Econ 805 – Advanced Micro Theory I
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Thus far, we've made four key assumptions that have greatly simplified our analysis:

1. Risk-neutral bidders
2. Ex-ante symmetric bidders
3. Independent types
4. Private values

These have bought us a lot – we proved revenue equivalence, and solved for the optimal auction out of every possible feasible auction mechanism. But they're also restrictive assumptions, and revenue equivalence fails unless we have all of them. There's a significant literature devoted to what happens when you relax each of these assumptions, generally one or two at a time. Since most common auctions are either first-price, second-price, or ascending auctions, much of this literature compares these formats to each other – in terms of revenue, efficiency, etc. – when each of these is relaxed.

When bidders are risk-averse, things get complicated. Revenue equivalence fails. There's a paper by Maskin and Riley, on the syllabus, characterizing the optimal auction under risk aversion, which is a pretty complicated object. (Basically, the seller can use the bidders' risk-aversion in two ways. First, he can effectively sell the bidders insurance – increase their payoffs when they lose the auction and decrease it when they win – and charge them for this. And second, he can use risk-aversion to extract more surplus from the high types. Since these are at odds which each other, he can't do either perfectly, but the optimal auction has aspects of both.)

There's also a general result that when bidders are risk-averse, first-price auctions outperform second-price auctions. (Intuition.) I may base a homework problem on this.

We already showed that when bidders are risk-neutral, risk-averse sellers prefer first-price auctions.

We've already seen the optimal auction with asymmetric bidders. There's also a nice paper by Maskin and Riley (also on the syllabus) comparing first- and second-price auctions when bidders are asymmetric. They find that when one bidder's types are drawn from a stochastically higher distribution than the other's, the "strong" bidder prefers second-price auctions and the "weak" bidder prefers first-price auctions; but which one raises more revenue depends.

The last two assumptions – independent signals and private values – are both relaxed in a fantastic paper by Milgrom and Weber (1982, Econometrica). They introduce the affiliated, interdependent values framework. It's very general, but in particular, it nests two special cases that have received a lot of attention:

- Private values, which we've been looking at already, but with values allowed to be positively correlated across bidders
- Common values – where ex post, the bidders all value the object the same; but this true value is uncertain, and each bidder has different information about it. (This is commonly used as a model of auctions for natural resource rights. Right to drill for oil on a tract of government-owned land is likely the same for every oil company, depends on how much oil is underground; each company might drill a couple of test holes to sample, so each has a different estimate of the value of the "object" up for bid.)

This week, we'll develop the Milgrom and Weber model (also covered in PATW section 5.4.) After we study this model, we'll then look at the special case of common values, some of which predated the Milgrom and Weber results.

Before we introduce the Milgrom and Weber model, however, we need one result that I had hoped to cover earlier but didn't get to.

We say a probability distribution F **first-order stochastically dominates** another one G if $F(t) \leq G(t)$ for every t.

Lemma 1. *Let X and Y be random variables with distributions F and G. If F first-order stochastically dominates G, then $Eu(X) \geq Eu(y)$ for any increasing function u.*

When u is differentiable, there's an elegant proof, similar to the one we used for second-order stochastic dominance. Define the step functions

$$1_k(t) = \begin{cases} 1 & if \quad t \geq k \\ 0 & if \quad t < k \end{cases}$$

Note that if F first-order stochastically dominates G, then

$$E1_k(X) = \int_{-\infty}^{\infty} 1_k(t)f(t)dt = \int_{-\infty}^{k} 0f(t)dt + \int_{k}^{\infty} 1f(t)dt = 1 - F(k) \geq 1 - G(k) = E1_k(Y)$$

so we have the result when u is one of these step functions. To prove it for general differentiable u, write

$$u(t) = u(-\infty) + \int_{-\infty}^{t} u'(s)ds = K + \int_{-\infty}^{\infty} u'(s)1_s(t)ds$$

so

$$\begin{aligned} Eu(x) &= \textstyle\int_{-\infty}^{\infty} u(t)f(t)dt \\ &= \textstyle\int_{-\infty}^{\infty} \left(K + \int_{-\infty}^{\infty} u'(s)1_s(t)ds\right) f(t)dt \\ &= K + \textstyle\int_{-\infty}^{\infty} u'(s) \left(\int_{-\infty}^{\infty} 1_s(t)f(t)dt\right) ds \\ &\geq K + \textstyle\int_{-\infty}^{\infty} u'(s) \left(\int_{-\infty}^{\infty} 1_s(t)g(t)dt\right) ds \\ &= Eu(y) \end{aligned}$$

So if X first-order stochastically dominates Y (that is, $F(t) \leq G(t)$ for all t), then $Eu(X) \geq Eu(Y)$ for any increasing function u.

