ECONOMETRICS

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Contents

Chapter 1

Preliminaries

This chapter is presented for review and reference. The matrix algebra results are largely taken from the Appendix of Muirhead (1982). The analysis results are taken from Rudin (1987).

1.1 Matrix Algebra

1.1.1 Terminology

A **scalar** a is a single number.

A **vector** a is a $k \times 1$ list of numbers, typically arranged in a column. We write this as

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}$$

Equivalently, a vector a is an element of Euclidean k space, hence $a \in R^k$. If $k = 1$ then a is a scalar.

A **matrix** A is a $k \times r$ rectangular array of numbers, written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kr} \end{bmatrix} = [a_{ij}]$$

By convention a_{ij} refers to the $i'th$ row and $j'th$ column of A. If $r = 1$ or $k = 1$ then A is a vector. If $r = k = 1$, then A is a scalar.

A matrix can be written as a set of column vectors or as a set of row vectors. That is,

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_r \end{bmatrix} = \begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_k' \end{bmatrix}$$

where

$$a_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ki} \end{bmatrix}$$

are column vectors and

$$\alpha_j' = \begin{bmatrix} a_{j1} & a_{j2} & \cdots & a_{jr} \end{bmatrix}$$

are row vectors.

The **transpose** of a matrix, denoted $B = A'$, is obtained by flipping the matrix on its diagonal.

$$B = A' = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & & \vdots \\ a_{1r} & a_{2r} & \cdots & a_{kr} \end{bmatrix}$$

Thus $b_{ij} = a_{ji}$ for all i and j. Note that if A is $k \times r$, then A' is $r \times k$. If a is a $k \times 1$ vector, then a' is a $1 \times k$ row vector.

A matrix is **square** if $k = r$. A square matrix is **symmetric** if $A = A'$, which implies $a_{ij} = a_{ji}$. A square matrix is **diagonal** if the only non-zero elements appear on the diagonal, so that $a_{ij} = 0$ if $i \neq j$. A square matrix is **upper (lower) diagonal** if all elements below (above) the diagonal equal zero.

A **partitioned matrix** takes the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kr} \end{bmatrix}$$

where the A_{ij} denote matrices, vectors and/or scalars.

1.1.2 Matrix Multiplication

If a and b are both $k \times 1$, then their inner product is

$$a'b = a_1b_1 + a_2b_2 + \cdots + a_kb_k = \sum_{j=1}^{k} a_jb_j$$

Note that $a'b = b'a$.

If A is $k \times r$ and B is $r \times s$, then we define their product AB by writing A as a set of row vectors and B as a set of column vectors (each which of length r). Then

$$\begin{aligned} AB &= \begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_k' \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_r \end{bmatrix} \\ &= \begin{bmatrix} a_1'b_1 & a_1'b_2 & \cdots & a_1'b_r \\ a_2'b_1 & a_2'b_2 & \cdots & a_2'b_r \\ \vdots & \vdots & & \vdots \\ a_k'b_1 & a_k'b_2 & \cdots & a_k'b_r \end{bmatrix} \end{aligned}$$

When the number of columns of A equals the number of rows of B, we say that A and B, or the product AB, is **conformable**, and this is the only case where this product is defined.

An alternative way to write the matrix product is to use matrix partitions. For example,

$$\begin{aligned} AB &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} AB &= \begin{bmatrix} A_1 & A_2 & \cdots & A_r \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_r \end{bmatrix} \\ &= A_1B_1 + A_2B_2 + \cdots + A_rB_r \\ &= \sum_{j=1}^{r} A_rB_r \end{aligned}$$

1.1.3 Identity Matrix

An important diagonal matrix is the **identity matrix**, which has ones on the diagonal. A $k \times k$ identity matrix is denoted as

$$I_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Important properties are that if A is $k \times r$, then $AI_r = A$ and $I_k A = A$.

We say that two vectors a and b are **orthogonal** if $a'b = 0$. The columns of $k \times r$ A, $r \leq k$, are said to be orthogonal if $A'A = I_r$. A square matrix A is called orthogonal if $A'A = I_k$.

1.1.4 Trace of a Matrix

The **trace** of a $k \times k$ square matrix A is the sum of its diagonal elements

$$\operatorname{tr}(A) = \sum_{i=1}^{k} a_{ii}$$

Some straightforward properties are

$$\begin{aligned}
\operatorname{tr}(cA) &= c \operatorname{tr}(A) \\
\operatorname{tr}\left(A'\right) &= \operatorname{tr}(A) \\
\operatorname{tr}(A+B) &= \operatorname{tr}(A) + \operatorname{tr}(B) \\
\operatorname{tr}(I_k) &= k \\
\operatorname{tr}(AB) &= \operatorname{tr}(BA)
\end{aligned}$$

The last result follows since

$$\begin{aligned}
\operatorname{tr}(AB) &= \operatorname{tr}\begin{bmatrix} a_1'b_1 & a_1'b_2 & \cdots & a_1'b_k \\ a_2'b_1 & a_2'b_2 & \cdots & a_2'b_k \\ \vdots & \vdots & & \vdots \\ a_k'b_1 & a_k'b_2 & \cdots & a_k'b_k \end{bmatrix} \\
&= \sum_{i=1}^{k} a_i'b_i \\
&= \sum_{i=1}^{k} b_i'a_i \\
&= \operatorname{tr}(BA).
\end{aligned}$$

1.1.5 Matrix Inversion

A $k \times k$ matrix A has **full rank,** or is **nonsingular**, if there is no $c \neq 0$ such that $Ac = 0$. In this case there exists a unique matrix B such that $AB = BA = I_k$. This matrix is called the **inverse** of A and is denoted by A^{-1}. Some properties include

$$\begin{aligned} AA^{-1} &= A^{-1}A = I_k \\ \left(A^{-1}\right)' &= \left(A'\right)^{-1} \\ (AC)^{-1} &= C^{-1}A^{-1} \\ (A+C)^{-1} &= A^{-1}\left(A^{-1}+C^{-1}\right)^{-1}C^{-1} \\ A^{-1}-(A+C)^{-1} &= A^{-1}\left(A^{-1}+C^{-1}\right)A^{-1} \end{aligned}$$

Also, if A is an orthogonal matrix, then $A^{-1} = A$.

The following fact about inverting partitioned matrices is sometimes useful

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} E^{-1} & -E^{-1}BD^{-1} \\ -D^{-1}CE^{-1} & F^{-1} \end{bmatrix}$$

where

$$\begin{aligned} E^{-1} &= A^{-1} + A^{-1}BF^{-1}CA^{-1} = \left(A - BD^{-1}C\right)^{-1} \\ F^{-1} &= D^{-1} + D^{-1}CE^{-1}BD^{-1} = \left(D - CA^{-1}B\right)^{-1} \end{aligned}$$

1.1.6 Generalized Inverse

Even if a matrix A does not possess an inverse, we can still define a **generalized inverse** A^- as a matrix which satisfies

$$AA^-A = A. \tag{1.1}$$

The matrix A^- is not necessarily unique. The **Moore-Penrose generalized inverse** A^- satisfies (1.1) plus the following three conditions

$$\begin{gathered} A^-AA^- = A^- \\ AA^- \text{ is symmetric} \\ A^-A \text{ is symmetric} \end{gathered}$$

For any matrix A, the Moore-Penrose generalized inverse A^- exists and is unique.

1.1.7 Determinant

The notion of a determinant is defined for square matrices. While the determinant is seen often in econometrics, the actual definition is rarely used. I give the definition here for completeness.

If A is 2×2, then its determinant is $\det A = a_{11}a_{22} - a_{12}a_{21}$.

For a general $k \times k$ matrix $A = [a_{ij}]$, we can define the determinant as follows. Let $\pi = (j_1, ..., j_k)$ denote a permutation of $(1, ..., k)$. There are $k!$ such permutations. There is a unique count of the number of inversions of the indices of such permutations (relative to the natural order

$(1, ..., k)$), and let $\varepsilon_\pi = +1$ if this count is even and $\varepsilon_\pi = -1$ if the count is odd. Then the general definition of the determinant is

$$\det A = \sum_\pi \varepsilon_\pi a_{1j_1} a_{2j_2} \cdots a_{kj_k}$$

Some properties include

$$\begin{aligned}
\det A &= \det A' \\
\det(\alpha A) &= \alpha^k \det A \\
\det(AB) &= (\det A)(\det B) \\
\det\left(A^{-1}\right) &= (\det A)^{-1} \\
\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} &= (\det D)\det\left(A - BD^{-1}C\right) \text{ if } \det D \neq 0.
\end{aligned}$$

Some additional properties include

- $\det A \neq 0$ if and only if A is nonsingular.
- If A is triangular (upper or lower), then $\det A = \prod_{i=1}^k a_{ii}$
- If A is orthogonal, then $\det A = \pm 1$

1.1.8 Eigenvalues

The characteristic equation of a square matrix A is

$$\det(A - \lambda I_k) = 0.$$

The left side is a polynomial of degree k in λ so has exactly k roots, which are not necessarily distinct and may be real or complex. They are called the **latent roots** or **characteristic roots** or **eigenvalues** of A. If λ_i is an eigenvalue of A, then $A - \lambda_i I_k$ is singular so there exists a non-zero vector h_i such that

$$(A - \lambda_i I_k) h_i = 0$$

The vector h_i is called a **latent vector** or **characteristic vector** or **eigenvector** of A corresponding to λ_i.

We now state some useful properties. Let λ_i and h_i, $i = 1, ..., k$ denote the k eigenvalues and eigenvectors of a square matrix A. Let Λ be a diagonal matrix with the characteristic roots in the diagonal, and let $H = [h_1 \cdots h_k]$.

- $\det(A) = \prod_{i=1}^k \lambda_i$
- $\mathrm{tr}(A) = \sum_{i=1}^k \lambda_i$

- A is non-singular if and only if all its characteristic roots are non-zero.
- If A has distinct characteristic roots, there exists a nonsingular matrix P such that $A = P^{-1}\Lambda P$ and $PAP^{-1} = \Lambda$.
- If A is symmetric, then $A = H\Lambda H'$ and $H'AH = \Lambda$, and the characteristic roots are all real.
- The characteristic roots of A^{-1} are λ_1^{-1}, λ_2^{-1}, ..., λ_k^{-1}.

The decomposition $A = H\Lambda H'$ is sometimes called the **spectral decomposition** of a matrix. We define the **rank** of a square matrix as the number of its non-zero characteristic roots.

1.1.9 Positive Definite Matrices

We say that a square matrix A is **positive semi-definite** if for all non-zero c, $c'Ac \geq 0$. This is written as $A \geq 0$. We say that A is **positive definite** if for all non-zero c, $c'Ac > 0$. This is written as $A > 0$.

If A is positive definite, then A is non-singular and A^{-1} exists. Furthermore, $A^{-1} > 0$.

We say that X is $n \times k$, $k < n$, has full rank k if there is no non-zero c such that $Xc = 0$. In this case, $X'X$ is symmetric and positive definite.

If A is symmetric, then $A > 0$ if and only if all its characteristic roots are positive.

If $A > 0$ we can find a matrix B such that $A = BB'$. We call B a **matrix square root** of A. The matrix B need not be unique. One way to construct B is to use the spectral decomposition $A = H\Lambda H'$ where Λ is diagonal, and then set $B = H\Lambda^{1/2}$.

1.1.10 Idempotent Matrices

A square matrix A is **idempotent** if $AA = A$.

If A is also symmetric (most idempotent matrices are) then all its characteristic roots equal either zero or one. To see this, note that we can write $A = H\Lambda H'$ where H is orthogonal and Λ contains the (real) characteristic roots. Then

$$A = AA = H\Lambda H'H\Lambda H' = H\Lambda^2 H'.$$

By the uniqueness of the characteristic roots, we deduce that $\Lambda^2 = \Lambda$ and $\lambda_i^2 = \lambda_i$ for $i = 1, ..., k$. Hence they must equal either 0 or 1.

It follows that if A is symmetric and idempotent, then $\mathrm{tr}(A) = rank(A)$.

1.1.11 Projection Matrices

Let X be an $n \times k$ matrix, $k < n$. Two **projection matrices** are

$$\begin{aligned} P &= X\left(X'X\right)^{-1}X' \\ M &= I_n - P \\ &= I_n - X\left(X'X\right)^{-1}X'. \end{aligned}$$

They are called projection matrices due to the property that for any matrix Z which can be written as $Z = X\Gamma$ for some matrix Γ, (we say that Z lies in the **range space** of X) then

$$PZ = PX\Gamma = X\left(X'X\right)^{-1}X'X\Gamma = X\Gamma = Z$$

and

$$MZ = \left(I_n - P\right)Z = Z - PZ = Z - Z = 0.$$

As an important example of this property, partition the matrix X into two matrices X_1 and X_2, so that

$$X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}.$$

Then $PX_1 = X_1$ and $MX_1 = 0$.

P and M are symmetric:

$$\begin{aligned} P' &= \left(X\left(X'X\right)^{-1}X'\right)' \\ &= \left(X'\right)'\left(\left(X'X\right)^{-1}\right)'\left(X\right)' \\ &= X\left(\left(X'X\right)'\right)^{-1}X' \\ &= X\left(\left(X\right)'\left(X'\right)'\right)^{-1}X' \\ &= P \end{aligned}$$

and

$$M' = \left(I_n - P\right)' = I_n' - P' = I_n - P = M.$$

The projection matrices P and M are idempotent:

$$\begin{aligned} PP &= \left(X\left(X'X\right)^{-1}X'\right)\left(X\left(X'X\right)^{-1}X'\right) \\ &= X\left(X'X\right)^{-1}X'X\left(X'X\right)^{-1}X' \\ &= X\left(X'X\right)^{-1}X' = P, \end{aligned}$$

and

$$\begin{aligned} MM &= \left(I_n - P\right)\left(I_n - P\right) \\ &= I_nI_n - PI_n - I_nP + PP \\ &= I_n - P - P + P \\ &= I_n - P = M. \end{aligned}$$

Furthermore,

$$\begin{aligned} M + P &= I_n - P + P = I_n \\ MP &= \left(I_n - P\right)P = P - PP = P - P = 0. \end{aligned}$$

Another useful property is that

$$\begin{aligned} \operatorname{tr} P &= k \quad (1.2) \\ \operatorname{tr} M &= n-k. \quad (1.3) \end{aligned}$$

Indeed,

$$\begin{aligned} \operatorname{tr} P &= \operatorname{tr}\left(X\left(X'X\right)^{-1}X'\right) \\ &= \operatorname{tr}\left(\left(X'X\right)^{-1}X'X\right) \\ &= \operatorname{tr}\left(I_k\right) \\ &= k, \end{aligned}$$

and

$$\operatorname{tr} M = \operatorname{tr}\left(I_n - P\right) = \operatorname{tr}\left(I_n\right) - \operatorname{tr}\left(P\right) = n-k.$$

From this, we deduce that the ranks of P and M are k and $n=k$, respectively. Since M is symmetric and idempotent, its spectral decomposition takes the form

$$M = H\begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix}H' \quad (1.4)$$

with $H'H = I_n$.

1.1.12 Kronecker Products and the Vec Operator

Let $A = [a_1\ a_2\ \cdots\ a_n] = [a_{ij}]$ be $m \times n$. The **vec** of A, denoted by $\operatorname{vec}(A)$, is the $mn \times 1$ vector

$$\operatorname{vec}(A) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Let B be any matrix. The **Kronecker product** of A and B, denoted $A \otimes B$, is the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

Some important properties are now summarized. These results hold for matrices for which all matrix multiplications are conformable.

- $(A+B)\otimes C = A\otimes C + B\otimes C$
- $(A\otimes B)(C\otimes D) = AC\otimes BD$
- $A\otimes(B\otimes C) = (A\otimes B)\otimes C$
- $(A\otimes B)' = A'\otimes B'$
- $\operatorname{tr}(A\otimes B) = \operatorname{tr}(A)\operatorname{tr}(B)$
- If A is $m\times m$ and B is $n\times n$, $\det(A\otimes B) = (\det(A))^n(\det(B))^m$
- $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$
- If $A>0$ and $B>0$ then $A\otimes B>0$
- $\operatorname{vec}(ABC) = (C'\otimes A)\operatorname{vec}(B)$
- $\operatorname{tr}(ABCD) = \operatorname{vec}(D')'(C'\otimes A)\operatorname{vec}(B)$

1.1.13 Matrix Calculus

Let $x=(x_1,...,x_k)$ be $k\times 1$ and $g(x)=g(x_1,...,x_k): R^k\to R$. The vector derivative is

$$\frac{\partial}{\partial x}g(x) = \begin{pmatrix} \frac{\partial}{\partial x_1}g(x) \\ \vdots \\ \frac{\partial}{\partial x_k}g(x) \end{pmatrix}$$

and

$$\frac{\partial}{\partial x'}g(x) = \begin{pmatrix} \frac{\partial}{\partial x_1}g(x) & \cdots & \frac{\partial}{\partial x_k}g(x) \end{pmatrix}.$$

Some properties are now summarized.

- $\frac{\partial}{\partial x}(a'x) = \frac{\partial}{\partial x}(x'a) = a$
- $\frac{\partial}{\partial x'}(Ax) = A$
- $\frac{\partial}{\partial x}(x'Ax) = (A+A')x$
- $\frac{\partial^2}{\partial x\partial x'}(x'Ax) = (A+A')$

$A=[a_{ij}]$ be $m\times n$ and $g(A): R^{mn}\to R$. We define

$$\frac{\partial}{\partial A}g(A) = \left[\frac{\partial}{\partial a_{ij}}g(A)\right]$$

Some properties are now summarized.

- $\frac{\partial}{\partial A}(x'Ax) = xx'$
- $\frac{\partial}{\partial A}\ln(A) = \left(A^{-1}\right)'$
- $\frac{\partial}{\partial A}\operatorname{tr}(AB) = B'$
- $\frac{\partial}{\partial A}\operatorname{tr}\left(A^{-1}B\right) = -A^{-1}BA^{-1}$

1.2 Probability

1.2.1 Random Vectors

A random vector X takes values in R^k according to a probability distribution defined by

$$F(x) = P(X \leq x).$$

When $k > 1$, the inequality $X \leq x$ means that $X_j \leq x_j$ for $j = 1, ..., k$. If the distribution is **discrete**, then there are a countable set of vectors $\tau_1, ..., \tau_r$ which X can take, with associated probabilities $\pi_1, ..., \pi_r$. Specifically,

$$\begin{aligned} P(X = \tau_j) &= \pi_j, \qquad j = 1, ..., r \\ \sum_{j=1}^{r} \pi_j &= 1 \end{aligned}$$

If the distribution is **continuous**, then $P(X = \tau) = 0$ for all $\tau \in R^k$. In this case we represent the relative probabilities by the density function

$$f(x) = \frac{d^k}{dx_1 \cdots dx_k} F(x).$$

1.2.2 Some Discrete Distributions

For reference, we now list some important discrete distribution functions.

