'\beta$ on

$$\{\beta \in \mathbb{R}^L : \bar{c}_l(x) \leq \tilde{p}(x)'(\beta - \tilde{\beta}) \leq \bar{c}_u(x) \text{ for all } x \in \mathcal{X}\},$$

where $\tilde{\beta} = B'\hat{\beta} \sim N(0, B'\Sigma B)$. Since we can always rewrite the problem using the L dimensional vector $\tilde{p}(x)$, it is sufficient to prove that there exists $C_s < \infty$ such that $S \subseteq S_u$ and $S_J \subseteq S_u$ when $B = I_{K \times K}$.

Now notice that from arguments in the proof of Corollary 1, the band resulting from the projection on $\{\beta : (\hat{\beta} - \beta)\Sigma^{-1}(\hat{\beta} - \beta) \leq c_{K,1-\alpha}\}$, where $c_{K,1-\alpha}$ is the $1 - \alpha$ quantile of the χ^2_K distribution, leads to a conservative band of the form $p(x)'\hat{\beta} \pm c_{K,1-\alpha}\sigma(x)$. The weighted area of this band is

$$2c_{K,1-\alpha} \int \sqrt{p(x)\Sigma p(x)} w_X(x) dx \leq 2c_{K,1-\alpha} \lambda_{max}(\Sigma^{1/2}) \int \|p(x)\| w_X(x) dx < \infty,$$

where $\lambda_{max}(\Sigma^{1/2})$ denotes the largest eigenvalue of $\Sigma^{1/2}$. Define

$$\bar{U} = 2c_{K,1-\alpha} \lambda_{max}(\Sigma^{1/2}) \int \|p(x)\| w_X(x) dx.$$

Let $\gamma \in S$. Then for each $x \in \mathcal{X}$ either $\bar{c}_u(x) \geq |p(x)'\gamma|$ or $\bar{c}_l(x) \leq -|p(x)'\gamma|$. Also notice that $\bar{c}_u(x) \geq 0$ and $\bar{c}_l(x) \leq 0$. It follows from Assumption 8 that for some $\varepsilon > 0$

$$\int (\bar{c}_u(x) - \bar{c}_l(x)) w_X(x) dx \geq \int |p(x)'\gamma| w_X(x) dx \geq \varepsilon \|\gamma\|.$$

Since we also established $\int (\bar{c}_u(x) - \bar{c}_l(x)) w_X(x) d(x) \leq \bar{U}$, it follows that $\|\gamma\| \leq \frac{\bar{U}}{\varepsilon}$ and therefore there exists C_s such that

$$S \subseteq \{\gamma \in \mathbb{R}^K : -C_s \leq \gamma_k \leq C_s \text{ for all } k = 1, \ldots, K\}.$$

Similarly, projecting on the set $\{\beta : (\hat{\beta} - \beta)\Sigma^{-1}(\hat{\beta} - \beta) \leq c_{K,1-\alpha}\}$ for all $p(x_j)$ yields a conservative band and the objective function h_J for this band is

$$2c_{K,1-\alpha} \sum_{j=1}^{J-1} \sqrt{p(x_j)\Sigma p(x_j)} w_X(x_j)(x_{j+1} - x_j).$$

Arguments as in the first part imply that

$$\left| 2c_{K,1-\alpha} \sum_{j=1}^{J-1} \sqrt{p(x_j)\Sigma p(x_j)} w_X(x_j)(x_{j+1} - x_j) - 2c_{K,1-\alpha} \int \sqrt{p(x)\Sigma p(x)} w_X(x) dx \right| \to 0.$$

Therefore, for J large enough

$$\sum_{j=1}^{J-1} (\bar{c}_u^J(x_j) - \bar{c}_l^J(x_j)) w_X(x_j)(x_{j+1} - x_j) \leq 2c_{K,1-\alpha} \sum_{j=1}^{J-1} \sqrt{p(x_j)\Sigma p(x_j)} w_X(x_j)(x_{j+1} - x_j) \leq \bar{U} + 1.$$

Now let $\gamma \in S_J$ and write $\bar{\gamma} = \alpha\gamma$. Then, similar as before there exists $\varepsilon > 0$ such that $\|\gamma\| \leq \frac{\bar{U}+1}{\varepsilon}$ for all $\bar{\gamma} \in S_J$. Thus, there exists a C_s such that

$$S_J \subseteq \{\gamma \in \mathbb{R}^K : -C_s \leq \gamma_k \leq C_s \text{ for all } k = 1, \ldots, K\}.$$

The conclusion now follows from the first part. $\square$

Proof of Lemma A2. As in the proof of Lemma A1, we can assume without loss of generality that $B = I_{K \times K}$ because otherwise we can simply work with a transformed problem. Analogously, we can assume without loss of generality that $\Sigma = I_{K \times K}$ (but $\hat{\Sigma} \neq I_{K \times K}$ in general).