Affiliated Interdependent Values Model

The general setup for the Milgrom-Weber model is that there are N risk-neutral bidders. Each bidder i gets a signal t_i; the value of the object to bidder i given these signals is

$$v_i(t_1, t_2, \ldots, t_N, t_0)$$

where t_0 indicates information that is not available to any of the bidders. (This could be information the seller has, or information that nobody has. t_0 is allowed to be multi-dimensional – that is, it could consist of several different attributes of the good – but this doesn't end up making a difference, so we'll treat it as a single variable for simplicity. As before, individual bidders' signals t_i must be one-dimensional.)

Milgrom and Weber make the following assumptions about v_i:

- v_i is nonnegative and continuous
- v_i is nondecreasing in all its arguments (so good news for one bidder is good news for all the bidders)
- v_i is symmetric, in the following way:

  $$v_i(t_i = x, t_{-i} = \mathbf{y}, t_0 = z) = v_j(t_j = x, t_{-j} = \mathbf{y}, t_0 = z)$$

  and

  $$v_i(t_i = x, t_{-i} = \mathbf{y}, t_0 = z) = v_i(t_i = x, t_{-i} = \sigma(\mathbf{y}), t_0 = z)$$

  where σ is any permutation. That is, each bidder's valuation responds in the same way to t_0; responds in the same way to his own signal; responds in the same way to the other bidders' signals; and does not respond to which of his opponents had which signal, just what they all are

Given this symmetry, we can rewrite bidder i's valuation as

$$v_i(t_i = x, t_{-i} = \mathbf{y}, t_0 = z) = v(x, y^{(1)}, y^{(2)}, \ldots, y^{(N-1)}, t_0)$$

where $y^{(i)}$ is the ith-highest of the $N-1$ elements of $\mathbf{y}$. That is, we can rewrite bidder i's valuation as a function of his own signal, t_0, and the order statistics of his opponents' signals; and by the symmetry assumption, this function is the same for every bidder.

(Note that private values is simply the special case where $v(x, y^{(1)}, y^{(2)}, \ldots, y^{(N-1)}, t_0) = g(x)$ for some function g, and pure common values is the special case where $v_i(t_i = x, t_j = y, t_{-ij} = z, t_0) = v_j(t_i = x, t_j = y, t_{-ij} = z, t_0)$.)

The final two assumptions in the setup of the Milgrom-Weber model are:

- The distribution of $(t_0, t_1, \ldots, t_N)$ is **symmetric in the last N arguments**
- $(t_0, t_1, \ldots, t_N)$ are **affiliated**

Affiliation is a technical condition which implies the variables are all positively correlated. We'll spend the rest of today on affiliation and its implications, and then come back to the auction model on Thursday.

(Sadly, that means the rest of today is going to be a lot of gruntwork to lay the foundations for the very clean, elegant stuff we'll do on Thursday. Sorry.)

Affiliation

Affiliation is easiest to define for random variables with a density function. The appendix of Milgrom and Weber gives a definition for the other cases as well, but we'll focus on this one.

Suppose you have n random variables jointly distributed in $\Re^n$ according to a distribution function f. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two generic points in $\Re^n$. Define

- $x \wedge y$, "x join y", as the componentwise min: $x \wedge y = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$
- $x \vee y$, "x meet y", as the componentwise max: $x \vee y = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$

(Draw it in $\Re^2$, with the corners of a rectangle.)

The n random variables with joint probability density f are **affiliated** if and only if for every x and y,

$$f(x \wedge y) f(x \vee y) \geq f(x) f(y)$$

(We'll show that this implies that a higher value of, say, x_1 leads to stochastically higher values of the other variables – but this is a result, not part of the definition.)