Bernoulli. $X \in \{0, 1\}$

$$\begin{aligned} P(X = 1) &= p \\ P(X = 0) &= 1 - p \end{aligned}$$

The parameter p is the probability of observing $X = 1$.

Binomial. $X \in \{0, 1, ..., n\}$

$$P(X = x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

The random variable X equals the number of "successes" out of n independent Bernoulli trials each with probability p of success.

Multinomial. $X = (X_1, ..., X_m)$ where $X_1 + \cdots X_m = n$

$$P(X_1 = x_1, X_2 = x_2, ..., X_m = x_m) = \frac{n!}{x_1! x_2! \cdots x_m!} p_1^{x_1} p_2^{x_2} \cdots p_m^{x_m}$$

This generalizes the Binomial distribution from 2 to m mutually exhaustive categories.

Poisson. $X = \{0, 1, 2, ...\}$

$$P(X = x) = \frac{\exp(-\lambda)\lambda^x}{x!}$$

The parameter $\lambda > 0$ is also the mean and variance of X.

1.2.3 Transformations

Suppose that $X \in R^k$ with continuous distribution function $F_X(x)$ and density $f_X(x)$. Let $Y = g(X)$ where $g(x) : R^k \to R^k$ is one-to-one and invertible. Let $h(y)$ denote the inverse of $g(x)$. The **Jacobian** is

$$J(y) = \det\left(\frac{\partial}{\partial y'} h(y)\right)$$

which we assume is non-zero.

Consider the univariate case $k = 1$. If $g(x)$ is an increasing function, then the distribution function of Y is

$$\begin{aligned} F_Y(y) &= P(g(X) \le y) \\ &= P(X \le h(Y)) \\ &= F_X(h(Y)) \end{aligned}$$

so the density of Y is

$$f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(h(Y)) \frac{d}{dy} h(y).$$

If $g(x)$ is a decreasing function, then

$$\begin{aligned} F_Y(y) &= P(g(X) \le y) \\ &= 1 - P(X \ge h(Y)) \\ &= 1 - F_X(h(Y)) \end{aligned}$$

so the density of Y is

$$f_Y(y) = -f_X(h(Y)) \frac{d}{dy} h(y).$$

We can write these two cases jointly as

$$f_Y(y) = f_X(h(Y)) |J(y)|. \tag{1.5}$$

This is known as the **change-of-variables** formula. This same formula (1.5) holds for $k > 1$, but its justification requires some deeper results from analysis.

1.2.4 Marginal and Conditional Densities

Let $(Y, X) \in R^m \times R^k$ have joint density $f(y, x)$.

The **marginal densities** of Y and X are

$$\begin{aligned} f_Y(y) &= \int_{R^k} f(y, x)dx \\ f_X(x) &= \int_{R^m} f(y, x)dy \end{aligned}$$

Since x is a vectors, the symbol dx represents $dx = dx_1 \cdots dx_k$, and similarly for dy. The marginal densities are valid density functions.

For $f_X(x) > 0$, the **conditional density** of Y given X is

$$f_{Y|X}(y \mid x) = \frac{f(y, x)}{f_X(x)}.$$

For fixed x, the conditional density function is a valid density function.

The random vectors Y and X are **independent** if and only if the joint density factors into the product of the two marginals

$$f(y, x) = f_Y(y)f_X(x).$$

If Y and X are independent, then $f_{Y|X}(y \mid x) = f_Y(y)$. Furthermore, if X and Y are independent, then $g(X)$ and $h(Y)$ are independent for any non-trivial functions g and h.

1.2.5 Expectation

For any $h : R^k \to R^q$, we define the **mean** or **expectation** $Eh(X)$ as follows. If X is discrete,

$$Eh(X) = \sum_{j=1}^{r} h(\tau_j)\pi_j,$$

and if X is continuous

$$Eh(X) = \int_{R^k} h(x)f(x)dx.$$

The **conditional mean** or **conditional expectation** is the function

$$m(x) = E(Y \mid X = x) = \int_{-\infty}^{\infty} y f_{Y|X}(y \mid x)\, dy.$$

While this definition requires the existence of densities, it can be extended so that $g(x)$ is well defined whenever $E|Y| < \infty$.

The conditional mean $m(x)$ is a function, meaning that when X equals x, then the expected value of Y is $m(x)$. If you ask what is the conditional mean of Y given the random vector X, the answer is

$$E(Y \mid X) = m(X) = \int_{-\infty}^{\infty} y f_{Y|X}(y \mid X)\, dy$$

This is a random variable as it is a function of the random vector X.

Expectations are linear operators, in the sense that

$$E(a + bX + cY) = a + bE(X) + cE(Y)$$

and similarly for conditional expectations.

Theorem 1.2.1 *Simple Law of Iterated Expectations:*

$$E(E(Y \mid X)) = E(Y)$$

Theorem 1.2.2 *Conditioning Theorem. For any function $g(x)$,*

$$E(g(X)Y \mid X) = g(X)\, E(Y \mid X)$$

Theorem 1.2.3 *If X and Y are independent, then*

$$E(XY) = E(X)\, E(Y)$$

1.2.6 Moment Generating and Characteristic Functions

The **moment generating function** (MGF) of a $k \times 1$ random vector X is

$$M(\lambda) = E \exp\left(\lambda' X\right).$$

The MGF does not necessarily exist. However, if it does then for $k = 1$ we have

$$\frac{d^m}{d\lambda^m} M(\lambda)\bigg|_{\lambda=0} = E(X^m),$$

the $m'th$ moment of X.

More generally, the **characteristic function** (CF) of X is

$$C(\lambda) = E \exp\left(i\lambda' X\right).$$

where $i = \sqrt{-1}$ is the imaginary unit. The CF always exists. For $k = 1$ we have

$$\frac{d^m}{d\lambda^m} C(\lambda)\bigg|_{\lambda=0} = i^m E(X^m).$$

Theorem 1.2.4 *If X and Y are independent with MGFs $M_X(\lambda)$ and $M_Y(\lambda)$, then the MGF of $X + Y$ is $M_X(\lambda)M_Y(\lambda)$.*

1.2.7 Normal and Related Distributions

The **standard normal** density is

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \qquad -\infty < x < \infty.$$

You may calculate by integration that $E(X) = 0$ and $Var(X) = 1$. It is conventional to write $X \sim N(0,1)$ when X has the standard normal distribution. It is also conventional to denote the standard normal density function by $\phi(x)$ and its distribution function by $\Phi(x)$. The latter has no closed-form solution.

If Z is standard normal and $X = \mu + \sigma Z$, then using the change-of-variables formula, X has density

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad -\infty < x < \infty.$$

which is the **univariate normal density**. The mean and variance of the distribution are μ and σ^2, and it is conventional to write $X \sim N(\mu, \sigma^2)$ when X has the univariate normal distribution.

The **multivariate normal density** is

$$f(x) = \frac{1}{(2\pi)^{k/2} \det(\Sigma)^{1/2}} \exp\left(-\frac{(x-\mu)' \Sigma^{-1} (x-\mu)}{2}\right), \qquad x \in R^k.$$

The mean and covariance matrix of the distribution are μ and Σ, and it is conventional to write $X \sim N(\mu, \Sigma)$.

It useful to observe that the MGF and CF of the multivariate normal are $\exp\left(\lambda'\mu + \lambda'\Sigma\lambda/2\right)$ and $\exp\left(i\lambda'\mu - \lambda'\Sigma\lambda/2\right)$, respectively.

If $X \in R^k$ is multivariate normal and the elements of X are mutually uncorrelated, then $\Sigma = diag\{\sigma_j^2\}$ is a diagonal matrix. In this case the density function can be written as

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^{k/2} \sigma_1 \cdots \sigma_k} \exp\left(-\left(\frac{(x_1-\mu_1)^2/\sigma_1^2 + \cdots + (x_k-\mu_k)^2/\sigma_k^2}{2}\right)\right) \\ &= \prod_{j=1}^{k} \frac{1}{(2\pi)^{1/2} \sigma_j} \exp\left(-\frac{(x_j-\mu_j)^2}{2\sigma_j^2}\right) \end{aligned}$$

which is the product of marginal univariate densities. This shows that if X is multivariate normal with uncorrelated elements, then they are mutually independent.

Another useful fact is that if $X \sim N(\mu, \Sigma)$ and we set $Y = a + BX$, with B an invertible matrix, then by the change-of-variables formula, we can calculate that the density of Y is

$$f(y) = \frac{1}{(2\pi)^{k/2} \det(\Sigma_Y)^{1/2}} \exp\left(-\frac{(y-\mu_Y)' \Sigma_Y^{-1} (y-\mu_Y)}{2}\right), \qquad x \in R^k.$$

where $\mu_Y = a + B\mu$ and $\Sigma_Y = B\Sigma B'$, where we used the fact that $\det(B\Sigma B')^{1/2} = \det(\Sigma)^{1/2}\det(B)$. This shows that linear transformations of normals are also normal.

Let $X \sim N(0, I_r)$ and set $Q = X'X$. We say that Q has a **chi-square distribution** with r **degrees of freedom**, conventionally written as χ^2_r

Theorem 1.2.5 *The χ^2_r pdf is*

$$f(y) = \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{r/2}} y^{r/2-1} \exp(-y/2), \qquad y \geq 0. \tag{1.6}$$

The symbol $\Gamma(a)$ is the **gamma function**

$$\Gamma(a) = \int_0^\infty y^{a-1} e^{-y} dy.$$

Let $Z \sim N(0,1)$ and $Q \sim \chi^2_r$ be independent. Set

$$t_r = \frac{Z}{\sqrt{Q/r}}.$$

The distribution of the random variable t_q is known as the **student's t distribution** with r degrees of freedom.

Theorem 1.2.6 *The density function for t_r is*

$$f(x) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\Gamma\left(\frac{r}{2}\right)\left(1 + \frac{x^2}{r}\right)^{\frac{r+1}{2}}}$$

1.2.8 Analysis

The **Euclidean norm** of an $m \times 1$ vector a is

$$|a| = \left(a'a\right)^{1/2} = \left(\sum_{i=1}^m a_i^2\right)^{1/2}.$$

If A is a $m \times n$ matrix, then its Euclidean norm is

$$|A| = \operatorname{tr}\left(A'A\right)^{1/2} = \left(\operatorname{vec}(A)'\operatorname{vec}(A)\right)^{1/2} = \left(\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2\right)^{1/2}.$$

Triangle inequality.

$$|X+Y| \leq |X| + |Y|\,.$$

C^r inequality.

$$|X+Y|^r \leq \begin{cases} |X|^r + |Y|^r\,. & 0 < r \leq 1 \\ 2^{r-1}\left(|X|^r + |Y|^r\right) & r \geq 1 \end{cases}.$$

Jensen's Inequality. If $\phi(\cdot) : R \to R$ is convex, then

$$\phi(E(X)) \leq E(\phi(X)).$$

Theorem of Geometric Means. If $\alpha_1 + \alpha_2 = 1$ and $A, B > 0$, then

$$A^{\alpha_1} B^{\alpha_2} \leq \alpha_1 A + \alpha_2 B.$$

The L^p norm, $p \geq 1$, of a random matrix X is

$$\|X\|_p = \left(E\,|X|^p\right)^{1/p}.$$

Holder's Inequality. If $p > 1$ and $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$E\,|XY| \leq \|X\|_p \|Y\|_q\,.$$

Cauchy-Schwarz Inequality.

$$E\,|XY| \leq \|X\|_2 \|Y\|_2$$

This is Holder's inequality with $p = q = 2$.

Minkowski's Inequality. For $p \geq 1$,

$$\|X+Y\|_p \leq \|X\|_p + \|Y\|_p\,.$$

Markov's Inequality. For any strictly increasing function $g(X) \geq 0$,

$$P(g(X) > \alpha) \leq \alpha^{-1} E g(X).$$

Let X_n, $n = 1, 2, ...$ be a sequence of random variables such that $X_n \to X$ as $n \to \infty$.

Monotone Convergence Theorem. If $0 \leq X_n \leq X_{n+1}$ for all n, then as $n \to \infty$

$$EX_n \to EX.$$

Dominated Convergence Theorem. If $|X_n| \leq Y$ for some random variable Y such that $E\,|Y| < \infty$, then as $n \to \infty$

$$EX_n \to EX.$$

1.2.9 Asymptotic Theory

A set of random vectors $\{X_1, ..., X_n\}$ are **independent and identically distributed** if they are mutually independent and are drawn from a common distribution F. This is typically abbreviated as **IID**. The most basic statistic constructed on a random sample is the sample mean

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

The purpose of asymptotic theory is to derive useful approximations to the distribution of sample statistics such as the sample mean. The three main components of asymptotic theory are convergence in probability, convergence in distribution, and the continuous mapping theorem.

We say that Z_n **converges in probability** to Z as $n \to \infty$ denoted $Z_n \to_p Z$ as $n \to \infty$, if for all $\delta > 0$,

$$\lim_{n\to\infty} P\left(|Z_n - Z| > \delta\right) = 0.$$

This is a probabilistic way of generalizing the mathematical definition of a limit.

Theorem 1.2.7 *Weak Law of Large Numbers (WLLN). If $X_i \in R^k$ is iid and $E|X_i| < \infty$, then $\overline{X}_n \to_p E(X)$ as $n \to \infty$.*

Definition 1.2.1 *Let Z_n be a random variable with distribution $F_n(x) = P\left(Z_n \leq x\right)$. We say that Z_n **converges in distribution** to Z as $n \to \infty$, denoted $Z_n \to_d Z$, where Z has distribution $F(x) = P\left(Z \leq x\right)$, if for all x at which $F(x)$ is continuous, $F_n(x) \to F(x)$ as $n \to \infty$.*

Theorem 1.2.8 *Central Limit Theorem (CLT). If $X_i \in R^k$ is iid and $E|X_i|^2 < \infty$, then as $n \to \infty$*

$$\sqrt{n}\left(\overline{X}_n - \mu\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(X_i - \mu\right) \to_d N\left(0, V\right).$$

where $\mu = EX$ and $V = E\left(X - \mu\right)\left(X - \mu\right)'$.

The two following results are referred to as the **Continuous Mapping Theorem** (CMT)

Theorem 1.2.9 *If $Z_n \to_p c$ as $n \to \infty$ and $g\left(\cdot\right)$ is continuous at c, then $g(Z_n) \to_p g(c)$ as $n \to \infty$.*

Theorem 1.2.10 *If $Z_n \to_d Z$ as $n \to \infty$ and $g\left(\cdot\right)$ is continuous, then $g(Z_n) \to_d g(Z)$ as $n \to \infty$.*

1.3 Maximum Likelihood

1.3.1 Likelihood

If the distribution of X_i is $F(x, \beta)$ where F is a known distribution function and β is an unknown $m \times 1$ vector, we say that the distribution is **parametric** and that β is the **parameter** of the distribution F. In this setting the **method of maximum likelihood** is the appropriate technique for estimation and inference on β.

If the distribution F is continuous then the density of X_i can be written as $f(x, \beta)$ and the joint density of the random sample $\tilde{X} = (X_1, ..., X_n)$ is

$$f\left(\tilde{X}, \beta\right) = \prod_{i=1}^{n} f\left(X_i, \beta\right).$$

The **likelihood** of the sample is this joint density evaluated at the observed sample values, viewed as a function of β. The **log-likelihood** function is its natural log

$$\mathcal{L}_n(\beta) = \sum_{i=1}^{n} \ln f\left(X_i, \beta\right).$$

If the distribution F is discrete, the likelihood and log-likelihood are constructed by setting $f\left(x, \beta\right) = P\left(X = x, \beta\right)$.

1.3.2 Information and Efficiency

Define the **information matrix**

$$H = -E\frac{\partial^2}{\partial\beta\partial\beta'} \ln f\left(X_i, \beta_0\right)$$

and the outer product matrix

$$\Omega = E\left(\frac{\partial}{\partial\beta} \ln f\left(X_i, \beta_0\right) \frac{\partial}{\partial\beta} \ln f\left(X_i, \beta_0\right)'\right).$$

Two important features of the likelihood are

Theorem 1.3.1

$$\frac{\partial}{\partial\beta} E \ln f\left(X_i, \beta\right)\Big|_{\beta=\beta_0} = 0 \tag{1.7}$$

$$H = \Omega \tag{1.8}$$

The equality (1.8) is often called the **information matrix equality**.

Theorem 1.3.2 *Cramer-Rao Lower Bound. If $\tilde{\beta}$ is an unbiased estimator of $\beta \in R$, then* $Var(\hat{\beta}) \geq (nV)^{-1}$.

The Cramer-Rao Theorem gives a lower bound for estimation. However, the restriction to unbiased estimators means that the theorem has little direct relevance for finite sample estimation.

1.3.3 Estimation

The **maximum likelihood estimator** or **MLE** $\hat{\beta}$ is the value of the parameter which maximizes the likelihood (equivalently, which maximizes the log-likelihood). We can write this as

$$\hat{\beta} = \underset{\beta}{\operatorname{argmax}} \mathcal{L}_n(\beta).$$

If the estimation problem is regular, the MLE is consistent for β and asymptotically normally distributed. Let β_0 denote the true value of β so that $f(x, \beta_0)$ is the true density of X_i.

Theorem 1.3.3 *Under regularity conditions,* $\sqrt{n}\left(\hat{\beta} - \beta_0\right) \to_d N\left(0, V^{-1}\right)$.

Thus in large samples, the approximate variance of the MLE is $(nV)^{-1}$ which is the Cramer-Rao lower bound. Thus in large samples the MLE has approximately the best possible variance. Therefore the MLE is called **asymptotically efficient**.

1.3.4 Quasi-Likelihood

Sometimes a parametric density function $f(x, \beta)$ is used to approximate the true unknown density $f(x)$, but it is not literally believed that the model $f(x, \beta)$ is necessarily the true density. In this case, we refer to $\mathcal{L}_n(\beta)$ as a **quasi-likelihood** and the its maximizer $\hat{\beta}$ as a **quasi-mle** or **QMLE**

In this case while there may not be a "true" value of the parameter β, we can define the **pseudo-true** value β_0 as the maximizer of

$$E \ln f\left(X_i, \beta\right) = \int f\left(x\right) \ln f\left(x, \beta\right) dx$$

which is the same as the minimizer of

$$KLIC = \int f\left(x\right) \ln \left(\frac{f(x)}{f\left(x, \beta\right)}\right) dx$$

the Kullback-Leibler information distance between the true density $f(x)$ and the parametric density $f(x, \beta)$. Thus the QMLE β_0 is the value which makes the parametric density "closest" to the true value according to this measure of distance. The QMLE is consistent for the pseudo-true value, but has a different covariance matrix than in the pure MLE case, since the information matrix equality (1.8) does not hold. A minor adjustment to Theorem (1.3.3) yields the asymptotic distribution of the QMLE.