The proof proceeds in several steps. First, we show that the bands are based on projections on bounded sets and that the weighted areas corresponding to the optimal solutions are bounded from above and below (in probability). We then show that the optimal bands are symmetric. Next we show that

$$\left| \int \bar{c}_u(x, \hat{\Sigma}) w_X(x) dx - \int \bar{c}_u(x, \Sigma) w_X(x) dx \right| \xrightarrow{p} 0.$$

Finally, we show a contradiction if for some $\varepsilon > 0$ and $\delta > 0$ and n large enough

$$P\left(\sup_{x \in \mathcal{X}} |\bar{c}_u(x, \hat{\Sigma}) - \bar{c}_u(x, \Sigma)| > \varepsilon \right) > \delta.$$

For the first step, arguments as in the proof of Lemma A1 imply that there exists a constant C_s such that $S(\Sigma) \subseteq S_u$, where

$$S_u = \{\gamma \in \mathbb{R}^K : |\gamma_k| \leq C_s \text{ for all } k = 1, \ldots, K\},$$

and

$$S(\Sigma) = \{\gamma \in \mathbb{R}^K : \bar{c}_l(x, \Sigma) \leq p(x)'\Sigma^{1/2}\gamma \leq \bar{c}_u(x, \Sigma) \text{ for all } x \in \mathcal{X}\}.$$

Analogous arguments imply that there is a constant C_s such that $S(\hat{\Sigma}) \subseteq S_u$ with probability approaching 1. From the proof Lemma A1 it also follows that $\int \left(\bar{c}_u(x, \Sigma) - \bar{c}_l(x, \Sigma)\right) w_X(x) dx$ is bounded and that $\int (\bar{c}_u(x, \hat{\Sigma}) - \bar{c}_l(x, \hat{\Sigma})) w_X(x) dx$ is bounded by a constant with probability approaching 1.

Now notice that $\bar{c}_u(x, \Sigma) \geq \sigma(x) c_{1-\alpha}$ and $\bar{c}_l(x, \Sigma) \leq -\sigma(x) c_{1-\alpha}$, where $c_{1-\alpha}$ denotes the $1 - \alpha$ quantile of the standard normal distribution. Therefore,

$$\int \left(\bar{c}_u(x, \Sigma) - \bar{c}_l(x, \Sigma)\right) w_X(x) dx \geq 2c_{1-\alpha} \int \sigma(x) w_X(x) dx > 0$$

and

$$\int \left(\bar{c}_u(x, \hat{\Sigma}) - \bar{c}_l(x, \hat{\Sigma})\right) w_X(x) dx \geq c_{1-\alpha} \int \sigma(x) w_X(x) dx + o_p(1).$$

We now prove that $\bar{c}_l(x, \Sigma) = -\bar{c}_u(x, \Sigma)$. Suppose $\bar{c}_u(x, \Sigma) \neq -\bar{c}_l(x, \Sigma)$ for some $x \in \mathcal{X}$. Let

$$\tilde{c}_l(x, \Sigma) = \bar{c}_l(x, \Sigma) - 1/2(\bar{c}_l(x, \Sigma) + \bar{c}_u(x, \Sigma)) = 1/2\bar{c}_l(x, \Sigma) - 1/2\bar{c}_u(x, \Sigma)$$

and

$$\tilde{c}_u(x, \Sigma) = \bar{c}_u(x, \Sigma) - 1/2(\bar{c}_l(x, \Sigma) + \bar{c}_u(x, \Sigma)) = 1/2\bar{c}_u(x, \Sigma) - 1/2\bar{c}_l(x, \Sigma).$$

Let

$$S_1 = \{z \in \mathbb{R}^K : \bar{c}_l(x, \Sigma) \leq p(x)'\Sigma^{1/2} z \leq \bar{c}_u(x, \Sigma) \text{ for all } x \in \mathcal{X}\}$$

and

$$S_2 = \{z \in \mathbb{R}^K : -\bar{c}_u(x, \Sigma) \leq p(x)'\Sigma^{1/2} z \leq -\bar{c}_l(x, \Sigma) \text{ for all } x \in \mathcal{X}\}.$$

By the symmetry of the normal measure, $P(Z \in S_1) = P(Z \in S_2) = 1 - \alpha$. It follows from Lemmas S3 and S4 that

$$P(\tilde{c}_l(x, \Sigma) \leq p(x)'\Sigma^{1/2} Z \leq \tilde{c}_u(x, \Sigma) \text{ for all } x \in \mathcal{X}) > 1 - \alpha.$$

But

$$\int \left(\tilde{c}_u(x, \Sigma) - \tilde{c}_l(x, \Sigma)\right) w_X(x) dx = \int \left(\bar{c}_u(x, \Sigma) - \bar{c}_l(x, \Sigma)\right) w_X(x) dx,$$

which contradicts that $(\bar{c}_l(x, \Sigma), \bar{c}_u(x, \Sigma))$ is optimal. Therefore, $\bar{c}_u(x, \Sigma) = -\bar{c}_l(x, \Sigma)$.