If we adopt the convention that $\log 0 = -\infty$ and take logs of both sides, this condition is the same as

$$\log f(x \wedge y) + \log f(x \vee y) \geq \log f(x) + \log f(y)$$

which is exactly the requirement that $\log f$ is supermodular. So random variables are affiliated if and only if their density function is log-supermodular.

For those of you who know supermodularity, you already know that supermodularity of a function of n variables is equivalent to pairwise increasing differences in any two variables, holding the others constant. We restate this result in terms of log-supermodularity:

Lemma 2. *The function f is log-supermodular if and only if, for any $x_i > x_i'$, $x_j > x_j'$, and $x_{-i,j} \in \Re^{n-2}$,*

$$f(x_i, x_j, x_{-i,j}) f(x_i', x_j', x_{-i,j}) \geq f(x_i', x_j, x_{-i,j}) f(x_i, x_j', x_{-i,j})$$

(The reason this has to do with increasing differences is that if we rewrite this with logs and rearrange, this is equivalent to

$$\log f(x_i, x_j, x_{-i,j}) - \log f(x_i', x_j, x_{-i,j}) \geq \log f(x_i, x_j', x_{-i,j}) - \log f(x_i', x_j', x_{-i,j})$$

which is the same as saying that the difference $\log f(x_i, x_j, x_{-i,j}) - \log f(x_i', x_j, x_{-i,j})$ is increasing in x_j.)

The pairwise condition we just gave can be rewritten as

$$\frac{f(x_i, x_j, x_{-i,j})}{f(x_i', x_j, x_{-i,j})} \geq \frac{f(x_i, x_j', x_{-i,j})}{f(x_i', x_j', x_{-i,j})}$$

That is, for $x_i > x_i'$, and holding everything else fixed, the ratio $\frac{f(x_i, x_j, x_{-i,j})}{f(x_i', x_j, x_{-i,j})}$ must be increasing in x_j.

To see why the pairwise condition implies the affiliation inequality, consider two arbitrary points x and x'. For convenience, reorder the arguments of f so that $x_i \geq x_i'$ for $i \leq k$, $x_i < x_i'$ for $i > k$.

Then letting x_i^* and x_i^o be the max and min of x_i and x_i', respectively, the affiliation inequality can be written as

$$f(x_1^*, x_2^*, \ldots, x_n^*) f(x_1^o, x_2^o, \ldots, x_n^o) \geq f(x_1^*, \ldots, x_k^*, x_{k+1}^o, \ldots, x_n^o) f(x_1^o, \ldots, x_k^o, x_{k+1}^*, \ldots, x_n^*)$$

or

$$\frac{f(x_1^*, \ldots, x_k^*, x_{k+1}^*, \ldots, x_n^*)}{f(x_1^*, \ldots, x_k^*, x_{k+1}^o, \ldots, x_n^o)} \geq \frac{f(x_1^o, \ldots, x_k^o, x_{k+1}^*, \ldots, x_n^*)}{f(x_1^o, \ldots, x_k^o, x_{k+1}^o, \ldots, x_n^o)}$$

Now, let $X^* = (x_1^*, \ldots, x_k^*)$, and for $j \geq k$, $X_j = (x_{k+1}^*, \ldots, x_j^*, x_{j+1}^o, \ldots, x_n^o)$. We can rewrite the left-hand ratio as

$$\frac{f(X^*, X_n)}{f(X^*, X_k)} = \frac{f(X^*, X_{k+1})}{f(X^*, X_k)} \frac{f(X^*, X_{k+2})}{f(X^*, X_{k+1})} \cdots \frac{f(X^*, X_n)}{f(X^*, X_{n-1})}$$

and similarly, if we let $X^o = (x_1^o, \ldots, x_k^o)$, we can rewrite the right-hand ratio as

$$\frac{f(X^o, X_n)}{f(X^o, X_k)} = \frac{f(X^o, X_{k+1})}{f(X^o, X_k)} \frac{f(X^o, X_{k+2})}{f(X^o, X_{k+1})} \cdots \frac{f(X^o, X_n)}{f(X^o, X_{n-1})}$$

Now, the pairwise condition we had above is exactly the condition that

$$\frac{f(x_1, x_2, \ldots, x_k, X_{j+1})}{f(x_1, x_2, \ldots, x_k, X_j)}$$

is increasing in x_i for $i \leq k$. Applying this k times gives

$$\frac{f(X^*, X_{j+1})}{f(X^*, X_j)} \geq \frac{f(X^o, X_{j+1})}{f(X^o, X_j)}$$

and the affiliation inequality follows.