Theorem 1.3.4 *Under regularity conditions,* $\sqrt{n}\left(\hat{\beta} - \beta_0\right) \to_d N\left(0, H^{-1}\Omega H^{-1}\right)$.

1.3.5 Likelihood Ratio Testing

A hypothesis is a restriction on the parameter space, typically written as $H_0 : h(\beta) = 0$ for some function $h : R^k \to R^q$ with $q \leq k$. The restricted estimator of β is the value which maximizes the likelihood subject to the constraint that it satisfies H_0. This can be written as

$$\tilde{\beta} = \underset{h(\beta)=0}{\operatorname{argmax}} \mathcal{L}_n(\beta).$$

The standard test statistic for H_0 is the likelihood ratio, the ratio of the two values of the likelihood, or equivalently, the difference in the two log-likelihood values. The statistic is

$$LR = 2\left(\mathcal{L}_n(\hat{\beta}) - \mathcal{L}_n(\tilde{\beta})\right).$$

Theorem 1.3.5 *Under H_0, $LR \to_d \chi^2_q$, the chi-square distribution with q degrees of freedom.*

An optimality property of the LR test concerns simple tests.

Theorem 1.3.6 *Neyman-Pearson. Among all tests of $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_1$ with size α, the LR test has the highest power.*

The proofs of these two results are rather lengthy and omitted.

1.4 Proofs

Proof of Theorem 1.2.1:

$$\begin{aligned} E\left(E\left(Y \mid X\right)\right) &= E\left(m(X)\right) \\ &= \int_{-\infty}^{\infty} m(x) f_X(x) dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{Y|X}\left(y \mid x\right) f_X(x) dy dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f\left(y, x\right) dy dx \\ &= E(Y). \end{aligned}$$

■

Proof of Theorem 1.2.2:

$$\begin{aligned} h(x) &= E\left(g(X)Y \mid X = x\right) \\ &= \int_{-\infty}^{\infty} g(x) y f_{Y|X}\left(y \mid x\right) dy \\ &= g(x)\int_{-\infty}^{\infty} y f_{Y|X}\left(y \mid x\right) dy \\ &= g(x)m(x) \end{aligned}$$

where $m(x) = E(Y \mid X = x)$. Thus $h(X) = g(X)m(X)$, which is the same as $E(g(X)Y \mid X) = g(X)E(Y \mid X)$. ■

Proof of Theorem 1.2.3: By the definition of independence,

$$\begin{aligned} E(XY) &= \int\int xyf(y,x)dydx \\ &= \int\int xyf_Y(y)f_X(x)dydx \\ &= \int xf_X(x)dx \int yf_Y(y)dy \\ &= E(X)E(Y). \end{aligned}$$

Proof of Theorem 1.2.4. By the properties of the exponential function and Theorem 1.2.3, the MGF of $X + Y$ is

$$E\exp\left(\lambda'(X+Y)\right) = E\left(\exp\left(\lambda'X\right)\exp\left(\lambda'Y\right)\right) = E\exp\left(\lambda'X\right)E\exp\left(\lambda'Y\right) = M_X(\lambda)M_Y(\lambda).$$

■

Proof of Theorem 1.2.5. Using a change-of-variable and the definition of the gamma function, we find that

$$\int_0^\infty y^{a-1}\exp(-by)\,dy = b^{-a}\Gamma(a). \tag{1.9}$$

Using (1.9), we find that the MGF for the density (1.6) is

$$\begin{aligned} E\exp(tQ) &= \int_0^\infty \frac{1}{\Gamma\left(\frac{r}{2}\right)2^{r/2}} y^{r/2-1}\exp(ty)\exp(-y/2)\,dy \\ &= (1-2t)^{-r/2}. \end{aligned} \tag{1.10}$$

Note that if $Z \sim N(0,1)$ the distribution of Z^2 is

$$\begin{aligned} P\left(Z^2 \le y\right) &= 2P\left(0 \le Z \le \sqrt{y}\right) \\ &= 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx \\ &= \int_0^y \frac{1}{\sqrt{2\pi}} s^{-1/2}\exp\left(-\frac{s}{2}\right)ds \end{aligned}$$

using the change–of-variables $s = x^2$. Thus the density of Z^2 is (1.6) with $r = 1$ (since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$). Given (1.10), we see that the MGF of Z^2 is $(1-2t)^{-1/2}$. Since we can write $Q = X'X = \sum_{j=1}^r Z_j^2$ where the Z_j are independent $N(0,1)$ variables and using Theorem 1.2.4, we deduce that the MGF

of Q is $(1-2t)^{-r/2}$. Since (1.10) shows that this is the MGF of (1.6), we conclude that Q has density (1.6). ■

Proof of Theorem 1.2.6. The distribution function for t_r is

$$F(x) = P\left(\frac{Z}{\sqrt{Q/r}} \leq x\right) = E\left[P\left(Z \leq x\sqrt{\frac{Q}{r}} \mid Q\right)\right] = E\Phi\left(x\sqrt{\frac{Q}{r}}\right)$$

Thus its density is

$$\begin{aligned} f(x) &= E\left(\phi\left(x\sqrt{\frac{Q}{r}}\right)\sqrt{\frac{Q}{r}}\right) \\ &= \int_0^\infty \left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{qx^2}{2r}\right)\right)\sqrt{\frac{q}{r}}\left(\frac{1}{\Gamma\left(\frac{r}{2}\right)2^{r/2}}q^{r/2-1}\exp\left(-q/2\right)\right)dq \\ &= \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{r\pi}\Gamma\left(\frac{r}{2}\right)}\left(1+\frac{x^2}{r}\right)^{-\left(\frac{r+1}{2}\right)} \end{aligned}$$

the final equality using (1.9). ■

Proof of Jensen's Inequality. By definition, ϕ is convex if and only if for all $s < t < u$,

$$\frac{\phi(t)-\phi(s)}{t-s} \leq \frac{\phi(u)-\phi(t)}{u-t}. \tag{1.11}$$

Set $t = E(X)$, and define

$$C = \sup_{s<t}\frac{\phi(t)-\phi(s)}{t-s}.$$

Hence for all $s < t$,

$$C \geq \frac{\phi(t)-\phi(s)}{t-s},$$

or re-arranging terms,

$$\phi(s)-\phi(t)-C(s-t) \geq 0. \tag{1.12}$$

By (1.11), for all $u > t$,

$$C \leq \frac{\phi(u)-\phi(t)}{u-t},$$

which implies that

$$\phi(u)-\phi(t)-C(u-t) \geq 0. \tag{1.13}$$

Equations (1.12) and (1.13) combine to yield that for all y,

$$\phi(y)-\phi(t)-C(y-t) \geq 0. \tag{1.14}$$

Let $y = X$, and using $t = E(X)$, (1.14) implies

$$\phi(X) - \phi(E(X)) - C(X - E(X)) \geq 0.$$

Taking expectations yields

$$E\phi(X) - E\phi(E(X)) - CE(X - E(X)) = E\phi(X) - E\phi(E(X)) \geq 0,$$

or

$$E\phi(X) \geq E\phi(E(X)).$$

■

Proof of Theorem of Geometric Means. Define a random variable X which takes the value $\ln A$ with probability α_1, and the value $\ln B$ with probability α_2. (Since $\alpha_1 + \alpha_2 = 1$ this is a valid probability measure). The exponential function is convex, so Jensen's inequality yields

$$\begin{aligned} A^{\alpha_1} B^{\alpha_2} &= \exp\left[\alpha_1 \ln A + \alpha_2 \ln B\right] \\ &= \exp(E(X)) \\ &\leq E\left(\exp(X)\right) \\ &= \alpha_1 \exp(\ln A) + \alpha_2 \exp(\ln B) \\ &= \alpha_1 A + \alpha_2 B. \end{aligned}$$

■

Proof of Holder's Inequality. By the theorem of geometric means

$$|XY| = \left(|X|^p\right)^{1/p} \left(|Y|^q\right)^{1/q} \leq \frac{1}{p}|X|^p + \frac{1}{q}|Y|^q.$$

Taking expectations yields

$$E|XY| \leq \frac{1}{p}E|X|^p + \frac{1}{q}E|Y|^q. \tag{1.15}$$

Now set

$$W = \frac{X}{\|X\|_p} \quad , \quad Z = \frac{Y}{\|Y\|_q}.$$

By (1.15),

$$E|WZ| \leq \frac{1}{p}E|W|^p + \frac{1}{q}E|Z|^q$$

which equals

$$\frac{1}{p}E\left|\frac{X}{\|X\|_p}\right|^p + \frac{1}{q}E\left|\frac{Y}{\|Y\|_q}\right|^q = \frac{1}{p}\frac{EX^p}{\|X\|_p^p} + \frac{1}{q}\frac{EY^q}{\|Y\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1. \tag{1.16}$$

But

$$E|WZ| = E\left|\frac{X}{\|X\|_p}\frac{Y}{\|Y\|_q}\right| = \frac{E|XY|}{\|X\|_p\|Y\|_q},$$

so (1.16) implies that

$$\frac{E|XY|}{\|X\|_p\|Y\|_q} \leq 1,$$

which is the desired result. ■

Proof of Minkowski's Inequality. If $p = 1$, the triangle inequality yields $|X+Y| \leq |X| + |Y|$, and then take expectations.

For $p > 1$, define its conjugate $q = p/(p-1)$ (so $1/p + 1/q = 1$). By the triangle inequality,

$$\begin{aligned}|X+Y|^p &= |X+Y|^{p-1}|X+Y| \leq |X+Y|^{p-1}(|X| + |Y|)\\ &= |X+Y|^{p-1}|X| + |X+Y|^{p-1}|Y|.\end{aligned}$$

Thus taking expectations,

$$\|X+Y\|_p^p = E|X+Y|^p \leq E\left(|X+Y|^{p-1}|X|\right) + E\left(|X+Y|^{p-1}|Y|\right),$$

then applying Holder's inequality,

$$\begin{aligned}&\leq \left\||X+Y|^{p-1}\right\|_q \|X\|_p + \left\||X+Y|^{p-1}\right\|_q \|Y\|_p = \|X+Y\|_{(p-1)q}^{(p-1)}\left(\|X\|_p + \|Y\|_p\right)\\ &= \|X+Y\|_p^{p-1}\left(\|X\|_p + \|Y\|_p\right),\end{aligned}$$

where the final inequality holds since $(p-1)q = p$. We have shown that

$$\|X+Y\|_p^p \leq \|X+Y\|_p^{p-1}\left(\|X\|_p + \|Y\|_p\right).$$

Multiplying both sides by $\|X+Y\|_p^{1-p}$ yields the result. ■

Proof of Markov's Inequality. Set $Y = g(X)$, and let $\{\cdot\}$ denote the indicator function. Then

$$\alpha P(Y > \alpha) = \alpha E\{Y > \alpha\} = \alpha\int\{Y > \alpha\}dP = \int_{\{Y>\alpha\}}\alpha dP,$$

and since for the entire region of integration $\alpha < Y$, this is bounded by

$$\leq \int_{\{Y>\alpha\}} Y dP \leq \int Y dP = E(Y).$$

Hence $P(Y > \alpha) \leq \alpha^{-1}E(Y)$ and we are done. ■

Proof of Weak Law of Large Numbers. Without loss of generality, we can set $E(X) = 0$ (by recentering X_i on its expectation). We need to show that for all $\delta > 0$ and $\eta > 0$ there is some $\overline{n} < \infty$ so that for all $n \geq \overline{n}$, $P\left(\left|\overline{X}_n\right| > \delta\right) \leq \eta$. Fix δ and η. Set $\varepsilon = \delta\eta/3$. Pick $C < \infty$ large enough so that

$$E\left(|X| 1\left(|X| > C\right)\right) \leq \varepsilon \tag{1.17}$$

(where $1(\cdot)$ is the indicator function) which is possible since $E|X| < \infty$. Then set

$$\overline{n} \geq 4C^2/\varepsilon^2. \tag{1.18}$$

Define the random vectors

$$\begin{aligned} W_i &= X_i 1\left(|X_i| \leq C\right) - E\left(X_i 1\left(|X_i| \leq C\right)\right) \\ Z_i &= X_i 1\left(|X_i| > C\right) - E\left(X_i 1\left(|X_i| > C\right)\right). \end{aligned}$$

Since X_i is iid, W_i and Z_i are also.

By Jensen's inequality and (1.17),

$$\left|E\left(X_i 1\left(|X_i| > C\right)\right)\right| \leq E\left(|X_i| 1\left(|X_i| > C\right)\right) \leq \varepsilon.$$

By the triangle inequality and (1.17),

$$E\left|\overline{Z}_n\right| \leq E|Z_i| \leq E|X_i| 1\left(|X_i| > C\right) + \left|E\left(X_i 1\left(|X_i| > C\right)\right)\right| \leq 2\varepsilon.$$

Note that $|W_i| \leq 2C$. Thus (crudely) $EW_i^2 \leq 4C^2$. Since the W_i are iid and mean zero,

$$E\overline{W}_n^2 = \frac{EW_i^2}{n} \leq \frac{4C^2}{n} \leq \varepsilon^2$$

the final inequality holding for $n \geq \overline{n}$ by (1.18). Thus by Jensen's inequality

$$\left(E\left|\overline{W}_n\right|\right)^2 \leq E\overline{W}_n^2 \leq \varepsilon^2.$$

Finally, by Markov's inequality, the fact that $\overline{X}_n = \overline{W}_n + \overline{Z}_n$, the triangle inequality, and these two bounds,

$$P\left(\left|\overline{X}_n\right| > \delta\right) \leq \frac{E\left|\overline{X}_n\right|}{\delta} \leq \frac{E\left|\overline{W}_n\right| + E\left|\overline{Z}_n\right|}{\delta} \leq \frac{3\varepsilon}{\delta} = \eta,$$

the equality by the definition of ε. We have shown that for any $\delta > 0$ and $\eta > 0$ there is some $\overline{n} < \infty$ so that for all $n \geq \overline{n}$, $P\left(\left|\overline{X}_n\right| > \delta\right) \leq \eta$, as needed. ■

Proof of Central Limit Theorem: Without loss of generality, it is sufficient to consider the case $\mu = 0$ and $V = I_k$. For $\lambda \in R^k$, let $C(\lambda) = E\exp\left(i\lambda' X\right)$ denote the characteristic function and set $g(\lambda) = \log C(\lambda)$. Then observe

$$\begin{aligned} \frac{\partial}{\partial\lambda} C(\lambda) &= iE\left(X\exp\left(i\lambda' X\right)\right) \\ \frac{\partial^2}{\partial\lambda\partial\lambda'} C(\lambda) &= i^2 E\left(XX'\exp\left(i\lambda' X\right)\right) \end{aligned}$$

so when evaluated at $\lambda = 0$

$$\begin{aligned} C(0) &= 1 \\ \frac{\partial}{\partial \lambda} C(0) &= iE(X) = 0 \\ \frac{\partial^2}{\partial \lambda \partial \lambda'} C(0) &= -E\left(XX'\right) = -I_k. \end{aligned}$$

Furthermore,

$$\begin{aligned} g_\lambda(\lambda) &= \frac{\partial}{\partial \lambda} g(\lambda) = C(\lambda)^{-1} \frac{\partial}{\partial \lambda} C(\lambda) \\ g_{\lambda\lambda}(\lambda) &= \frac{\partial^2}{\partial \lambda \partial \lambda'} g(\lambda) = C(\lambda)^{-1} \frac{\partial^2}{\partial \lambda \partial \lambda'} C(\lambda) - C(\lambda)^{-2} \frac{\partial}{\partial \lambda} C(\lambda) \frac{\partial}{\partial \lambda'} C(\lambda) \end{aligned}$$

so when evaluated at $\lambda = 0$

$$\begin{aligned} g(0) &= 0 \\ g_\lambda(0) &= 0 \\ g_{\lambda\lambda}(0) &= -I_k. \end{aligned}$$

By a second-order Taylor series expansion of $g(\lambda)$ about $\lambda = 0$,

$$g(\lambda) = g(0) + g_\lambda(0)'\lambda + \frac{1}{2}\lambda' g_{\lambda\lambda}(\lambda^*)\lambda = \frac{1}{2}\lambda' g_{\lambda\lambda}(\lambda^*)\lambda$$

where λ^* lies on the line segment joining 0 and λ.

We now compute the log CF of $\sqrt{n}\overline{X}_n$. For $\alpha \in R^k$, by the properties of the exponential function, the independence of the X_i, the definition of $g(\lambda)$

$$\begin{aligned} G_n(\alpha) &= \log E \exp\left(i\alpha'\sqrt{n}\overline{X}_n\right) \\ &= \log E \exp\left(i\frac{1}{\sqrt{n}}\sum_{j=1}^n \alpha' X_j\right) \\ &= \log E \prod_{j=1}^n \exp\left(i\frac{1}{\sqrt{n}}\alpha' X_j\right) \\ &= \log \prod_{i=1}^n E \exp\left(i\frac{1}{\sqrt{n}}\alpha' X_j\right) \\ &= ng\left(\frac{\alpha}{\sqrt{n}}\right) \\ &= n\frac{1}{2}\left(\frac{\alpha}{\sqrt{n}}\right)' g_{\lambda\lambda}(\alpha_n)\left(\frac{\alpha}{\sqrt{n}}\right) \\ &= \frac{1}{2}\alpha' g_{\lambda\lambda}(\alpha_n)\alpha \end{aligned}$$

where α_n lies on the line segment joining 0 and $\alpha/\sqrt{n}$. Since $\alpha_n \to 0$ as $n \to \infty$, then $g_{\lambda\lambda}(\alpha_n) \to g_{\lambda\lambda}(0) = -I_k$. We see that as $n \to \infty$,

$$G_n(\alpha) \to -\frac{1}{2}\alpha'\alpha.$$

It follows that the CF of $\sqrt{n}\overline{X}_n$ tends to $\exp\left(-\frac{1}{2}\alpha'\alpha\right)$ as $n \to \infty$, which is the CF of the $N(0, I_k)$ distribution. This is sufficient to establish the theorem. ■

Proof of Theorem 1.3.1. To see (1.7),

$$\begin{aligned}
\frac{\partial}{\partial\beta} E \ln f(X_i, \beta)\Big|_{\beta=\beta_0} &= \frac{\partial}{\partial\beta}\int \ln f(x,\beta) f(x,\beta_0)\, dx\Big|_{\beta=\beta_0} \\
&= \int \frac{\partial}{\partial\beta} f(x,\beta)\frac{f(x,\beta_0)}{f(x,\beta)} dx\Big|_{\beta=\beta_0} \\
&= \frac{\partial}{\partial\beta}\int f(x,\beta)\, dx\Big|_{\beta=\beta_0} \\
&= \frac{\partial}{\partial\beta} 1\Big|_{\beta=\beta_0} = 0.
\end{aligned}$$

Similarly, we can show that

$$E\left(\frac{\frac{\partial^2}{\partial\beta\partial\beta'} f(X_i,\beta_0)}{f(X_i,\beta_0)}\right) = 0.$$

By direction computation,

$$\begin{aligned}
\frac{\partial^2}{\partial\beta\partial\beta'} \ln f(X_i,\beta_0) &= \frac{\frac{\partial^2}{\partial\beta\partial\beta'} f(X_i,\beta_0)}{f(X_i,\beta_0)} - \frac{\frac{\partial}{\partial\beta} f(X_i,\beta_0)\frac{\partial}{\partial\beta} f(X_i,\beta_0)'}{f(X_i,\beta_0)^2} \\
&= \frac{\frac{\partial^2}{\partial\beta\partial\beta'} f(X_i,\beta_0)}{f(X_i,\beta_0)} - \frac{\partial}{\partial\beta}\ln f(X_i,\beta_0)\frac{\partial}{\partial\beta}\ln f(X_i,\beta_0)'.
\end{aligned}$$

Taking expectations yields (1.8). ■

Proof of Cramer-Rao Lower Bound. Let

$$S = \frac{\partial}{\partial\beta}\ln f(\tilde{x},\beta_0) = \sum_{i=1}^{n}\frac{\partial}{\partial\beta}\ln f(x_i,\beta_0)$$

which by Theorem (1.3.1) has mean zero and variance nH. Write the estimator $\tilde{\beta} = \tilde{\beta}(\tilde{X})$ as a function of the data. Since $\tilde{\beta}$ is unbiased for any β,

$$\beta = E\tilde{\beta} = \int \tilde{\beta}(\tilde{x}) f(\tilde{x},\beta)\, d\tilde{x}.$$

Differentiating with respect to β and evaluating at β_0 yields

$$1 = \int \tilde{\beta}(\tilde{x}) \frac{\partial}{\partial \beta} f(\tilde{x}, \beta)\, d\tilde{x} = \int \tilde{\beta}(\tilde{x}) \frac{\partial}{\partial \beta} \ln f(\tilde{x}, \beta)\, f(\tilde{x}, \beta_0)\, d\tilde{x} = E\left(\tilde{\beta} S'\right).$$

By the Cauchy-Schwarz inequality

$$1 = \left| E\left(\tilde{\beta} S'\right) \right|^2 \leq Var(S)\, Var\left(\tilde{\beta}\right)$$

so

$$Var\left(\tilde{\beta}\right) \geq \frac{1}{Var(S)} = \frac{1}{nH}.$$

■

Proof of Theorem 1.3.3 This is only a quick sketch of the proof, and omits many formal details.