Next we prove that

$$\left| \int \bar{c}_u(x, \hat{\Sigma}) w_X(x) dx - \int \bar{c}_u(x, \Sigma) w_X(x) dx \right| \xrightarrow{p} 0.$$

Let c_1 be a constant such that

$$P(-c_1 \bar{c}_u(x, \Sigma) \leq p(x)'\hat{\Sigma}^{1/2} Z \leq c_1 \bar{c}_u(x, \Sigma) \text{ for all } x \in \mathcal{X} \mid \hat{\Sigma}) = 1 - \alpha.$$

Since

$$|p(x)'(\hat{\Sigma}^{1/2} - \Sigma^{1/2})Z| \leq \|p(x)\| \|\hat{\Sigma}^{1/2} - \Sigma^{1/2}\| \|Z\| = o_p(1),$$

it holds that $|c_1 - 1| = o_p(1)$. Since $\bar{c}_u(x, \hat{\Sigma})$ is optimal and $c_1 \bar{c}_u(x, \Sigma)$ is feasible, it holds that

$$\begin{aligned}
\int \bar{c}_u(x, \hat{\Sigma}) w_X(x) dx &\leq \int \bar{c}_u(x, \Sigma) w_X(x) dx + (c_1 - 1) \int \bar{c}_u(x, \Sigma) w_X(x) dx \\
&\leq \int \bar{c}_u(x, \Sigma) w_X(x) dx + o_p(1).
\end{aligned}$$

Similarly, $|c_2 - 1| = o_p(1)$, where c_2 is such that

$$P(-c_2 \bar{c}_u(x, \hat{\Sigma}) \leq p(x)'\Sigma^{1/2} Z \leq c_2 \bar{c}_u(x, \hat{\Sigma}) \text{ for all } x \in \mathcal{X} \mid \hat{\Sigma}) = 1 - \alpha.$$

Thus,

$$\begin{aligned}
\int \bar{c}_u(x, \Sigma) w_X(x) dx &\leq \int \bar{c}_u(x, \hat{\Sigma}) w_X(x) dx + (c_2 - 1) \int \bar{c}_u(x, \hat{\Sigma}) w_X(x) dx \\
&\leq \int \bar{c}_u(x, \hat{\Sigma}) w_X(x) dx + o_p(1).
\end{aligned}$$

Together

$$\left| \int \bar{c}_u(x, \hat{\Sigma}) w_X(x) dx - \int \bar{c}_u(x, \Sigma) w_X(x) dx \right| \xrightarrow{p} 0.$$

Since we assume that $\inf_{x \in \mathcal{X}} \sigma(x) > 0$, it is sufficient to prove that

$$\sup_{x \in \mathcal{X}} |\bar{c}_u(x, \hat{\Sigma}) - \bar{c}_u(x, \Sigma)| = o_p(1).$$

Let $\varepsilon > 0$ and $\delta > 0$ and suppose that

$$P\left(\sup_{x\in\mathcal{X}} |\bar{c}_u(x,\hat{\Sigma}) - \bar{c}_u(x,\Sigma)| > \varepsilon\right) > \delta.$$

Define

$$C(\Sigma) = \{\gamma \in \mathbb{R}^K : -\bar{c}_u(x,\Sigma) \leq p(x)'\gamma \leq \bar{c}_u(x,\Sigma) \text{ for all } x \in \mathcal{X}\}.$$

Also let $\lambda \in (0,1)$ and define

$$\bar{c}_u^\lambda(x,\hat{\Sigma}) = \lambda\bar{c}_u(x,\hat{\Sigma}) + (1-\lambda)\bar{c}_u(x,\Sigma)$$

and

$$\hat{C}^\lambda(\hat{\Sigma}) = \{\gamma \in \mathbb{R}^K : -\bar{c}_u^\lambda(x,\hat{\Sigma}) \leq p(x)'\gamma \leq \bar{c}_u^\lambda(x,\hat{\Sigma}) \text{ for all } x \in \mathcal{X}\}.$$