(The only-if is trivial; set $x = (x_i', x_j, x_{-ij})$ and $y = (x_i, x_j', x_{-ij})$.)

Note that this pairwise property is

$$\log f(x_i, x_j, x_{-i,j}) - \log f(x_i', x_j, x_{-i,j}) \geq \log f(x_i, x_j', x_{-i,j}) - \log f(x_i', x_j', x_{-i,j})$$

for $x_i > x_i'$ and $x_j > x_j'$. Rewriting x_i as $x_i' + \epsilon$ and dividing both sides by ϵ gives

$$\frac{\log f(x_i' + \epsilon, x_j, x_{-i,j}) - \log f(x_i', x_j, x_{-i,j})}{\epsilon} \geq \frac{\log f(x_i' + \epsilon, x_j', x_{-i,j}) - \log f(x_i', x_j', x_{-i,j})}{\epsilon}$$

which, taking ϵ to 0, is the same as saying $\frac{\partial \log f}{\partial x_i}$ is increasing in x_j. So when f is twice differentiable, the variables are affiliated if and only if

$$\frac{\partial^2 \log f}{\partial x_i \partial x_j} \geq 0$$

everywhere.

Next, as with supermodularity, we point out that affiliation is preserved by any order-preserving transformation of the variables:

Lemma 3. *If $g_1, \ldots, g_n$ are strictly increasing functions from $\Re$ to $\Re$, then $x_1, \ldots, x_n$ are affiliated if and only if $g_1(x_1), g_2(x_2), \ldots, g_n(x_n)$ are affiliated.*

This will be useful for us, since equilibrium bid functions will be increasing: it will imply that if bidder signals are affiliated, then bids are affiliated as well. To prove it, let F_g be the joint cumulative distribution function of all the $g_i(x_i)$. Since each g_i is strictly increasing, The probability that $g_i(x_i) \leq z_i$ for all i is the probability that $x_i \leq g_i^{-1}(z_i)$ for all i; or

$$F_g(z_1, z_2, \ldots, z_n) = F(g_1^{-1}(z_1), \ldots, g_n^{-1}(z_n))$$

If we assume the g_i are all differentiable and differentiate n times, once with respect to each z_i, then

$$f_g(z_1, z_2, \ldots, z_n) = f(g_1^{-1}(z_1), \ldots, g_n^{-1}(z_n))(g_1^{-1})'(z_1)(g_2^{-1})'(z_2) \cdots (g_n^{-1})'(z_n)$$

For simplicity, we'll assume f is twice differentiable. (The result still holds without it, but is harder to prove.) Then $g_1(x_1), \ldots, g_n(x_n)$ are affiliated if and only if $\log f_g$ has positive mixed partials. But

$$\log f_g(z_1, z_2, \ldots, z_n) =$$

$$\log f(g_1^{-1}(z_1), \ldots, g_n^{-1}(z_n)) + \log(g_1^{-1})'(z_1) + \log(g_2^{-1})'(z_2) + \cdots + \log(g_n^{-1})'(z_n)$$

The last n terms are functions of one variable and therefore have no mixed partial, so

$$\frac{\partial^2}{\partial z_i \partial z_j} \log f_g(z_1, z_2, \ldots, z_n) = \frac{\partial}{\partial z_j} \left(\frac{f_i}{f} (g_i^{-1})'(z_i) \right)$$

But $(g_i^{-1})'(z_i)$ is positive, and does not depend on z_j; so this is positive if and only if f_i/f is increasing in z_j. So the transformed variables are affiliated if and only if the original variables are affiliated.

Lemma 4. *If $f : \Re_+^2 \to \Re$ is log-supermodular, then*

$$g(x_1, x_2) = \int_0^{x_1} f(s, x_2) ds$$

is log-supermodular.