We first show that $\hat{\beta} \to_p \beta_0$. By the WLLN, for all β,

$$\frac{1}{n} \mathcal{L}_n(\beta) = \frac{1}{n} \sum_{i=1}^{n} \ln f(X_i, \beta) \to_p E \ln f(X_i, \beta) = \mathcal{L}(\beta),$$

say. Furthermore, it can be shown that this convergence is uniform in β. Thus for large n, $n^{-1}\mathcal{L}_n(\beta)$ is uniformly very close to $\mathcal{L}(\beta)$. It therefore seems reasonable to expect that the MLE $\hat{\beta}$ which maximizes $\mathcal{L}_n(\beta)$ will be close to the β which maximizes $\mathcal{L}(\beta)$, and indeed this can be shown. The maximizer of $\mathcal{L}(\beta)$ is β_0, since (1.7) shows that $\frac{\partial}{\partial \beta} \mathcal{L}(\beta_0) = 0$. Together, this argument establishes that $\hat{\beta} \to_p \beta_0$.

Taking the first-order condition for maximization of $\mathcal{L}_n(\beta)$, and making a first-order Taylor series expansion,

$$\begin{aligned} 0 &= \frac{\partial}{\partial \beta} \mathcal{L}_n(\beta) \Big|_{\beta = \hat{\beta}} \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial \beta} \ln f\left(X_i, \hat{\beta}\right) \\ &\simeq \sum_{i=1}^{n} \frac{\partial}{\partial \beta} \ln f(X_i, \beta_0) + \sum_{i=1}^{n} \frac{\partial^2}{\partial \beta \partial \beta'} \ln f(X_i, \beta_0) \left(\hat{\beta} - \beta_0\right). \end{aligned}$$

The final approximation makes use of the fact that $\hat{\beta} \to_p \beta_0$. Rewriting this equation, we find

$$\left(\hat{\beta} - \beta_0\right) = \left(- \sum_{i=1}^{n} \frac{\partial^2}{\partial \beta \partial \beta'} \ln f(X_i, \beta_0) \right)^{-1} \left(\sum_{i=1}^{n} \frac{\partial}{\partial \beta} \ln f(X_i, \beta_0) \right).$$

An application of the WLLN yields

$$-\frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2}{\partial\beta\partial\beta'}\ln f\left(X_i,\beta_0\right)\to H.$$

Since $\frac{\partial}{\partial\beta}\ln f\left(X_i,\beta_0\right)$ is mean-zero with covariance matrix Ω, an application of the CLT yields

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial}{\partial\beta}\ln f\left(X_i,\beta_0\right)\to_d N\left(0,\Omega\right).$$

Together,

$$\sqrt{n}\left(\hat{\beta}-\beta_0\right)\to_d H^{-1}N\left(0,\Omega\right)=N\left(0,H^{-1}\Omega H^{-1}\right)=N\left(0,H^{-1}\right),$$

the final equality using Theorem 1.3.1 . ■

Chapter 2

Method of Moments Estimation

2.1 Random Sampling

An econometrician has the observational data

$$\{w_1, w_2, ..., w_i, ..., w_n\},$$

where each w_i is a vector of data on an individual (e.g., household or firm).

If this data is **cross-sectional** (each observation is a different individual) it is considered reasonable to assume they are mutually independent. If the data are symmetrically gathered (e.g., randomly), it is reasonable to model each observation as a random draw from the same probability distribution. Thus, the data are **independent and identically distributed**, or **IID**. We call this a **random sample**.

The observed data $\{w_1, ..., w_n\}$ are called the **sample**, as they are treated as random draws. We call their distribution F the **population**. The goal in statistical inference is to learn about characteristics of the population F from a random sample. That is, we want to learn about the true distribution of the random variables. This "population" is infinitely large. Sometimes this is a source of confusion, but it is merely an abstraction.

2.2 Moment Equations

Many econometric models are completely summarized by a finite set of parameters and moment conditions. We will collect the parameters in a $k \times 1$ vector β. It will sometimes be convenient to let β_0 denote the true value of the parameters and to let β denote a generic value. Let $g(w, \beta)$ be an $\ell \times 1$ vector of functions for which the expectation $Eg(w_i, \beta)$ is well defined. The moment conditions take the form

$$Eg(w_i, \beta_0) = 0. \tag{2.1}$$

and this equality only holds at the true value β_0, not when evaluated at other values.

We will focus for now on the **just-identified** case where $\ell = k$. If $\ell < k$ then the parameter β is **underidentified** and cannot be estimated without additional information. If $\ell > k$ then the parameters are **overidentified** and estimation becomes more complicated.

For example, suppose $w_i \in R$ and we are interested in the mean and variance of w_i. Then

$$\beta = \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}$$

and

$$g(w, \beta) = \begin{pmatrix} w - \mu \\ (w - \mu)^2 - \sigma^2 \end{pmatrix}. \tag{2.2}$$

Clearly, $Eg(w_i, \beta) = 0$ only if evaluated at the true mean and variance.

As another example, suppose we are interested in the median of w_i, the number β such that $P(W \leq \beta) = \frac{1}{2}$. Then we can set

$$g(w, \beta) = 1(w \leq \beta) - \frac{1}{2}.$$

Finally, many rational expectations models have implications of the form $E\left(w_i^{\beta}\right) = \theta$. If θ is known then we can set

$$g(w, \beta) = w^{\beta} - \theta.$$

2.3 Empirical Likelihood Estimation

If all that is known about β is (2.1), how should the parameter β be estimated from a random sample? The classical method of maximum likelihood does not apply, since (2.1) does not fully characterize the distribution of w_i. The solution is to construct a nonparametric analog of the likelihood function which imposes the equation (2.1) but no additional structure.

A general solution known as **empirical likelihood** or **EL** was introduced by Art Owen (1988, 2001). The idea is to construct a multinomial distribution $F(p_1, ..., p_n)$ which places probability p_i at each observation w_i, $i = 1, ..., n$. To be a valid multinomial distribution, these probabilities must satisfy the requirements that $p_i \geq 0$ and

$$\sum_{i=1}^{n} p_i = 1. \tag{2.3}$$

Otherwise they are unconstrained.

For any set of the probabilities $(p_1, ..., p_n)$ the moment condition (2.1) is

$$0 = \sum_{i=1}^{n} p_i g(w_i, \beta). \tag{2.4}$$

By the implicit function theorem we can solve for β as a function of the probabilities, i.e. $\beta = h(p_1, ..., p_n)$. That is, for any $p_1, ..., p_r$ we can solve the k equations (2.4) to find the $k \times 1$ vector β. As the p_i change the vector β changes. The function h is implicitly defined by this mapping.

In some cases the solution is explicit. For example, in the mean-variance example above,

$$\begin{aligned} \mu(p_1, ..., p_n) &= \sum_{i=1}^{n} p_i w_i \\ \sigma^2(p_1, ..., p_n) &= \sum_{i=1}^{n} p_i w_i^2 - \left(\sum_{i=1}^{n} p_i w_i \right)^2. \end{aligned}$$

Since each of the values w_i is observed once in the sample, the log-likelihood function for this multinomial distribution is simply

$$\mathcal{L}_n(p_1, ..., p_n) = \sum_{i=1}^{n} \log(p_i).$$

The maximum likelihood principle suggests picking $(p_1, ..., p_n)$ to maximize this function subject to the constraint (2.3). This is equivalent to maximizing

$$\sum_{i=1}^{n} \log(p_i) - \lambda \left(\sum_{i=1}^{n} p_i - 1 \right)$$

where λ is a Lagrange multiplier. This has the n first order conditions $0 = p_i^{-1} - \lambda$. Combined with the constraint (2.3) we find that the optimal solution is $p_i = n^{-1}$ yielding the log-likelihood $-n \log(n)$.

Recalling that the parameter β is a function of the probabilities, this means that the **empirical likelihood estimator** of β is obtained by substituting these optimized values. This is $\hat{\beta} = h(n^{-1}, ..., n^{-1})$, so that $\hat{\beta}$ solves the equation

$$0 = \frac{1}{n} \sum_{i=1}^{n} g\left(w_i, \hat{\beta}\right) = \overline{g}_n(\hat{\beta}) \tag{2.5}$$

where

$$\overline{g}_n(\beta) = \frac{1}{n} \sum_{i=1}^{n} g\left(w_i, \beta\right)$$

is the sample moment analog of the population expectation $Eg(w_i, \beta)$.

The estimator which solves (2.5) is also known as the **method of moments estimator**. We could have simply defined this estimator of β directly, but the empirical likelihood method will be easier to generalize to broader contexts. It is a general principle which tends to yield estimators with excellent properties.

Returning to the mean-variance example, the EL estimator of the parameters turns out to be the sample mean and variance.

2.4 Efficiency

Is the EL estimator efficient, in the sense of achieving the smallest possible mean-squared error among feasible estimators? The answer was affirmatively provided by Chamberlain (1987).

Suppose that the distribution of w_i is discrete. That is, for finite r,

$$P(w_i = \tau_j) = \pi_j, \qquad j = 1, ..., r$$

for some constant vectors τ_j and constants π_j. Assume that the τ_j are known, but the π_j are unknown. (We know the values w_i can take, but we don't know the probabilities.)

In this discrete setting, the moment condition (2.1) can be rewritten as

$$\sum_{j=1}^{r} \pi_j g(\tau_j, \beta_0) = 0. \tag{2.6}$$

By the implicit function theorem, we can write $\beta = h(\pi_1, ..., \pi_r)$.

As the data are multinomial, the maximum likelihood estimator (MLE) is

$$\hat{\pi}_j = \frac{1}{n} \sum_{i=1}^{n} 1(w_i = \tau_j)$$

for $j = 1, ..., r$, where $1(\cdot)$ is the indicator function. That is, $\hat{\pi}_j$ is the percentage of the observed w_i which equal τ_j. The MLE for β is then $\hat{\beta} = h(\hat{\pi}_1, ..., \hat{\pi}_r)$. By the definition of the function h, this means that $\hat{\beta}$ solves the set of equations

$$\sum_{j=1}^{r} \hat{\pi}_j g(\tau_j, \hat{\beta}) = 0.$$

Substituting in the expressions for $\hat{\pi}_j$,

$$\begin{aligned} 0 &= \sum_{j=1}^{r} \left(\frac{1}{n} \sum_{i=1}^{n} 1(w_i = \tau_j) \right) g(\tau_j, \hat{\beta}) \\ &= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{r} 1(w_i = \tau_j) g(\tau_j, \hat{\beta}) \\ &= \frac{1}{n} \sum_{i=1}^{n} g(w_i, \hat{\beta}) \\ &= \overline{g}_n(\hat{\beta}). \end{aligned}$$

Thus the MLE equals the MME, which is the EL estimator. The population equation (2.1) exhausts all that is known about the parameter. Thus in this setting, MLE, MME and EL are

identical. Since this is a regular parametric model the MLE is asymptotically efficient, and thus so is the EL estimator. The only restriction we made is that the data w_i come from a discrete distribution.

Chamberlain (1987) extends this argument to the case of continuously-distributed w_i. He observes that the above argument holds for all multinomial distributions, and any continuous distribution can be arbitrarily well approximated by a multinomial distribution. He then shows that this allows us to prove that generically the MME estimator is asymptotically efficient.

Chapter 3

Multivariate Regression

3.1 Regression

Partition

$$w_i = (y_i, x_i)$$

where $y_i \in R$, $x_i \in R^k$.

In regression, we want to find the central tendency of the conditional distribution of y_i given x_i. A standard measure of central tendency is the mean. The conditional analog is the **conditional mean** $m(x) = E(y_i \mid x_i = x)$ (see section 1.2.5). In general, $m(x)$ can take any form.

The regression error e_i is defined to be the difference between y_i and its conditional mean:

$$e_i = y_i - m(x_i).$$

By construction, this yields the formula

$$y_i = m(x_i) + e_i. \tag{3.1}$$

It is worth emphasizing that no assumptions have been used to develop (3.1), other than that (y_i, x_i) have a joint distribution and $E|y_i| < \infty$.

Proposition 3.1.1 *Properties of the regression error e_i*

1. $E(e_i \mid x_i) = 0$.
2. $E(e_i) = 0$.
3. $E(h(x_i)e_i) = 0$ *for any function* $h(\cdot)$.
4. $E(x_i e_i) = 0$.

Proof:

1. By the definition of e_i and the linearity of conditional expectations,

$$\begin{aligned} E\left(e_i \mid x_i\right) &= E\left(\left(y_i - m(x_i)\right) \mid x_i\right) \\ &= E\left(y_i \mid x_i\right) - E\left(m(x_i) \mid x_i\right) \\ &= m(x_i) - m(x_i) \\ &= 0. \end{aligned}$$

2. By the law of iterated expectations (Theorem 1.2.1) and the first result,

$$\begin{aligned} E(e_i) &= E\left(E\left(e_i \mid x_i\right)\right) \\ &= E(0) \\ &= 0. \end{aligned}$$

3. By a similar argument, and using the conditioning theorem (Theorem 1.2.2),

$$\begin{aligned} E(h(x_i)e_i) &= E\left(E\left(h(x_i)e_i \mid x_i\right)\right) \\ &= E\left(h(x_i)E\left(e_i \mid x_i\right)\right) \\ &= E(h(x_i) \bullet 0) \\ &= 0. \end{aligned}$$

4. Follows from the third result setting $h(x_i) = x_i$. ■

The final result is very useful. This implies that e_i and x_i are **uncorrelated**. It is important to understand that despite being uncorrelated, in general e_i need not be independent of x_i. We discuss this implication further in the next section.

Equation (3.1) plus the first result of Proposition 3.1.1 are often stated jointly as the regression framework:

$$\begin{aligned} y_i &= m(x_i) + e_i \\ E\left(e_i \mid x_i\right) &= 0. \end{aligned} \tag{3.2}$$

It is important to understand that this is a framework, not a model, because no restrictions have been placed on the joint distribution of the data. These equations hold true by definition. A regression model, as we explore in the following sections, imposes further restrictions on the joint distribution; most typically, restrictions on the permissible class of regression functions $m(x)$.

3.2 Conditional Variance

The conditional variance of y_i given $x_i = x$ is

$$Var\left(y_i \mid x_i = x\right) = E\left(e_i^2 \mid x_i = x\right) = \sigma^2(x).$$

Generally, this is a function of x. Just as the conditional mean function may take any form, so may the conditional variance function (other than the restriction that it is non-negative). Given the random variable x_i, the conditional variance is $\sigma_i^2 = \sigma^2(x_i)$.

It is possible that $\sigma^2(x)$ is a constant, or that

$$E\left(e_i^2 \mid x_i\right) = \sigma^2. \tag{3.3}$$

In this case we say that the error e_i is **homoskedastic**. In the general case where $\sigma^2(x)$ is not necessarily a constant function, so σ_i^2 may different across i, we say that the error e_i is **heteroskedastic**.

3.3 Linear Models

While $m(x)$ in general can take any shape, we typically pick a parametric family $\{m(x, \beta) : \beta \in R^l\}$ to simplify estimation and interpretation. Sometimes, the form of $m(x, \theta)$ is given by an economic theory or model. Most often, however, we a linear form for convenience and data coherence.

For convenience, we revise the notation so that the vector x_i is presumed to include a constant, that is, the first element of x_i equals 1. We denote the $k-$element regressor x_i as

$$x_i = \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ki} \end{pmatrix} = \begin{pmatrix} 1 \\ x_{2i} \\ \vdots \\ x_{ki} \end{pmatrix}.$$

A linear model for $g(x)$ then takes the form

$$m(x_i) = \beta_1 + x_{2i}\beta_2 + \cdots + x_{ki}\beta_k$$

where β_1 through β_k are parameters. The parameter vector β is written as

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}.$$

In vector notation,

$$m(x_i) = x_i'\beta.$$

The regression model is then

$$\begin{aligned} y_i &= x_i'\beta + e_i \\ E\left(e_i \mid x_i\right) &= 0. \end{aligned} \tag{3.4}$$

This is a model because m has been restricted to the linear form.

While linearity is substantively restrictive, it has still a great deal of flexibility. For example, if x_i is real-valued and

$$m(x_i) = \beta_1 + x_i\beta_2 + x_i^2\beta_3 + \cdots + x_i^{k-1}\beta_k$$

is a polynomial, then a linear regression model still holds, by the redefinition of x_i as $(1, x_i, x_i^2, ..., x_i^{k-1})$.