Let $Z \sim N(0, I_{K\times K})$. Since $P(\sup_{x\in\mathcal{X}} |\bar{c}_u(x,\hat{\Sigma}) - \bar{c}_u(x,\Sigma)| > \varepsilon) > \delta$, by Lemmas S3 and S4 there exists a constant $\eta > 0$ such that with probability at least $\delta/2$ and for n large enough

$$P(\hat{\Sigma}^{1/2}Z \in \hat{C}^\lambda(\hat{\Sigma}) \mid \hat{\Sigma}) \geq P(\hat{\Sigma}^{1/2}Z \in C(\hat{\Sigma}) \mid \hat{\Sigma})^\lambda P(\hat{\Sigma}^{1/2}Z \in C(\Sigma) \mid \hat{\Sigma})^{1-\lambda} + \eta$$

Notice that $P(\hat{\Sigma}^{1/2}Z \in C(\hat{\Sigma}) \mid \hat{\Sigma}) = 1-\alpha$. Moreover, since $P(\Sigma^{1/2}Z \in C(\Sigma)) = 1-\alpha$ and

$$|p(x)'(\hat{\Sigma}^{1/2} - \Sigma^{1/2})Z| \leq \|p(x)\|\|\hat{\Sigma}^{1/2} - \Sigma^{1/2}\|\|Z\| = o_p(1),$$

$P(\hat{\Sigma}^{1/2}Z \in C(\Sigma) \mid \hat{\Sigma}) = 1-\alpha+o_p(1)$. It follows that with probability at least $\delta/2$ and for n large enough

$$P(\hat{\Sigma}^{1/2}Z \in \hat{C}^\lambda \mid \hat{\Sigma}) \geq 1-\alpha+\eta+o_p(1).$$

Next let $c = \frac{1-\alpha+\eta/2}{1-\alpha+\eta} \in (0,1)$ and let

$$\tilde{c}^\lambda(x,\hat{\Sigma}) = c(\lambda\bar{c}_u(x,\hat{\Sigma}) + (1-\lambda)\bar{c}_u(x,\Sigma)).$$

Then with probability at least $\delta/2$ and for n large enough

$$\begin{aligned}
&P(-\tilde{c}^\lambda(x,\hat{\Sigma}) \leq p(x)'\hat{\Sigma}^{1/2}Z \leq \tilde{c}^\lambda(x,\hat{\Sigma}) \text{ for all } x \in \mathcal{X} \mid \hat{\Sigma}) \\
&\quad \geq cP(-\bar{c}_u^\lambda(x,\hat{\Sigma}) \leq p(x)'\hat{\Sigma}^{1/2}Z \leq \bar{c}_u^\lambda(x,\hat{\Sigma}) \text{ for all } x \in \mathcal{X} \mid \Sigma) \\
&\quad \geq c(1-\alpha+\eta+o_p(1)) \\
&\quad = 1-\alpha+\eta/2+o_p(1).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int \tilde{c}(x,\hat{\Sigma})w_X(x)dx \\
&\quad = c\int \bar{c}_u(x,\hat{\Sigma})w_X(x)dx + c(1-\lambda)\left(\int \bar{c}_u(x,\Sigma)w_X(x)dx - \int \bar{c}_u(x,\hat{\Sigma})w_X(x)dx\right) \\
&\quad = c\int \bar{c}_u(x,\hat{\Sigma})w_X(x)dx + o_p(1).
\end{aligned}$$

It follows that with probability at least $\delta/4$ and for n large enough

$$P(-\tilde{c}^{\lambda}(x,\hat{\Sigma}) \leq p(x)'\hat{\Sigma}^{1/2}Z \leq \tilde{c}^{\lambda}(x,\hat{\Sigma}) \text{ for all } x \in \mathcal{X} \mid \hat{\Sigma}) > 1-\alpha$$

and

$$\int \tilde{c}(x,\hat{\Sigma})w_X(x)dx < \int \bar{c}_u(x,\hat{\Sigma})w_X(x)dx,$$

which would contradict that $\bar{c}_u(x,\hat{\Sigma})$ is optimal.

It therefore has to hold that

$$P\left(\sup_{x\in\mathcal{X}} |\bar{c}_u(x,\hat{\Sigma}) - \bar{c}_u(x,\Sigma)| > \varepsilon\right) \to 0$$

which means that

$$\sup_{x\in\mathcal{X}} |\bar{c}_u(x,\hat{\Sigma}) - \bar{c}_u(x,\Sigma)| = o_p(1).$$

□

References

Bogachev, V. I. (1998). *Gaussian Measures*. Mathematical surveys and monographs. American Mathematical Society.