Again, this is the same as showing that

$$\frac{g(x_1, x_2)}{g(x_1', x_2)}$$

is increasing in x_2 for any $x_1 > x_1'$. We can write

$$\frac{g(x_1, x_2)}{g(x_1', x_2)} = \frac{\int_0^{x_1} f(s, x_2) ds}{\int_0^{x_1'} f(s, x_2) ds} = 1 + \frac{\int_{x_1'}^{x_1} f(s, x_2) ds}{\int_0^{x_1'} f(s, x_2) ds} = 1 + \frac{\int_{x_1'}^{x_1} \frac{f(s, x_2)}{f(x_1', x_2)} ds}{\int_0^{x_1'} \frac{f(s, x_2)}{f(x_1', x_2)} ds}$$

Now here's the fun part. Since f is log-supermodular, the fraction $\frac{f(s,x_2)}{f(x_1',x_2)}$ is increasing in x_2 whenever $s > x_1'$, and decreasing in x_2 whenever $s < x_1'$. So the numerator in this last expression is increasing in x_2, and the denominator is decreasing, so the whole expression is increasing in x_2, which is what we wanted to show.

Now, here's why we wanted that last lemma:

Lemma 5. *If $f(x_1, x_2)$ is a log-supermodular density function on $\Re_+^2$, then*

1. *the conditional density $f(x_1|x_2)$ is log-supermodular*
2. *the conditional cumulative distribution function $F(x_1|x_2)$ is log-supermodular*
3. *the conditional cumulative distribution function $F(x_1|x_2)$ is nonincreasing in x_2*

1. The conditional density function $f(x_1|x_2)$ can be written as $f(x_1, x_2)/f_2(x_2)$, where f_2 is the unconditional marginal density of x_2. Then $\log f(x_1|x_2) = \log f(x_1, x_2) - \log f_2(x_2)$. Supermodularity is not affected by additive terms that do not include both variables; so $\log f(x_1|x_2)$ is supermodular if and only if $\log f(x_1, x_2)$ is supermodular.

2. $F(x_1|x_2) = \int_0^{x_1} f(s|x_2)ds$, which we just showed was log-supermodular in our last result.

3. Let $x_2 > x_2'$. Since $F(x_1|x_2)$ is log-supermodular,

$$\frac{F(x_1|x_2)}{F(x_1|x_2')} \leq \frac{F(x_1^*|x_2)}{F(x_1^*|x_2')}$$

for $x_1^* > x_1$. Take $x_1^* \to +\infty$ and the right-hand side goes to 1, since $F(x_1^*|x_2)$ eventually goes to 1 regardless of the value of x_2. So

$$\frac{F(x_1|x_2)}{F(x_1|x_2')} \leq 1$$

or $F(x_1|x_2) \leq F(x_1|x_2')$ for $x_2 > x_2'$.

This means that if x_1 and x_2 are affiliated, then a higher value of x_2 implies a higher distribution of x_1, in the sense of first-order stochastic dominance. That is, the distribution of x_1 conditional on a high value of x_2 first-order stochastically dominates the distribution of x_1 conditional on a lower value of x_2. This is the sense in which affiliation implies positive correlation; and it's pretty powerful, since we already know that if one variable first-order stochastically dominates another, it gives a higher expected value of any increasing function. We'll come back to this in a bit.

Next, we show that if a set of random variables are affiliated, then any subset of them are affiliated as well.

Lemma 6. *If $f(x_1, \ldots, x_n)$ is a log-supermodular probability density, then*

$$g(x_1, \ldots, x_{n-1}) = \int f(x_1, \ldots, x_{n-1}, s) ds$$

is a log-supermodular probability density as well.