3.4 Projection

The linear regression model (3.4) implies $E(x_i e_i) = 0$. It is interesting to observe that in linear models, there is always a vector β such that this equation holds. This vector β may be called the linear projection coefficient or linear predictor.

Proposition 3.4.1 *For any random variables (y_i, x_i), let*

$$\beta = \left(E\left(x_i x_i'\right)\right)^{-1} E\left(x_i y_i\right) \tag{3.5}$$

and

$$e_i = y_i - x_i'\beta.$$

Then

$$E(x_i e_i) = 0.$$

The result follows from the simple calculation

$$E(x_i e_i) = E(x_i \left(y_i - x_i' \left(E\left(x_i x_i'\right)\right)^{-1} E\left(x_i y_i\right)\right)) = 0.$$

While $E(x_i e_i) = 0$ holds true by construction, it does not necessarily follow that $E\left(e_i \mid x_i\right) = 0$. This only holds if the true conditional mean of y_i is $x_i'\beta$, which is a substantive restriction. Thus the linear regression assumption that $E\left(e_i \mid x_i\right) = 0$ is more restrictive than the linear projection construction.

It turns out that for most issues in statistical inference, the projection assumption is sufficient. Therefore we will adopt the more general assumption $E(x_i e_i) = 0$ wherever possible for our analysis.

For econometric practice, however, it is typically desired to estimate a regression equation rather than a projection equation. That is, it is typically desirable for $x_i'\beta$ to represent the conditional mean of y_i, rather than a simple linear projection. So while it is not necessary for inference on β, it may be necessary for inference on an economic relationship of interest.

3.5 Regression Error

While the regression motivation leads naturally to the model (3.4), at times it is more convenient to adopt assumptions which are either more restrictive or less restrictive. We now describe the standard types of models considered by econometricians, and discuss their strengths and weaknesses. All the models are based on the decomposition

$$y_i = x_i'\beta + e_i. \tag{3.6}$$

In addition, all models normalize the error so that $E(e_i) = 0$ and presume a finite variance $E\left(e_i^2\right) = \sigma^2 < \infty$.

Definition 3.5.1 *The Linear Projection Model is (3.6) plus*

$$E\left(x_i e_i\right) = 0. \tag{3.7}$$

The advantage of the linear projection model is that it is true by construction, and many inferential results hold under this broad condition. The disadvantage is that the coefficients β may not have useful economic interpretations without additional structure.

Definition 3.5.2 *The Linear Regression Model is (3.6) plus*

$$E\left(e_i \mid x_i\right) = 0.$$

This model leads naturally from the derivation of the conditional mean function. The primary advantage is that the parameter β is easily interpretable.

Definition 3.5.3 *The Homoskedastic Regression Model is the Linear Regression model plus (3.3). [Note: Hayashi calls this the Classical Model.]*

The homoskedasticity assumption (3.3) greatly simplifies many theoretical arguments and calculations, and is therefore very useful in pedagogical (illustrative) arguments. Many formulae simplify under this assumption, and as a result, alternative estimators and techniques are utilized. The danger in this assumption is that these simplifications result in incorrect answers and inferences if the homoskedasticity assumption is false. As a result, we try to avoid making this assumption wherever possible, since the empirical validity of conditional homoskedasticity is always doubtful.

Another meaningful justification for assumption (3.3) is that while it may not be precisely true in the data, it may be approximately true, and in some applications the cost of imposing homoskedasticity on the estimates may be less than the cost of using the more general techniques appropriate for the linear regression model.

Definition 3.5.4 *The Independent Error Model is (3.6) plus that e_i is independent of x_i.*

This model is more restrictive than the homoskedastic regression model, and is a common starting point in classical econometrics textbooks.

Definition 3.5.5 *The Normal Regression Model is (3.6) plus that e_i is independent of x_i and distributed as $N(0, \sigma^2)$.*

This model is a further restriction, and is also described in classical econometrics textbooks. The primary advantage of this model is that exact distributional results are available for the OLS estimators and test statistics. These exact results are less popular in current econometric practice, since the independence and normality assumptions are rarely considered viable.

The five models listed above are strictly nested, with the first (the linear projection model) the least restrictive, and the last (the normal regression model) the most restrictive. In this class, we will typically restrict attention to the first three models, with most attention being devoted to the first and second. At many points in the analysis, we will discuss the implications of the homoskedasticity assumption (3.3), as many results are substantively different under this condition.

3.6 Model in Matrix Notation

The linear regression model of equation (3.4) is a system of n equations, one for each observation. We can stack these n equations together as

$$\begin{aligned} y_1 &= x_1'\beta + e_1 \\ y_2 &= x_2'\beta + e_2 \\ &\vdots \\ y_n &= x_n'\beta + e_n. \end{aligned}$$

Let

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \qquad X = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}, \qquad e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}.$$

Observe that Y and e are $n \times 1$ vectors, and X is an $n \times k$ matrix. Using this matrix notation, the n equations may be written in the single equation

$$Y = X\beta + e.$$

3.7 Estimation

For the general projection model, the only information we have on β is (3.7). This can be written as the moment condition

$$Eg(w_i, \beta_0) = 0$$

where

$$g(w_i, \beta) = x_i \left(y_i - x_i'\beta\right)$$

In this setting asymptotically efficient estimation is achieved by the EL or MME estimator $\hat{\beta}$ which solves the equation

$$0 = \overline{g}_n(\hat{\beta}) \tag{3.8}$$

where

$$\begin{aligned} \overline{g}_n(\beta) &= \frac{1}{n}\sum_{i=1}^{n} g(w_i, \beta) \\ &= \frac{1}{n}\sum_{i=1}^{n} x_i \left(y_i - x_i'\beta\right) \\ &= \frac{1}{n}\left(X'Y - X'X\beta\right) \end{aligned}$$

(3.8) implies

$$0 = \frac{1}{n}\left(X'Y - X'X\hat{\beta}\right).$$

Solving for $\hat{\beta}$ we find explicitly:

Proposition 3.7.1 *The EL or MME estimator is*

$$\hat{\beta} = \left(X'X\right)^{-1}\left(X'Y\right).$$

We define the **predicted value** $\hat{y}_i = x_i'\hat{\beta}$ and the **residual**

$$\begin{aligned} \hat{e}_i &= y_i - \hat{y}_i \\ &= y_i - x_i'\hat{\beta}. \end{aligned}$$

Another way of writing the relationship between the predicted value, residual, and observation is

$$y_i = \hat{y}_i + \hat{e}_i.$$

In vector notation, $\hat{Y} = X\hat{\beta}$, $\hat{e} = Y - X\hat{\beta}$, and $Y = \hat{Y} + \hat{e}$.

Note that by definition,

$$0 = \overline{g}_n(\hat{\beta}) = \frac{1}{n} X'\hat{e}$$

Thus

$$X'\hat{e} = 0$$

and the residual vector $\hat{e}$ is orthogonal to the columns of the regressor matrix X.

Since the first column of X is a vector of ones, $X'\hat{e} = 0$ implies that $\sum_{i=}^{n} \hat{e}_i = 0$.

Using the projection matrices of section 1.1.11,

$$\hat{Y} = X\hat{\beta} = X\left(X'X\right)^{-1} X'Y = PY$$

and

$$\hat{e} = Y - X\hat{\beta} = Y - PY = (I_n - P)\, Y = MY.$$

Another way of writing this is

$$\begin{aligned} Y &= (P + M)\, Y \\ &= PY + MY \\ &= \hat{Y} + \hat{e}. \end{aligned}$$

This decomposition is **orthogonal**, that is

$$\begin{aligned} \hat{Y}'\hat{e} &= (PY)'\,(MY) \\ &= Y'PMY \\ &= 0. \end{aligned}$$

3.8 Least Squares

The MME estimator $\hat{\beta}$ is also known as the **Ordinary Least Squares** or **OLS** estimator. The **sum-of-squared errors** (SSE) function is

$$\begin{aligned} S_n(\beta) &= \sum_{i=1}^{n} \left(y_i - x_i'\beta\right)^2 \\ &= Y'Y - 2Y'X\beta + \beta'X'X\beta \end{aligned}$$

The definition of OLS is the vector $\hat{\beta}$ which minimizes $S_n(\beta)$. Vector calculus (see section 1.1.13) gives the first-order conditions for minimization:

$$0 = \frac{\partial}{\partial \beta} S_n(\hat{\beta}) = -2X'Y + 2X'X\hat{\beta}$$

which is the same as for the MME estimator.

Proposition 3.8.1 *The solution to* $\min_\beta S_n(\beta)$ *is* $\hat{\beta} = (X'X)^{-1}(X'Y)$.

Following convention, we will call $\hat{\beta}$ the OLS estimator of β.

3.9 Gaussian QMLE

In this section, we relate a traditional motivation for the OLS estimator. The normal regression model is $y_i = x_i'\beta + e_i$ with $e_i \sim N(0, \sigma^2)$. The log-likelihood function is

$$\begin{aligned}
\mathcal{L}_n(\beta, \sigma^2) &= \sum_{i=1}^{n} \log\left(\frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{1}{2\sigma^2}\left(y_i - x_i'\beta\right)^2\right)\right) \\
&= -\frac{n}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - x_i'\beta\right)^2 \\
&= -\frac{n}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}S_n(\beta).
\end{aligned}$$

The MLE $(\hat{\beta}, \hat{\sigma}^2)$ maximize $\mathcal{L}_n(\beta, \sigma^2)$. But since $\mathcal{L}_n(\beta, \sigma^2)$ is a function of β only through $S_n(\beta)$, the MLE minimizes $S_n(\beta)$.

Proposition 3.9.1 *In the normal regression model, the MLE is* $\hat{\beta} = (X'X)^{-1}(X'Y)$

Due to this equality, the OLS estimator $\hat{\beta}$ is frequently referred to as the "Gaussian MLE" or the "Gaussian Quasi-MLE". The term "quasi" is used to refer the context where the normality assumption has been used to construct the likelihood and the estimator, but the normality assumption is not believed to be true (see section 1.3.4).

3.10 Frisch-Waugh-Lovell (FWL) Theorem

Partition

$$X = [X_1 \quad X_2]$$

and

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$

Then the regression model can be rewritten as

$$Y = X_1\beta_1 + X_2\beta_2 + e. \tag{3.9}$$

Observe that the OLS estimator of $\beta = (\beta_1', \beta_2')'$ can be obtained by regression of Y on $X = [X_1 \; X_2]$. OLS estimation can be written as

$$Y = X_1\hat{\beta}_1 + X_2\hat{\beta}_2 + \hat{e} \tag{3.10}$$

where $\hat{e} = MY$ with $M = I - P$ and $P = X(X'X)^{-1}X'$.

Now define $P_1 = X_1(X_1'X_1)^{-1}X_1'$ and $M_1 = I - P_1$. Observe that since P_1 lies in the range space of X, $PP_1 = P_1$, so $P_1P = P_1$ and

$$\begin{aligned} M_1M &= (I - P_1)(I - P) \\ &= I - P_1 - P + P_1P \\ &= I - P = M. \end{aligned}$$

Now pre-multiply both sides of (3.10) by M_1 (and note that $M_1X_1 = 0$) to obtain

$$\begin{aligned} M_1Y &= M_1X_1\hat{\beta}_1 + M_1X_2\hat{\beta}_2 + M_1\hat{e} \\ &= M_1X_2\hat{\beta}_2 + M_1MY \\ &= M_1X_2\hat{\beta}_2 + MY. \end{aligned} \tag{3.11}$$

Next pre-multiply by X_2', and note that since $MX_2 = 0$, we obtain

$$\begin{aligned} X_2'M_1Y &= X_2'M_1X_2\hat{\beta}_2 + X_2'MY \\ &= X_2'M_1X_2\hat{\beta}_2. \end{aligned}$$

Inverting $(X_2'M_1X_2)$, we find a new expression for the OLS estimate $\hat{\beta}_2$:

$$\hat{\beta}_2 = \left(X_2'M_1X_2\right)^{-1}\left(X_2'M_1Y\right). \tag{3.12}$$

Since M_1 is idempotent, we can rewrite this as

$$\begin{aligned} \hat{\beta}_2 &= \left(X_2'M_1M_1X_2\right)^{-1}\left(X_2'M_1M_1Y\right) \\ &= \left(\tilde{X}_2'\tilde{X}_2\right)^{-1}\left(\tilde{X}_2'\tilde{Y}\right), \end{aligned}$$

where $\tilde{X}_2 = M_1X_2$ and $\tilde{Y} = M_1Y$. We have shown that the OLS estimate $\hat{\beta}_2$ can alternatively be computed from a simple OLS regression of $\tilde{Y}$ on $\tilde{X}_2$. Interestingly, $\tilde{Y}$ is the OLS residual from a regression of Y on X_1, and $\tilde{X}_2$ is the OLS residual from a regression of X_2 on X_1.

Furthermore, (3.11) says that

$$\tilde{Y} = \tilde{X}_2\hat{\beta}_2 + \hat{e}. \tag{3.13}$$

Since we have shown that $\hat{\beta}_2$ is the OLS estimate from a regression of $\tilde{Y}$ on $\tilde{X}_2$, the OLS residual from this regression is $\tilde{Y} - \tilde{X}_2\hat{\beta}_2$, and (3.13) shows that this equals $\hat{e}$, the OLS residual from regression on Y on X. We have proven the following theorem.

Theorem 3.10.1 *(Frisch-Waugh-Lovell). In the model (3.9), the OLS estimator of β_2 and the OLS residuals $\hat{e}$ may be equivalently computed by either the OLS regression (3.10) or via the following algorithm:*

1. *Regress Y on X_1, obtain residuals $\tilde{Y}$;*

2. *Regress X_2 on X_1, obtain residuals $\tilde{X}_2$;*

3. *Regress $\tilde{Y}$ on $\tilde{X}_2$, obtain LS estimates $\hat{\beta}_2$ and residuals $\hat{e}$.*

In some contexts, the FWL theorem can be used to speed computation, but in most cases there is little computational advantage to using the two-step algorithm. Rather, the theorem's primary use is theoretical.

A common application of the FWL theorem, which you may have seen in an introductory econometrics course, is the demeaning formula for regression.

Partition $X = [X_1 \ X_2]$ where $X_1 = \iota$ is a vector of ones, and X_2 is the vector of observed regressors. In this case,

$$M_1 = I - \iota \left(\iota'\iota\right)^{-1} \iota'.$$

Observe that

$$\begin{aligned} \tilde{X}_2 &= M_1 X_2 = X_2 - \iota \left(\iota'\iota\right)^{-1} \iota' X_2 \\ &= X_2 - \overline{X}_2 \end{aligned}$$

and

$$\begin{aligned} \tilde{Y} &= M_1 Y = Y - \iota \left(\iota'\iota\right)^{-1} \iota' Y \\ &= Y - \overline{Y}, \end{aligned}$$

which are "demeaned".

The FWL theorem says that $\hat{\beta}_2$ is the OLS estimate from a regression of $\tilde{Y}$ on $\tilde{X}_2$, or $y_i - \overline{y}$ on $x_{2i} - \overline{x}_2$:

$$\hat{\beta}_2 = \left(\sum_{i=1}^{n} \left(x_{2i} - \overline{x}_2\right)\left(x_{2i} - \overline{x}_2\right)'\right)^{-1} \left(\sum_{i=1}^{n} \left(x_{2i} - \overline{x}_2\right)\left(y_i - \overline{y}\right)\right).$$

Thus the OLS estimator for the slope coefficients is a regression with demeaned data.

3.11 Consistency

Assumption 3.11.1 *$E\left(x_i e_i\right) = 0$, $\sigma^2 = Ee_i^2 < \infty$, $Ex_i'x_i < \infty$, and $Q = Ex_i x_i' > 0$.*

The assumptions that $Ex_i'x_i < \infty$ and $\sigma^2 < \infty$ mean that all elements of x_i and e_i have finite second moments, and all cross-moments are finite. To see this, first observe that since

$$Ex_i'x_i = Ex_{1i}^2 + \cdots + Ex_{ki}^2 < \infty,$$

then it is the case that for all $j = 1, ..., k$, $Ex_{ji}^2 < \infty$. By the Cauchy-Schwarz inequality (section 1.2.8), for each j and l,

$$E\left|x_{ji}x_{li}\right| \leq E\left|x_{ji}^2\right|^{1/2} E\left|x_{li}^2\right|^{1/2} < \infty. \tag{3.14}$$

And for each j,

$$E\left|x_{ji}e_i\right| \leq E\left|x_{ji}^2\right|^{1/2}\sigma < \infty. \tag{3.15}$$

The following decomposition is quite useful.

Proposition 3.11.1 *If $Y = X\beta + e$, then $\hat{\beta} - \beta = (X'X)^{-1}X'e$.*

Proof: Since $Y = X\beta + e$,

$$\begin{aligned}
\hat{\beta} &= \left(X'X\right)^{-1}X'y \\
&= \left(X'X\right)^{-1}X'\left(X\beta + e\right) \\
&= \left(X'X\right)^{-1}X'X\beta + \left(X'X\right)^{-1}X'e \\
&= \beta + \left(X'X\right)^{-1}X'e.
\end{aligned}$$

■

We can now deduce the consistency of $\hat{\beta}$

Theorem 3.11.1 *Under Assumption 3.11.1, as $n \to \infty$*

1. $\frac{1}{n}X'X \to_p Q$;
2. $\frac{1}{n}X'e \to_p 0$;
3. $\hat{\beta} \to_p \beta$.

Proof: The bound (3.14) and the WLLN (section 1.2.9) directly imply that for all j and l

$$\frac{1}{n}\sum_{i=1}^{n} x_{ji}^2 \to_p Ex_{ji}^2$$

and

$$\frac{1}{n}\sum_{i=1}^{n} x_{ji}x_{li} \to_p Ex_{ji}x_{li}.$$

Hence

$$\begin{aligned}
\frac{1}{n}X'X &= \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n x_{1i}^2 & \frac{1}{n}\sum_{i=1}^n x_{1i}x_{2i} & \cdots & \frac{1}{n}\sum_{i=1}^n x_{1i}x_{ki} \\ \frac{1}{n}\sum_{i=1}^n x_{2i}x_{1i} & \frac{1}{n}\sum_{i=1}^n x_{2i}^2 & \cdots & \frac{1}{n}\sum_{i=1}^n x_{2i}x_{ki} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{n}\sum_{i=1}^n x_{ki}x_{1i} & \frac{1}{n}\sum_{i=1}^n x_{ki}x_{2i} & \cdots & \frac{1}{n}\sum_{i=1}^n x_{ki}^2 \end{pmatrix} \\
&\to_p \begin{pmatrix} Ex_{1i}^2 & Ex_{1i}x_{2i} & \cdots & Ex_{1i}x_{ki} \\ Ex_{2i}x_{1i} & Ex_{2i}^2 & \cdots & Ex_{2i}x_{ki} \\ \vdots & \vdots & \cdots & \vdots \\ Ex_{ki}x_{1i} & Ex_{ki}x_{2i} & \cdots & Ex_{ki}^2 \end{pmatrix} \\
&= E\begin{pmatrix} x_{1i}^2 & x_{1i}x_{2i} & \cdots & x_{1i}x_{ki} \\ x_{2i}x_{1i} & x_{2i}^2 & \cdots & x_{2i}x_{ki} \\ \vdots & \vdots & \cdots & \vdots \\ x_{ki}x_{1i} & x_{ki}x_{2i} & \cdots & x_{ki}^2 \end{pmatrix} \\
&= E\left(x_i x_i'\right) = Q,
\end{aligned}$$

which is the first result.

The bound (3.15) and the WLLN imply that for all j

$$\frac{1}{n}\sum_{i=1}^n x_{ji}e_i \to_p E\left(x_{ji}e_i\right) = 0.$$

Hence

$$\frac{1}{n}X'e = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n x_{1i}e_i \\ \frac{1}{n}\sum_{i=1}^n x_{2i}e_i \\ \vdots \\ \frac{1}{n}\sum_{i=1}^n x_{ki}e_i \end{pmatrix} \to_p 0,$$

which is the second result.

Finally, by Proposition 3.11.1

$$\begin{aligned}
\hat{\beta} &= \beta + \left(X'X\right)^{-1}\left(X'e\right) \\
&= \beta + \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}X'e\right) \\
&= \beta + g\left(\frac{1}{n}X'X, \frac{1}{n}X'e\right)
\end{aligned}$$

where $g(A,b) = A^{-1}b$ is a continuous function of A and b, at all values of the arguments such that A^{-1} exist. Now by the first two parts of this theorem,

$$\left(\frac{1}{n}X'X, \frac{1}{n}X'e\right) \to_p (Q, 0).$$

Under Assumption 3.11.1, $Q > 0$, so Q^{-1} exists and $g(\cdot,\cdot)$ is continuous at $(Q, 0)$. Hence by the CMT (section 1.2.9),

$$g\left(\frac{1}{n}X'X, \frac{1}{n}X'e\right) \to_p g(Q,0) = Q^{-1}0 = 0$$

so $\hat{\beta} = \beta + g\left(\frac{1}{n}X'X, \frac{1}{n}X'e\right) \to_p \beta + 0 = 0$, which is the third result. ■

3.12 Asymptotic Normality

Except in special cases, the exact distribution of $\hat{\beta}$ is unknown. Therefore we rely on approximations, and use a variety of techniques to assess the accuracy of these approximations. The dominate approximation technique is asymptotic theory, and is based on calculating the limiting distribution of a normalized version of $\hat{\beta}$ as the sample size n tends to positive infinity. The heuristic idea is that this approximation is likely to be accurate in contexts where the sample size n is "large", which is common in many econometric studies.

We need a stronger set of assumptions than for consistency. Define

$$\Omega = E\left(x_i x_i' e_i^2\right). \tag{3.16}$$

Ω has finite elements if x_i and e_i have finite fourth moments.

Assumption 3.12.1 $E(x_i e_i) = 0$, $Ee_i^4 < \infty$, $E|x_i|^4 < \infty$, $Q = Ex_i x_i' > 0$, *and* $\Omega > 0$.

The final statement that $\Omega > 0$ is not needed for many results, but is important for the development of test statistics. If $Q > 0$ and $\sigma^2 > 0$, it is typically believed that requiring $\Omega > 0$ is not meaningfully restrictive.

The following result is helpful.

Theorem 3.12.1 *Under Assumption 3.12.1, for* $u_i = x_i e_i$,

1. u_i *is iid,* $Eu_i = 0$, $Eu_i' u_i < \infty$, *and* $Eu_i u_i' = \Omega$;
2. $\frac{1}{\sqrt{n}} \sum_{i=1}^n u_i \to_d N(0, \Omega)$.

Proof: First, since (y_i, x_i) is iid, so is any function of (y_i, x_i), including $u_i = x_i(y_i - x_i\beta)$. Second, $Eu_i = Ex_i e_i = 0$ under Assumption 3.12.1. Third, by the Cauchy-Schwarz inequality and Assumption 3.12.1,

$$\begin{aligned} E\left|u_i' u_i\right| &= E\left|u_i u_i'\right| = E\left|x_i x_i' e_i^2\right| \\ &\le \left(E\left|x_i x_i'\right|^2\right)^{1/2} \left(E\left|e_i^4\right|\right)^{1/2} \\ &= \left(E|x_i|^4\right)^{1/2} \left(E\left|e_i^4\right|\right)^{1/2} < \infty. \end{aligned}$$

Fourth, $Eu_iu_i' = Ex_ix_i'e_i^2 = \Omega$ by (3.16). This establishes the first statement. The second statement follows by the CLT (section 1.2.9). ■

We now can give the asymptotic distribution of the OLS estimator.

Theorem 3.12.2 *Under Assumption 3.12.1,*

$$\sqrt{n}\left(\hat{\beta} - \beta\right) \to_d N(0, V)$$

where $V = Q^{-1}\Omega Q^{-1}$. *If in addition the homoskedasticity restriction (3.3) holds, then* $V = \sigma^2 Q^{-1} \equiv V^0$.

Proof: Using Proposition 3.11.1, Theorem 3.11.1, the CLT, and the CMT,

$$\begin{aligned}\sqrt{n}\left(\hat{\beta} - \beta\right) &= \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{\sqrt{n}}X'e\right) \\ &\to_d Q^{-1}N(0, \Omega) \\ &= N\left(0, Q^{-1}\Omega Q^{-1}\right).\end{aligned}$$

If in addition (3.3) holds, then

$$\Omega = E\left(x_ix_i'\sigma_i^2\right) = E\left(x_ix_i'\sigma^2\right) = \sigma^2 E\left(x_ix_i'\right) = \sigma^2 Q,$$

so

$$V = Q^{-1}\Omega Q^{-1} = Q^{-1}\left(\sigma^2 Q\right)Q^{-1} = \sigma^2 Q^{-1}.$$

■

As V is the variance of the asymptotic distribution of $\sqrt{n}\left(\hat{\beta} - \beta\right)$, V is often referred to as the **asymptotic covariance matrix** of $\hat{\beta}$. The form $V = Q^{-1}\Omega Q^{-1}$ is called a **sandwich** form. The theorem shows that V is the general form of the covariance matrix, but that under homoskedasticity it simplifies to the special form V^0. In general, however, the two asymptotic variance expressions differ.

3.13 Estimation of Error Variance

The regression error e_i is mean-zero and has unconditional variance

$$Ee_i^2 = \sigma^2.$$

A method of moments estimator for σ^2 is the sample average

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\hat{e}_i^2 = \frac{1}{n}\hat{e}'\hat{e}.$$

Since $Y = X\beta + e$ and $MX = 0$,

$$\hat{e} = MY = M(X\beta + e) = Me.$$

We see therefore that

$$\hat{\sigma}^2 = \frac{1}{n}e'MMe = \frac{1}{n}e'Me = \frac{1}{n}e'e - \left(\frac{1}{n}e'X\right)\left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}X'e\right). \tag{3.17}$$

An application of the WLLN and the CMT yields the consistency of $\hat{\sigma}^2$ for σ^2.

Theorem 3.13.1 *Under Assumption 3.11.1, as $n \to \infty$*

1. $\frac{1}{n}e'e \to_p \sigma^2$;
2. $\hat{\sigma}^2 \to_p \sigma^2$;

Proof: Assumption 3.11.1 specifies that $Ee_i^2 = \sigma^2 < \infty$, so an application of the WLLN yields

$$\frac{1}{n}\sum_{i=1}^{n} e_i^2 \to_p Ee_i^2 = \sigma^2$$

as $n \to \infty$, which is the first result.

Observe that (3.17) can be written as

$$\hat{\sigma}^2 = g\left(\frac{1}{n}e'e, \frac{1}{n}X'e, \frac{1}{n}X'X\right),$$

where $g(a, b, c) = a - b'c^{-1}b$ is a continuous function for all values of its arguments such that c^{-1} exists. Theorem 3.11.1 and the first result of this theorem show that as $n \to \infty$

$$\left(\frac{1}{n}e'e, \frac{1}{n}X'e, \frac{1}{n}X'X\right) \to_p \left(\sigma^2, 0, Q\right).$$

Since Q^{-1} exists under Assumption 3.11.1, the function g is continuous at the limit $\left(\sigma^2, 0, Q\right)$, so we can apply the CMT to discover that

$$\hat{\sigma}^2 = g\left(\frac{1}{n}e'e, \frac{1}{n}X'e, \frac{1}{n}X'X\right) \to_p g\left(\sigma^2, 0, Q\right) = \sigma^2 - 0'Q^{-1}0 = \sigma^2$$

as $n \to \infty$, which is the second stated result. ■

σ^2 measures the variation in the "unexplained" part of the regression. Forecasting from a regression is likely to be good if the variation in the unexplained part is small relative to the total. This is one motivation for the **coefficient of determination** or **R-squared**.

$$R^2 = 1 - \frac{\hat{\sigma}^2}{\hat{\sigma}_y^2}$$

where

$$\hat{\sigma}_y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \overline{y})^2$$

is the MME of the unconditional variance of y_i. This is a commonly reported statistic in regression analysis even though its usefulness outside of forecasting is minimal.

A very common alternative estimator for σ^2 is

$$\begin{aligned} s^2 &= \frac{n}{n-k}\hat{\sigma}^2 \\ &= \frac{1}{n-k}\hat{e}'\hat{e}. \end{aligned}$$

The estimator s^2 is frequently referred to as "the unbiased estimator" for reasons which are given in chapter 4. In practice, the difference is minimal unless k/n is large.

3.14 Covariance Matrix Estimation

In Theorem 3.12.2 we showed that the asymptotic variance of $\sqrt{n}\left(\hat{\beta} - \beta\right)$ is either $V^0 = \sigma^2 Q^{-1}$ or $V = Q^{-1}\Omega Q^{-1}$ depending on whether or not the error is homoskedastic. Thus the approximate variance of $\hat{\beta}$ is either $n^{-1}V^0$ or $n^{-1}V$, respectively.

A common choice to estimate $n^{-1}V^0$ is

$$\hat{V}_n^0 = s^2\left(X'X\right)^{-1}. \tag{3.18}$$

Theorem 3.14.1 *As $n \to \infty$, $n\hat{V}_n^0 \to V^0$.*

Proof: By Theorem 3.11.1, $n^{-1}X'X \to_p Q$, and by Theorem 3.13.1, $s^2 \to_p \sigma^2$. Thus by the CMT

$$n\hat{V}_n^0 = s^2\left(\frac{1}{n}X'X\right)^{-1} \to_p \sigma^2 Q^{-1} = V^0$$

as stated. ■

For V, we need an estimate of $\Omega = E\left(x_i x_i' e_i^2\right)$. The MME estimator is

$$\hat{\Omega} = \frac{1}{n}\sum_{i=1}^{n} x_i x_i' \hat{e}_i^2$$

where $\hat{e}_i$ are the OLS residuals. Another way of writing this is

$$\hat{\Omega} = \frac{1}{n}X'\hat{D}X$$

where

$$\hat{D} = \begin{bmatrix} \hat{e}_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \hat{e}_2^2 & 0 & \cdots & 0 \\ 0 & 0 & \hat{e}_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \hat{e}_n^2 \end{bmatrix}.$$

This equation is inappropriate for computation as the $n \times n$ matrix $\hat{D}$ is too large for storage in practice when n is large. A better computational formula is obtained by defining $\hat{u}_i = x_i \hat{e}_i$ and the $n \times k$ matrix

$$\hat{u} = \begin{pmatrix} \hat{u}_1' \\ \hat{u}_2' \\ \vdots \\ \hat{u}_n' \end{pmatrix}.$$

Then

$$\Omega = \frac{1}{n} \hat{u}' \hat{u}.$$

Our estimator for $n^{-1}V$ is then

$$\hat{V}_n = \left(X'X\right)^{-1} \left(\hat{u}'\hat{u}\right) \left(X'X\right)^{-1}. \tag{3.19}$$

This estimator was introduced to the econometrics literature by White (1980).

The estimator $\hat{V}_n^0$ was the dominate covariance estimator used before 1980, and was still the standard choice in the 1980s. From my reading of the literature, the White estimate $\hat{V}_n$ started to come in common use in the early 1990s, and by the late 1990s is quite commonly used, especially by younger researchers. When reading and reporting applied work, it is important to pay attention to the distinction between $\hat{V}_n^0$ and $\hat{V}_n$, as it is not always clear which has been used. When $\hat{V}_n$ is used rather than the traditional choice $\hat{V}_n^0$, many authors will state that "their standard errors have been corrected for heteroskedasticity", or that they use a "heteroskedasticity-robust covariance matrix estimator", or that they use the "White formula", the "Eicker-White formula", the "Huber formula", the "Huber-White formula" or the "GMM covariance matrix". In most cases, these all mean the same thing.

Theorem 3.14.2 *Under Assumption 3.12.1, as* $n \to \infty$,

1. $\hat{\Omega} \to_p \Omega$;
2. $n\hat{V} \to_p V$.

Proof: The main job is to show that $\hat{\Omega} \to_p \Omega$. Since

$$\begin{aligned} \hat{e}_i &= y_i - x_i' \hat{\beta} \\ &= e_i - x_i' \left(\hat{\beta} - \beta\right), \end{aligned}$$

then

$$\begin{aligned}\hat{e}_i^2 &= \left(e_i - x_i'\left(\hat{\beta}-\beta\right)\right)^2 \\ &= e_i^2 - 2\left(\hat{\beta}-\beta\right)' x_i e_i + \left(\hat{\beta}-\beta\right)' x_i x_i \left(\hat{\beta}-\beta\right).\end{aligned}$$

Hence

$$\begin{aligned}\hat{\Omega} &= \frac{1}{n}\sum_{i=1}^n x_i x_i' \hat{e}_i^2 \\ &= \frac{1}{n}\sum_{i=1}^n x_i x_i' e_i^2 - \frac{2}{n}\sum_{i=1}^n x_i x_i'\left(\hat{\beta}-\beta\right)' x_i e_i + \frac{1}{n}\sum_{i=1}^n x_i x_i'\left(\hat{\beta}-\beta\right)' x_i x_i \left(\hat{\beta}-\beta\right). \quad (3.20)\end{aligned}$$

First, Theorem 3.12.1 showed that $E\left|x_i x_i' e_i^2\right| < \infty$, so we can apply the WLLN to find that

$$\frac{1}{n}\sum_{i=1}^n x_i x_i' e_i^2 \to_p E\left(x_i x_i' e_i^2\right) = \Omega.$$

Second, by Holder's inequality (section 1.2.8)

$$E\left(|x_i|^3 |e_i|\right) \le \left(E|x_i|^4\right)^{3/4}\left(E\left|e_i^4\right|\right)^{1/4} < \infty,$$

so

$$\frac{1}{n}\sum_{i=1}^n |x_i|^3 |e_i| \to_p E\left(|x_i|^3 |e_i|\right),$$

and thus since $\left|\hat{\beta}-\beta\right| \to_p 0$,

$$\left|\frac{1}{n}\sum_{i=1}^n x_i x_i'\left(\hat{\beta}-\beta\right)' x_i e_i\right| \le \left|\hat{\beta}-\beta\right|\left(\frac{1}{n}\sum_{i=1}^n |x_i|^3 |e_i|\right) \to_p 0.$$

Third, by the WLLN

$$\frac{1}{n}\sum_{i=1}^n |x_i|^4 \to_p E|x_i|^4,$$

so

$$\left|\frac{1}{n}\sum_{i=1}^n x_i x_i'\left(\hat{\beta}-\beta\right)' x_i x_i \left(\hat{\beta}-\beta\right)\right| \le \left|\hat{\beta}-\beta\right|^2 \frac{1}{n}\sum_{i=1}^n |x_i|^4 \to_p 0.$$

Together, we find that the RHS of (3.20) converges in probability to Ω, which is the first result.

Then applying Theorem 3.11.1,

$$n\hat{V} = \left(\frac{1}{n}X'X\right)^{-1}\hat{\Omega}\left(\frac{1}{n}X'X\right)^{-1} \to_p Q^{-1}\Omega Q^{-1} = V,$$

as stated. ■

3.15 Standard Errors

The variance estimators $\hat{V}_n^0$ and $\hat{V}_n$ are estimates of the variance of the distribution of $\hat{\beta}$ (the first under homoskedasticity). A more easily interpretable measure of spread is its square root – the standard deviation. This motivates the definition of a standard error.

Definition 3.15.1 *A **standard error** $s(\hat{\beta})$ for an estimator $\hat{\beta}$ is an estimate of the standard deviation of the distribution of $\hat{\beta}$.*

When β is scalar, and $\hat{V}(\hat{\beta})$ is an estimator of the variance of $\hat{\beta}$, we set $s(\hat{\beta}) = \sqrt{\hat{V}(\hat{\beta})}$. When β is a vector, we focus on individual elements of β one-at-a-time, vis., β_j, $j = 1, ..., k$. Given our estimator (3.19), the standard error for $\hat{\beta}_j$ is

$$s(\hat{\beta}_j) = \sqrt{\left[\hat{V}_n\right]_{jj}}$$

and similarly if (3.18) is used.

Generically, standard errors are not unique, as there may be more than one estimator of the variance of the estimator. It is therefore important to understand what formula and method is used by an author when studying their work. It is also important to understand that a particular standard error may be relevant under one set of model assumptions, but not under another set of assumptions, just as any other estimator.

From a computational standpoint, the standard method to calculate the standard errors is to first calculate $\hat{V}_n$, then take the diagonal elements, and then the square roots.

3.16 Functions of Parameters

Sometimes we are interested in some function of the parameter vector. Let $h : R^k \to R^q$, and

$$\theta = h(\beta).$$

We will assume from now on that $h(\beta)$ is continuously differentiable at the true value of β.

The estimate of θ is

$$\hat{\theta} = h(\hat{\beta}).$$

What is an appropriate standard error for $\hat{\theta}$?

The following result is often referred to as "the delta method".

Theorem 3.16.1 *If $\sqrt{n}\left(\hat{\beta} - \beta\right) \to_d N(0, V)$ then*

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \to_d N(0, V_\theta)$$

where

$$V_\theta = H_\beta' V H_\beta$$

and

$$H_\beta = \frac{\partial}{\partial \beta} h(\beta). \qquad k \times q.$$

Proof. By a first-order Taylor series approximation:

$$h(\hat{\beta}) \simeq h(\beta) + H_\beta' \left(\hat{\beta} - \beta \right).$$

Thus

$$\begin{aligned} \sqrt{n}\left(\hat{\theta} - \theta\right) &= \sqrt{n}\left(h(\hat{\beta}) - h(\beta)\right) \\ &\simeq H_\beta' \sqrt{n}\left(\hat{\beta} - \beta\right) \\ &\to_d H_\beta' N(0, V) \\ &= N(0, H_\beta' V H_\beta). \end{aligned}$$

■

If $\hat{V}$ is the estimated covariance matrix for $\hat{\beta}$, then the natural estimate for the variance of $\hat{\theta}$ is

$$\hat{V}_\theta = \hat{H}_\beta' \hat{V} \hat{H}_\beta$$

where

$$\hat{H}_\beta = \frac{\partial}{\partial \beta} h(\hat{\beta}).$$

In many cases, the function $h(\beta)$ is linear:

$$h(\beta) = R'\beta$$

for some $k \times q$ matrix R. In this case, $H_\beta = R$ and $\hat{H}_\beta = R$, so $\hat{V}_\theta = R'\hat{V}R$.