Again, we'll show it for the case where f is twice differentiable, so that log-supermodularity is the same as positive mixed partials. The result holds more generally.

g is log-supermodular if $\frac{\partial}{\partial x_i} \log g$ is increasing in x_j. Taking the derivative,

$$\frac{\partial}{\partial x_i} \log g(x_1, \ldots, x_{n-1}) = \frac{\frac{\partial}{\partial x_i} \int f(x_1, \ldots, x_{n-1}, s) ds}{\int f(x_1, \ldots, x_{n-1}, s) ds} = \frac{\int f_i(x_1, \ldots, x_{n-1}, s) ds}{\int f(x_1, \ldots, x_{n-1}, s) ds}$$

$$= \frac{\int \frac{f_i(x_1, \ldots, x_{n-1}, s) ds}{f(x_1, \ldots, x_{n-1}, s) ds} f(x_1, \ldots, x_{n-1}, s) ds}{\int f(x_1, \ldots, x_{n-1}, s) ds}$$

$$= E\left\{ \frac{f_i(x_1, \ldots, x_{n-1}, s)}{f(x_1, \ldots, x_{n-1}, s)} \middle| x_1, \ldots, x_{n-1} \right\}$$

where the expectation is taken over s, conditional on the first $n-1$ variables. We need to show this is increasing in x_j.

An increase in x_j has two effects. First, since $(x_1, \ldots, x_{n-1}, s)$ are affiliated, an increase in x_j increases the ratio $\frac{f_i}{f}$ directly.

When we fix the values of all variables but x_j and x_n, it's clear that these remaining two are affiliated (since $\log f$ has positive mixed partials when other variables are held fixed). That means that an increase in x_j leads to an increase in the conditional distribution of s (in the first-order stochastic dominance sense) by the last result. But by affiliation, $\frac{f_i}{f}$ is increasing in x_n, and therefore its expectation over x_n is increasing in x_j.

So an increase in x_j increases the whole expression, which implies that g is log-supermodular.

The next result is that conditional expectations of increasing functions of affiliated variables behave predictably. That is:

Lemma 7. *Suppose $x_1, \ldots, x_n$ are affiliated. For any function $g : \Re^n \to \Re$ which is bounded and isotone (increasing in all its arguments),*

$$h(x) \equiv E(g(x_1, \ldots, x_n)|x_1 = x)$$

is increasing in x.

The proof is by induction on n and iterated expectations. First, suppose $n = 2$. Then for $x > x'$,

$$\begin{aligned} h(x) &= E(g(x, x_2)|x_1 = x) \\ &\geq E(g(x', x_2)|x_1 = x) \\ &\geq E(g(x', x_2)|x_1 = x') \\ &= h(x') \end{aligned}$$

The first inequality is because $x > x'$ and g is increasing in both its arguments. The second is because the distribution of x_2 conditional on $x_1 = x$ first-order stochastically dominates the distribution of x_2 conditional on $x_1 = x'$; and we just showed, the expected value of any increasing function is higher over a stochastically dominant distribution.

Now suppose the lemma is known to hold for functions of $n - 1$ affiliated variables. Define

$$j(x, y) = E(g(x_1, x_2, \ldots, x_n)|x_1 = x, x_2 = y)$$

and rewrite $h(x)$ as

$$h(x) = E_{x_2|x_1=x} j(x, x_2)$$

By the inductive assumption, for a fixed x, $j(x, y)$ is increasing in y; and the distribution of x_2, conditional on $x_1 = x$, is increasing in x by FOSD. So by the same logic, for $x > x'$,

$$\begin{aligned} h(x) &= E(j(x, x_2)|x_1 = x) \\ &\geq E(j(x', x_2)|x_1 = x) \\ &\geq E(j(x', x_2)|x_1 = x') \\ &= h(x') \end{aligned}$$

(The Milgrom and Weber paper actually give a different formulation of this result: if $x_1, \ldots, x_n$ are affiliated and g is any increasing function, then the expectation

$$E\left(g(x_1, \ldots, x_n)|x_1 \in [a_1, b_1], x_2 \in [a_2, b_2], \ldots, x_n \in [a_n, b_n]\right)$$

is increasing in all $2n$ of its arguments a_i and b_i.)

Finally, we will show that a bunch of random variables are affiliated if and only if their order statistics are affiliated.