For example, if R is a "selector matrix"

$$R = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

so that if $\beta = (\beta_1, \beta_2)$, then $\theta = R'\beta = \beta_1$ and

$$\hat{V}_\theta = \begin{pmatrix} I & 0 \end{pmatrix} \hat{V} \begin{pmatrix} I \\ 0 \end{pmatrix} = \hat{V}_{11},$$

the upper-left block of $\hat{V}$.

When $q = 1$ (so $h(\beta)$ is real-valued), the standard error for $\hat{\theta}$ is the square root of $\hat{V}_\theta$, that is, $s(\hat{\theta}) = \sqrt{\hat{H}_\beta' \hat{V} \hat{H}_\beta}$. An asymptotic 95% confidence interval for θ is

$$\left[\hat{\theta} \pm 2s(\hat{\theta})\right] = \left[h(\hat{\beta}) \pm 2\sqrt{\hat{H}_\beta' \hat{V} \hat{H}_\beta}\right].$$

Example: Conditional Mean

In the linear regression model,

$$g(x) = E\left(y_i \mid x_i = x\right) = x'\beta.$$

In some cases, we want to estimate $g(x)$ at a particular point x. Notice that this is a (linear) function of β. Letting $h(\beta) = x'\beta$ and $\theta = h(\beta)$, we see that $\hat{g}(x) = \hat{\theta} = x'\hat{\beta}$ and $H_\beta = x$, so $s(\hat{\theta}) = \sqrt{x'\hat{V}x}$. Thus an asymptotic 95% confidence interval for $g(x)$ is

$$\left[x'\hat{\beta} \pm 2\sqrt{x'\hat{V}x}\right].$$

It is interesting to observe that if this is viewed as a function of x, the width of the confidence set is dependent on x.

Example: Forecast Intervals

For a given value of $x_i = x$, we may want to forecast (guess) y_i out-of-sample. A reasonable guess is the conditional mean $g(x)$, and indeed this is the mean-square-minimizing decision rule. Thus a point forecast is $\hat{g}(x) = x'\hat{\beta}$, the estimated conditional mean, as discussed in the previous sub-section. We would also like a measure of uncertainty for the forecast.

The forecast error is $\hat{e}_i = y_i - \hat{g}(x) = e_i - x'\left(\hat{\beta} - \beta\right)$. As the out-of-sample error e_i is independent of the in-sample estimate $\hat{\beta}$, this has variance

$$\begin{aligned} E\hat{e}_i^2 &= E\left(e_i^2 \mid x_i = x\right) + x'E\left(\hat{\beta} - \beta\right)\left(\hat{\beta} - \beta\right)' x \\ &= \sigma^2(x) + x'V_n x. \end{aligned}$$

Assuming $E\left(e_i^2 \mid x_i\right) = \sigma^2$, the natural estimate of this variance is $\hat{\sigma}^2 + x'\hat{V}x$, so a standard error for the forecast is $\sqrt{\hat{\sigma}^2 + x'\hat{V}x}$. Notice that this is different from the standard error for the conditional mean.

It would appear natural to conclude that an asymptotic 95% forecast interval for y_i is

$$\left[x'\hat{\beta} \pm 2\sqrt{\hat{\sigma}^2 + x'\hat{V}x}\right],$$

but this turns out to be incorrect. In general, the validity of an asymptotic confidence interval is based on the asymptotic normality of the studentized ratio. In the present case, this would require the asymptotic normality of the ratio

$$\frac{e_i - x'\left(\hat{\beta} - \beta\right)}{\sqrt{\hat{\sigma}^2 + x'\hat{V}x}}.$$

But no such asymptotic approximation can be made. The only special exception is the case where e_i has the exact distribution $N(0, \sigma^2)$, which is generally invalid.

To get an accurate forecast interval, we need to estimate the conditional distribution of e_i given $x_i = x$, which is a much more difficult task. Given the difficulty, most applied forecasters focus on the simple and unjustified interval $\left[x'\hat{\beta} \pm 2\sqrt{\hat{\sigma}^2 + x'\hat{V}x}\right]$.

3.17 Studentized Statistic

For simplicity, suppose that $\theta = h(\beta)$ is real-valued.

The asymptotic distribution of $\hat{\theta}$ depends on the unknown variance V_θ. Thus the distribution of $\hat{\theta}$ is not suitable for testing or construction of confidence intervals. Gosset had the brilliant suggestion to study the studentized statistic

$$t_n(\theta) = \frac{\hat{\theta} - \theta}{s(\hat{\theta})}$$

where $s(\hat{\theta}) = \sqrt{\hat{V}_\theta}$ is a standard error for $\hat{\theta}$, since this statistic has a distribution which does not (asymptotically) depend on unknowns. In special cases (such as the normal regression model, see chapter 4), the statistic t_n has an exact t distribution, and is therefore exactly free of unknowns. In this case, we say that t_n is a **pivotal** statistic. More generally, the exact distribution of t_n is unknown, but its asymptotic distribution is known. In this case, we say that t is **asymptotically pivotal**.

Formally, if $\sqrt{n}\left(\hat{\theta} - \theta\right) \to_d N(0, V_\theta)$ and $n\hat{V}_\theta \to_p V_\theta$ as $n \to \infty$, then

$$\begin{aligned} t_n(\theta) &= \frac{\hat{\theta} - \theta}{s(\hat{\theta})} \\ &= \frac{\sqrt{n}\left(\hat{\theta} - \theta\right)}{\sqrt{n\hat{V}_\theta}} \\ &\to_d \frac{N(0, V_\theta)}{\sqrt{V_\theta}} = N(0, 1). \end{aligned}$$

3.18 Asymptotic Confidence Interval

A confidence interval C_n is an interval estimate of β, and is a function of the data and hence is random. It is designed to cover β with high probability. Either $\beta \in C_n$ or $\beta \notin C_n$. The coverage probability is $P(\beta \in C_n)$.

We typically cannot calculate the exact coverage probability $P(\beta \in C_n)$. However we often can calculate $\lim_{n\to\infty} P(\beta \in C_n)$. We call this the asymptotic coverage probability. We say that C_n has asymptotic $(1 - \alpha)\%$ coverage for β if $P(\beta \in C_n) \to 1 - \alpha$ as $n \to \infty$.

When $\hat{\beta}$ is asymptotically normally distributed and has standard error $s(\hat{\beta})$, the standard $(1-\alpha)\%$ asymptotic confidence interval for β is

$$\begin{aligned} C_n &= \left[\hat{\beta} - z_{\alpha/2}s(\hat{\beta}), \hat{\beta} + z_{\alpha/2}s(\hat{\beta})\right] \\ &= \left[\hat{\beta} \pm z_{\alpha/2}s(\hat{\beta})\right] \end{aligned} \tag{3.21}$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution. That is, if $Z \sim N(0,1)$, $P(Z > z_{\alpha/2}) = \alpha/2$. For example, $z_{.025} = 1.96$ and $z_{.05} = 1.645$.

Theorem 3.18.1 *If C_n is defined by (3.21), then $P(\beta \in C_n) \to 1-\alpha$ as $n \to \infty$.*

Proof. Since $t_n(\beta) \to_d Z \sim N(0,1)$,

$$\begin{aligned} P\left(\beta \in C_n\right) &= P\left(\beta \in \left[\hat{\beta} - z_{\alpha/2}s(\hat{\beta}), \hat{\beta} + z_{\alpha/2}s(\hat{\beta})\right]\right) \\ &= P\left(\hat{\beta} - z_{\alpha/2}s(\hat{\beta}) \le \beta \le \hat{\beta} + z_{\alpha/2}s(\hat{\beta})\right) \\ &= P\left(-z_{\alpha/2} \le \frac{\beta - \hat{\beta}}{s(\hat{\beta})} \le z_{\alpha/2}\right) \\ &= P\left(|t_n(\beta)| \le z_{\alpha/2}\right) \\ &\to P\left(|Z| \le z_{\alpha/2}\right) \\ &= 1 - 2P\left(Z > z_{\alpha/2}\right) \\ &= 1-\alpha. \end{aligned}$$

■

While there is no hard-and-fast guideline for the coverage probability $1-\alpha$, the most common professional choice is 95%, or $\alpha = .05$. This corresponds to selecting the confidence interval $\left[\hat{\beta} \pm 1.96s(\hat{\beta})\right] \approx \left[\hat{\beta} \pm 2s(\hat{\beta})\right]$. Thus values of β within two standard errors of the estimated $\hat{\beta}$ are considered "reasonable" candidates for the true value β, and values of β outside two standard errors of the estimated $\hat{\beta}$ are considered unlikely or unreasonable candidates for the true value.

3.19 t tests

A simple null and composite hypothesis takes the form

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta \neq \theta_0 \end{aligned}$$

where θ_0 is some pre-specified value, and $\theta = h(\beta)$ is some function of the parameter vector. (For example, θ could be a single element of β).

The standard test for H_0 against H_1 is the t-statistic (or studentized statistic)

$$t_n = t_n(\theta_0) = \frac{\hat{\theta} - \theta_0}{s(\hat{\theta})}.$$

Under H_0, $t_n \to_d N(0,1)$. So a test of asymptotic size α rejects H_0 if $|t_n| \geq z_{\alpha/2}$, else does not reject, or "accepts" H_0. This is because

$$\begin{aligned} P\left(\text{reject } H_0 \mid H_0\right) &= P\left(|t_n| \geq z_{\alpha/2} \mid \theta = \theta_0\right) \\ &\to P\left(|Z| \geq z_{\alpha/2}\right) = \alpha. \end{aligned}$$

The rejection/acceptance dichotomy is associate with the Neyman-Pearson approach to hypothesis testing.

An alternative approach, associate with Fisher, is to report an asymptotic p-value. The asymptotic p-value for the above statistic is constructed as follows. Define the tail probability, or asymptotic p-value function

$$p(t) = P\left(|Z| \geq |t|\right) = 2\left(1 - \Phi(|t|)\right).$$

Then the asymptotic p-value of the statistic t_n is

$$p_n = p(t_n).$$

If the p-value p_n is small (close to zero) then the evidence against H_0 is strong. In a sense, p-values and hypothesis tests are equivalent since $p_n \leq \alpha$ if and only if $|t_n| \geq z_{\alpha/2}$, thus an equivalent statement of a Neyman-Pearson test is to reject at the $\alpha\%$ level iff $p_n \leq \alpha$. The p-value is more general, however, in that the reader is allowed to pick the level of significance (α), in contrast to Neyman-Pearson rejection/acceptance reporting, where the researcher picks the level.

Another helpful observation is that the p-value function has simply made a unit-free transformation of the test statistic. That is, under H_0, $p_n \to_d U[0,1]$, so the "unusualness" of the test statistic can be compared to the easy-to-understand uniform distribution, regardless of the complication of the distribution of the original test statistic. To see this fact, note that the asymptotic distribution of $|t_n|$ is $F(x) = 1 - p(x)$. Thus

$$\begin{aligned} P\left(1 - p_n \leq u\right) &= P\left(1 - p(t_n) \leq u\right) \\ &= P\left(F(t_n) \leq u\right) \\ &= P\left(|t_n| \leq F^{-1}(u)\right) \\ &\to F\left(F^{-1}(u)\right) = u, \end{aligned}$$

establishing that $1 - p_n \to_d U[0,1]$, from which it follows that $p_n \to_d U[0,1]$.

It may be helpful to note that in the GAUSS language, the function $p(t)$ may be computed by the expression $p = 2 * cdfnc(t)$.

3.20 Wald Tests

Sometimes $\theta = h(\beta)$ is a $q \times 1$ vector, and it is desired to test the joint restrictions simultaneously. In this case the t-statistic approach does not work. We still have the null and alternative

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta \neq \theta_0. \end{aligned}$$

The natural estimate of θ is $\hat{\theta} = h(\hat{\beta})$ and has asymptotic covariance matrix estimate

$$\hat{V}_\theta = \hat{H}_\beta' \hat{V} \hat{H}_\beta$$

where

$$\hat{H}_\beta = \frac{\partial}{\partial \beta} h(\hat{\beta}).$$

The Wald statistic for H_0 against H_1 is

$$\begin{aligned} W_n &= \left(\hat{\theta} - \theta_0\right)' \hat{V}_\theta^{-1} \left(\hat{\theta} - \theta_0\right) \\ &= \left(h(\hat{\beta}) - \theta_0\right)' \left(\hat{H}_\beta' \hat{V} \hat{H}_\beta\right)^{-1} \left(h(\hat{\beta}) - \theta_0\right). \end{aligned}$$

When h is a linear function of β, $h(\beta) = R'\beta$, then the Wald statistic takes the form

$$W_n = \left(R'\hat{\beta} - \theta_0\right)' \left(R' \hat{V} R\right)^{-1} \left(R'\hat{\beta} - \theta_0\right).$$

The following fact is useful.

Theorem 3.20.1 *If $Z \sim N(0, A)$ and $A > 0$, then $Z' A^{-1} Z \sim \chi^2_q$, where $q = \dim(Z)$.*

Proof. The fact that $A > 0$ means that we can write $A = CC'$ where C is non-singular. Then $A^{-1} = C^{-1\prime} C^{-1}$ and

$$\begin{aligned} Z^* &= C^{-1} Z \\ &\sim N(0, C^{-1} A C^{-1\prime}) \\ &= N(0, C^{-1} C C' C^{-1\prime}) \\ &= N(0, I_q). \end{aligned}$$

Thus

$$Z' A^{-1} Z = Z' C^{-1\prime} C^{-1} Z = Z^{*\prime} Z^* \sim \chi^2_q.$$

■

Theorem 3.20.2 *Under H_0 and Assumption 3.12.1, if rank$(H_\beta) = q$, then $W_n \to_d \chi^2_q$, a chi-square random variable with q degrees of freedom.*

Proof. Theorem 3.16.1 showed that $\sqrt{n}\left(\hat{\theta} - \theta\right) \to_d Z \sim N(0, V_\theta)$, and Theorem 3.14.2 showed that $n\hat{V} \to_p V$. Furthermore, $H_\beta(\beta)$ is a continuous function of β, so by the continuous mapping theorem, $H_\beta(\hat{\beta}) \to_p H_\beta$. Thus $n\hat{V}_\theta = \hat{H}'_\beta \left(n\hat{V}\right) \hat{H}_\beta \to_p H'_\beta V H_\beta = V_\theta$ and the latter has full rank q. Hence

$$\begin{aligned} W_n &= \left(\hat{\theta} - \theta_0\right)' \hat{V}_\theta^{-1} \left(\hat{\theta} - \theta_0\right) \\ &= \sqrt{n}\left(\hat{\theta} - \theta_0\right)' \left(n\hat{V}_\theta\right)^{-1} \sqrt{n}\left(\hat{\theta} - \theta_0\right) \\ &\to_d Z' V_\theta^{-1} Z = \chi^2_q, \end{aligned}$$

by Theorem 3.20.1.

■

An asymptotic Wald test rejects H_0 in favor of H_1 if W_n exceeds $\chi^2_q(\alpha)$, the upper-α quantile of the χ^2_q distribution. For example, $\chi^2_1(.05) = 3.84 = z^2_{.025}$. The Wald test fails to reject if W_n is less than $\chi^2_q(\alpha)$. The asymptotic p-value for W_n is $p_n = p(W_n)$, where $p(x) = P\left(\chi^2_q \geq x\right)$ is the tail probability function of the χ^2_q distribution. As before, the test rejects at the $\alpha\%$ level iff $p_n \leq \alpha$, and p_n is asymptotically $U[0,1]$ under H_0. In addition, it may be helpful to note that in the GAUSS language, the function $p(t)$ may be computed by the expression $p = cdfchic(t)$.

3.21 F Tests

Take the linear model

$$Y = X_1\beta_1 + X_2\beta_2 + e$$

where X_1 is $n \times k_1$ and X_2 is $n \times k_2$ and $k = k_1 + k_2$. The null hypothesis is

$$H_0 : \beta_2 = 0.$$

In this case, $\theta = \beta_2$, and there are $q = k_2$ restrictions. Also $h(\beta) = R'\beta$ is linear with $R = \begin{pmatrix} 0 \\ I \end{pmatrix}$ a selector matrix. We know that the Wald statistic takes the form

$$\begin{aligned} W_n &= \hat{\theta}' \hat{V}_\theta^{-1} \hat{\theta} \\ &= \hat{\beta}'_2 \left(R'\hat{V}R\right)^{-1} \hat{\beta}_2. \end{aligned}$$

What we will show in this section is that if $\hat{V}$ is replaced with $\hat{V}^0 = s^2 (X'X)^{-1}$, the covariance matrix estimator valid under homoskedasticity, then the Wald statistic can be written in the form

$$W_n = (n-k)\frac{\left(\tilde{\sigma}^2 - \hat{\sigma}^2\right)}{\hat{\sigma}^2} \tag{3.22}$$

where

$$\tilde{\sigma}^2 = \frac{1}{n}\tilde{e}'\tilde{e}, \qquad \tilde{e} = Y - X_1\tilde{\beta}_1, \qquad \tilde{\beta}_1 = \left(X_1'X_1\right)^{-1} X_1'Y$$

are from OLS of Y on X_1, and

$$\hat{\sigma}^2 = \frac{1}{n}\hat{e}'\hat{e}, \qquad \hat{e} = Y - X\hat{\beta}, \qquad \hat{\beta} = \left(X'X\right)^{-1} X'Y$$

are from OLS of Y on $X = (X_1, X_2)$.

The elegant feature about (3.22) is that it is directly computable from the standard output from two simple OLS regressions, as the sum of square errors RSS is a typical output from statistical packages. This statistic is typically reported as an "F-statistic" which is defined as

$$F = \frac{W_n}{k_2} = \frac{\left(\tilde{\sigma}^2 - \hat{\sigma}^2\right)/k_2}{\hat{\sigma}^2/(n-k)}.$$

While it should be emphasized that equality (3.22) only holds if $\hat{V}^0 = s^2 (X'X)^{-1}$, still this formula often finds good use in reading applied papers.