Lemma 8. *Suppose $(x_1, \ldots, x_n, y)$ have a joint density f which is symmetric in the first n arguments. Then $(x_1, \ldots, x_n, y)$ are affiliated if and only if $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}, y)$ are affiliated, where $x^{(i)}$ is the ith-highest of $\{x_1, \ldots, x_n\}$.*

The density of the $n+1$ random variables $x^{(1)}, x^{(2)}, \ldots, x^{(n)}, y$ is

$$\hat{f}(x_1, x_2, \ldots, x_n, y) = n! f(x_1, x_2, \ldots, x_n, y) \mathbf{1}_{x_1 > x_2 > \ldots > x_n}$$

(This is because the density is 0 unless $x_1 > x_2 > \ldots > x_n$, and $\sum f(\sigma(x), y)$ when this does hold, where the sum is taken over the permutations of the elements of x; but there are $n!$ of these permutations, and by assumption, f takes the same value at all of them.)

We need to show that $\hat{f}$ satisfies the affiliation inequality everywhere if and only if f does. We'll show the reverse. First, suppose $\hat{f}$ violates the affiliation inequality somewhere, that is, suppose there exist (x, y) and (x', y') such that

$$\hat{f}((x, y) \wedge (x', y'))\hat{f}((x, y) \vee (x', y')) < \hat{f}(x, y)\hat{f}(x', y')$$

There are two possible cases: either one of the indicator functions $\mathbf{1}$ is zero on the left-hand side, or not. (Since the right-hand side is strictly positive, the indicator functions must both be 1 on the right-hand side.)

If not, then $\hat{f}(\cdot) = n! f(\cdot)$ at all four points where it is being evaluated, and so the affiliation inequality is also violated by f.

If so, this means that although $x_1 > x_2 > \ldots > x_n$ and $x'_1 > x'_2 > \ldots > x'_n$, either the meet or the join are ordered incorrectly. It is easy to show that this cannot occur; so if $\hat{f}$ violates the affiliation inequality, so does f.

On the other hand, suppose f violates the affiliation inequality somewhere. Then there is some point where

$$\frac{f(x_i, x_j, x_{-i,j})}{f(x'_i, x_j, x_{-i,j})} < \frac{f(x_i, x'_j, x_{-i,j})}{f(x'_i, x'_j, x_{-i,j})}$$

By continuity, we can find such points like this such that x_j and x'_j are very close together; and similarly, we can find such points where x_i and x'_i are very close together. By symmetry of f, then, reorder its arguments so that $x_1 > x_2 > \ldots > x_n$, and we'll get a violation of the affiliation inequality for $\hat{f}$ at the same point.

So, to sum up what we've shown about affiliation...

- Random variables $x_1, x_2, \ldots, x_n$ are affiliated if and only if their joint density function is log-supermodular
- This is equivalent to

$$f(x_i, x_j, x_{-i,j}) f(x_i', x_j', x_{-i,j}) \geq f(x_i', x_j, x_{-i,j}) f(x_i, x_j', x_{-i,j})$$

  for $x_i > x_i'$, $x_j > x_j'$, any $x_{-i,-k}$; which is equivalent to the ratio

$$\frac{f(x_i, x_j, x_{-i,j})}{f(x_i', x_j, x_{-i,j})}$$

  increasing in x_j for any $x_i > x_i'$
- Affiliation is preserved by any order-preserving transformation: $x_1, x_2, \ldots, x_n$ are affiliated if and only if $g_1(x_1), g_2(x_2), \ldots, g_n(x_n)$ are affiliated, where g_i are any strictly-increasing functions
- If x_1 and x_2 are affiliated, then the conditional distribution of x_2 when $x_1 = x$ first-order stochastically dominates the conditional distribution of x_2 when $x_1 = x' < x$
- If $x_1, x_2, \ldots, x_n$ are affiliated, then any subset of them are affiliated as well
- If $x_1, x_2, \ldots, x_n$ are affiliated and g is any isotone function, then $E(g(x)|x_1)$ is increasing in x_1
- Suppose the density of $x_1, x_2, \ldots, x_n, y$ is symmetric in its first n arguments. Then $x_1, x_2, \ldots, x_n, y$ are affiliated if and only if $x^{(1)}, x^{(2)}, \ldots, x^{(n)}, y$ are affiliated.

That gives us the results we'll need to analyze the common auction types, and prove some very general results, in a world with affiliated signals and interdependent valuations. That's what's coming on Thursday.