We now derive expression (3.22). First, note that

$$R'\hat{V}^0 R = s^2 \left[(X'X)^{-1}\right]_{22},$$

where $[A]_{22}$ refers to the lower-right block of the matrix A. I claim that

$$\left[(X'X)^{-1}\right]_{22} = \left(X_2'M_1X_2\right)^{-1}, \tag{3.23}$$

where $M_1 = I - X_1(X_1'X_1)^{-1}X_1'$. We prove (3.23) below. But taking (3.23) as given, it implies that

$$\left(R'\hat{V}^0 R\right)^{-1} = \left(s^2\left(X_2'M_1X_2\right)^{-1}\right)^{-1} = s^{-2}\left(X_2'M_1X_2\right),$$

so

$$\begin{aligned} W_n &= \hat{\beta}_2'\left(R'\hat{V}^0 R\right)^{-1}\hat{\beta}_2 \\ &= \frac{\hat{\beta}_2'\left(X_2'M_1X_2\right)\hat{\beta}_2}{s^2}. \end{aligned}$$

To simplify this expression further, note that if we regress Y on X_1 alone, the residual is $\tilde{e} = M_1 Y$. Now consider the residual regression of $\tilde{e}$ on $\tilde{X}_2 = M_1 X_2$. By the FWL theorem, $\tilde{e} = \tilde{X}_2 \hat{\beta}_2 + \hat{e}$ and $\tilde{X}_2' \hat{e} = 0$. Thus

$$\begin{aligned} \tilde{e}'\tilde{e} &= \left(\tilde{X}_2\hat{\beta}_2 + \hat{e}\right)' \left(\tilde{X}_2\hat{\beta}_2 + \hat{e}\right) \\ &= \hat{\beta}_2'\tilde{X}_2'\tilde{X}_2\hat{\beta}_2 + \hat{e}'\hat{e} \\ &= \hat{\beta}_2' X_2' M_1 X_2 \hat{\beta}_2 + \hat{e}'\hat{e}, \end{aligned}$$

or alternatively,

$$\hat{\beta}_2' X_2' M_1 X_2 \hat{\beta}_2 = \tilde{e}'\tilde{e} - \hat{e}'\hat{e}.$$

Also, since

$$s^2 = \frac{1}{n-k}\hat{e}'\hat{e}$$

we conclude that

$$W_n = \frac{\tilde{e}'\tilde{e} - \hat{e}'\hat{e}}{\frac{1}{n-k}\hat{e}'\hat{e}} = (n-k)\left(\frac{\tilde{e}'\tilde{e} - \hat{e}'\hat{e}}{\hat{e}'\hat{e}}\right),$$

as claimed.

It remains to show (3.23). This can be done by partitioned matrix inversion. We use an alternative trick based on the FWL theorem. Suppose that $E\left(e_i^2 \mid x_i\right) = 1$. Then we know that $Var(\hat{\beta} \mid X) = (X'X)^{-1}$. Thus

$$\begin{aligned} Var(\hat{\beta}_2 \mid X) &= Var(R'\hat{\beta} \mid X) \\ &= R'(X'X)^{-1}R \\ &= \left[(X'X)^{-1}\right]_{22}. \end{aligned}$$

By the FWL theorem, we also know that

$$\begin{aligned} \hat{\beta}_2 &= \left(X_2'M_1X_2\right)^{-1} X_2'M_1Y \\ &= \left(X_2'M_1X_2\right)^{-1} X_2'M_1\left(X_1\beta_1 + X_2\beta_2 + e\right) \\ &= \beta_2 + \left(X_2'M_1X_2\right)^{-1} X_2'M_1e. \end{aligned}$$

Thus

$$\begin{aligned} Var(\hat{\beta}_2 \mid X) &= Var(\left(X_2'M_1X_2\right)^{-1} X_2'M_1e \mid X) \\ &= \left(X_2'M_1X_2\right)^{-1} X_2'M_1M_1X_2 \left(X_2'M_1X_2\right)^{-1} \\ &= \left(X_2'M_1X_2\right)^{-1}. \end{aligned}$$

Since these two expressions must be the same, we can conclude that (3.23) is true. This completes our proof of (3.22).

In many statistical packages, when an OLS regression is reported, an "F statistic" is reported. This is

$$F = \frac{\left(\tilde{\sigma}_y^2 - \hat{\sigma}^2\right)/(k-1)}{\hat{\sigma}^2/(n-k)}.$$

where

$$\tilde{\sigma}_y^2 = \frac{1}{n}\left(y - \overline{y}\right)'\left(y - \overline{y}\right)$$

is the sample variance of y_i, equivalently the residual variance from an intercept-only model. This special F statistic is testing the hypothesis that *all* slope coefficients (other than the intercept) are zero. This was a popular statistic in the early days of econometric reporting, when sample sizes were very small and researchers wanted to know if there was "any explanatory power" to their regression. This is rarely an issue today, as sample sizes are typically sufficiently large that this F statistic is highly "significant". Certainly, there are special cases where this F statistic is useful, but these cases are no longer typical.

3.22 Quasi-LR Tests

We showed before that the Gaussian quasi-MLE $\hat{\beta}$ is OLS. Recall, the Gaussian log-likelihood evaluated at the MLE $\hat{\beta}$ is

$$l_n(\hat{\beta}, \sigma^2) = -\frac{n}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\hat{e}'\hat{e}.$$

Thus the MLE for σ^2 is found by solving the first-order-conditions:

$$\frac{\partial}{\partial\sigma^2} l_n(\hat{\beta}, \hat{\sigma}^2) = -\frac{n}{2\hat{\sigma}^2} + \frac{1}{2\left(\hat{\sigma}^2\right)^2}\hat{e}'\hat{e} = 0.$$

Hence

$$\hat{\sigma}^2 = \frac{1}{n}\hat{e}'\hat{e}.$$

The log-likelihood evaluated at the parameter estimates is

$$\begin{aligned}\hat{l}_n &= -\frac{n}{2}\log\left(2\pi\hat{\sigma}^2\right) - \frac{1}{2\hat{\sigma}^2}\hat{e}'\hat{e} \\ &= -\frac{n}{2}\log\left(\hat{\sigma}^2\right) - \frac{n}{2}\log\left(2\pi\right) - \frac{n}{2}.\end{aligned}$$

Similarly, the MLE of the model under the restriction

$$H_0 : \beta_2 = 0.$$

is OLS of y_i on x_{1i} with log-likelihood

$$\tilde{l}_n = -\frac{n}{2}\log\left(\tilde{\sigma}^2\right) - \frac{n}{2}\log\left(2\pi\right) - \frac{n}{2}.$$

The LR statistic for H_0 is

$$\begin{aligned} LR &= 2\left(\hat{l}_n - \tilde{l}_n\right) \\ &= n\left(\log\left(\tilde{\sigma}^2\right) - \log\left(\hat{\sigma}^2\right)\right) \\ &= n\log\left(\frac{\tilde{\sigma}^2}{\hat{\sigma}^2}\right). \end{aligned}$$

Note that both LR and the Wald statistic computed under homoskedasticity:

$$W_n = (n-k)\left(\frac{\tilde{\sigma}^2}{\hat{\sigma}^2} - 1\right)$$

are functions of the data only through $\tilde{\sigma}^2/\hat{\sigma}^2$. They are numerically close, since

$$LR = n\log\left(1 + \frac{\tilde{\sigma}^2}{\hat{\sigma}^2} - 1\right) \simeq n\left(\frac{\tilde{\sigma}^2}{\hat{\sigma}^2} - 1\right) \simeq W_n.$$

Chapter 4

Small Sample Theory (Optional)

This chapter explores some special small sample results for the regression model where $E(e_i \mid x_i) = 0$.

4.1 Bias

Proposition 4.1.1 *In the linear regression model,* $E(e \mid X) = 0$.

Proof: Since the observations are independent, $E(e_i \mid X) = E(e_i \mid x_i) = 0$. Thus

$$E(e \mid X) = \begin{pmatrix} E(e_1 \mid X) \\ E(e_2 \mid X) \\ \vdots \\ E(e_n \mid X) \end{pmatrix} = \begin{pmatrix} E(e_1 \mid x_1) \\ E(e_2 \mid x_2) \\ \vdots \\ E(e_n \mid x_n) \end{pmatrix} = 0.$$

■

Proposition 4.1.2 *In the linear regression model,* $E\left(\hat{\beta} - \beta \mid X\right) = 0$ *and* $E\hat{\beta} = \beta$.

Proof: By Proposition (4.1.1),

$$\begin{aligned} E\left(\hat{\beta} - \beta \mid X\right) &= E\left(\left(X'X\right)^{-1} X'e \mid X\right) \\ &= \left(X'X\right)^{-1} X'E\left(e \mid X\right) \\ &= 0. \end{aligned}$$

Then applying the law of iterated expectations (section 1.2.5), $E\hat{\beta} = E\left(E\left(\hat{\beta} \mid X\right)\right) = \beta$. ■

Thus $\hat{\beta}$ is *unbiased* for β. Indeed, it is *conditionally unbiased*, conditional upon X, which is a stronger result. This result only holds for the regression model where $E(e_i \mid x_i) = 0$. It does not hold for the projection model where we only have the orthogonality $E(x_i e_i) = 0$.

4.2 Variance-Covariance Matrix of Regression Error

The conditional variance-covariance matrix of the regression error vector e is

$$D = E\left(ee' \mid X\right).$$

This $n \times n$ matrix plays an important role in our theory, so we start by exploring its structure.

Writing out the matrix ee', we find

$$ee' = \begin{bmatrix} e_1^2 & e_1e_2 & e_1e_3 & \cdots & e_1e_n \\ e_2e_1 & e_2^2 & e_2e_3 & \cdots & e_2e_n \\ e_3e_1 & e_3e_2 & e_3^2 & \cdots & e_3e_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_ne_1 & e_ne_2 & e_ne_3 & \cdots & e_n^2 \end{bmatrix}.$$

If the data are a random sample then (x_i, e_i) is independent of (x_j, e_j) for $i \neq j$, so

$$E\left(e_i^2 \mid X\right) = E\left(e_i^2 \mid x_i\right) = \sigma_i^2,$$

where σ_i^2 was introduced at the end of Section 1. Again by the independence of (x_i, e_i) from (x_j, e_j) for $i \neq j$,

$$E\left(e_ie_j \mid X\right) = E\left(e_i \mid x_i\right) E\left(e_j \mid x_j\right) = 0.$$

Thus in general

$$\begin{aligned} D &= E\left(ee' \mid X\right) \\ &= \begin{bmatrix} E\left(e_1^2 \mid X\right) & E\left(e_1e_2 \mid X\right) & E\left(e_1e_3 \mid X\right) & \cdots & E\left(e_1e_n \mid X\right) \\ E\left(e_2e_1 \mid X\right) & E\left(e_2^2 \mid X\right) & E\left(e_2e_3 \mid X\right) & \cdots & E\left(e_2e_n \mid X\right) \\ E\left(e_3e_1 \mid X\right) & E\left(e_3e_2 \mid X\right) & E\left(e_3^2 \mid X\right) & \cdots & E\left(e_3e_n \mid X\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ E\left(e_ne_1 \mid X\right) & E\left(e_ne_2 \mid X\right) & E\left(e_ne_3 \mid X\right) & \cdots & E\left(e_n^2 \mid X\right) \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}, \end{aligned} \tag{4.1}$$

which is a diagonal matrix with the conditional variances on the diagonal.

This is a case where the homoskedasticity restriction (3.3) implies a simplification. Under (3.3),

$E\left(e_i^2 \mid x_i\right) = \sigma^2$ for all i, so D simplifies to

$$D = \begin{bmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma^2 \end{bmatrix} = I_n \sigma^2. \tag{4.2}$$

4.3 Covariance Matrix of OLS Estimator

Since $\hat{\beta} - \beta = (X'X)^{-1} X'e$ (Proposition 3.11.1), the conditional covariance matrix for $\hat{\beta}$ is

$$\begin{aligned} Var\left(\hat{\beta} \mid X\right) &= E\left[\left(\hat{\beta} - \beta\right)\left(\hat{\beta} - \beta\right)' \mid X\right] \\ &= E\left[\left(X'X\right)^{-1} X'ee'X\left(X'X\right)^{-1} \mid X\right] \\ &= \left(X'X\right)^{-1} X'E\left[ee' \mid X\right] X\left(X'X\right)^{-1} \\ &= \left(X'X\right)^{-1} X'DX\left(X'X\right)^{-1}, \end{aligned}$$

where D is defined in (4.1). It may be helpful to observe that the central portion of this expression may be written alternatively as

$$X'DX = \sum_{i=1}^{n} x_i x_i' \sigma_i^2.$$

In the special case of (3.3), then $\sigma_i^2 = \sigma^2$, so $X'DX = X'X\sigma^2$ and the covariance matrix simplifies to

$$\left(X'X\right)^{-1} X'X\sigma^2 \left(X'X\right)^{-1} = \sigma^2 \left(X'X\right)^{-1}.$$

We have shown the following.

Theorem 4.3.1 *In the linear regression model,*

$$Var\left(\hat{\beta} \mid X\right) = \left(X'X\right)^{-1} X'DX\left(X'X\right)^{-1}. \tag{4.3}$$

If (3.3) holds,

$$Var\left(\hat{\beta} \mid X\right) = \sigma^2 \left(X'X\right)^{-1}. \tag{4.4}$$

4.4 Unbiased Estimation of Error Variance

We can calculate that

$$\begin{aligned} E\left(n\hat{\sigma}^2 \mid X\right) &= E\left(e'Me \mid X\right) \\ &= E\left(\operatorname{tr}\left(e'Me\right) \mid X\right) \\ &= E\left(\operatorname{tr}\left(Mee'\right) \mid X\right) \\ &= \operatorname{tr}\left[E\left(Mee' \mid X\right)\right] \\ &= \operatorname{tr}\left[ME\left(ee' \mid X\right)\right] \\ &= \operatorname{tr}\left[MD\right], \end{aligned}$$

where D is defined in (4.1).

In the general case of a heteroskedastic regression model, this expression does not appear to simplify further, so we do not have a clear formula for the bias of $\hat{\sigma}^2$. However, in the special case of the homoskedastic regression (3.3), then D simplifies to $D = I_n\sigma^2$ (see (4.2)) and then

$$\begin{aligned} E\left(n\hat{\sigma}^2 \mid X\right) &= \operatorname{tr}\left[MD\right] \\ &= \operatorname{tr}\left[MI_n\sigma^2\right] \\ &= \sigma^2 \operatorname{tr}\left[M\right] \\ &= \sigma^2(n-k). \end{aligned}$$

The final equality holds by (1.3). We have proved that:

Theorem 4.4.1 *In the homoskedastic regression model,*

$$\begin{aligned} E\hat{\sigma}^2 &= \frac{(n-k)}{n}\sigma^2. \\ Es^2 &= \sigma^2 \end{aligned}$$

Since $\frac{(n-k)}{n} < 1$, $\hat{\sigma}^2$ is a biased estimator for σ^2 and s^2 is unbiased. Because of Theorem 4.4.1 the estimator s^2 is frequently referred to as "the unbiased estimator." It is important to understand, however, that s^2 is unbiased only in the special case of a homoskedastic regression, and not generally.

4.5 Normal Regression Model

Some of the distribution results simplify dramatically under the assumption that e_i is independent of x_i and normally distributed. Much of classical econometric theory was derived under this set of assumptions.

Fairly directly, this implies that the vector e is independent of the matrix X, and has the distribution $N\left(0, I_n\sigma^2\right)$. Since linear functions of normals are also normal, this implies that conditional on X

$$\begin{pmatrix} \hat{\beta}-\beta \\ \hat{e} \end{pmatrix} = \begin{pmatrix} (X'X)^{-1}X' \\ M \end{pmatrix} e \sim N\left(0, \begin{pmatrix} \sigma^2 (X'X)^{-1} & 0 \\ 0 & \sigma^2 M \end{pmatrix}\right)$$

where $M = I - X(X'X)^{-1}X'$. Since uncorrelated normal variables are independent, it follows that $\hat{\beta}$ is independent of any function of the OLS residuals, including the estimated error variance s^2.

The spectral decomposition of M yields

$$M = H \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix} H'$$

(see equation (1.4)) where $H'H = I_n$. Let $u = \sigma^{-1}H'e \sim N(0, H'H) \sim N(0, I_n)$. Then

$$\begin{aligned} \frac{(n-k)s^2}{\sigma^2} &= \frac{1}{\sigma^2}\hat{e}'\hat{e} \\ &= \frac{1}{\sigma^2}e'Me \\ &= \frac{1}{\sigma^2}e'H \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix} H'e \\ &= u' \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix} u \\ &\sim \chi^2_{n-k}, \end{aligned}$$

a chi-square distribution with $n-k$ degrees of freedom. Furthermore, if standard errors are calculated using the homoskedastic formula (3.18)

$$\frac{\hat{\beta}_j - \beta_j}{s(\hat{\beta}_j)} = \frac{\hat{\beta}_j - \beta_j}{s\sqrt{\left[(X'X)^{-1}\right]_{jj}}} \sim \frac{N\left(0, \sigma^2\left[(X'X)^{-1}\right]_{jj}\right)}{\sqrt{\frac{\sigma^2}{n-k}\chi^2_{n-k}}\sqrt{\left[(X'X)^{-1}\right]_{jj}}} = \frac{N(0,1)}{\sqrt{\frac{\chi^2_{n-k}}{n-k}}} \sim t_{n-k}$$

a t distribution with $n-k$ degrees of freedom.

We summarize these findings

Theorem 4.5.1 *If e_i is independent of x_i and distributed $N(0, \sigma^2)$, and standard errors are calculated using the homoskedastic formula (3.18) then*

- $\hat{\beta} \sim N\left(0, \sigma^2 (X'X)^{-1}\right)$
- $\frac{(n-k)s^2}{\sigma^2} \sim \chi^2_{n-k}$,

- $\frac{\hat{\beta}_j - \beta_j}{s(\hat{\beta}_j)} \sim t_{n-k}$

Theorem **??** showed that generally in large samples, $\hat{\beta}$ and t are approximately normally distributed. In contrast, Theorem 4.5.1 shows that under the strong assumption of normality, $\hat{\beta}$ has an exact normal distribution and t has an exact t distribution. As inference (confidence intervals) are based on the t-ratio, the notable distinction is between the $N(0,1)$ and t_{n-k} distributions. The critical values are quite close if $n - k \geq 30$, so as a practical matter it does not matter which distribution is used. (Unless the sample size is unreasonably small.)

While Theorem 4.5.1 is quite remarkable, the result is difficult to generalize. Econometric models and questions typically fall outside the class of problems covered by this Theorem.

4.6 GLS and the Gauss-Markov Theorem

The linear regression model

$$\begin{aligned} y_i &= x_i'\beta + e_i \\ E(e_i \mid x_i) &= 0 \end{aligned}$$

imposes the condition of zero conditional mean. This is stronger than the orthogonality condition $E(x_i e_i) = 0$. This stronger condition can be exploited to improve estimation efficiency.

The **Generalized Least Squares** (GLS) estimator of β is

$$\tilde{\beta} = \left(X'D^{-1}X\right)^{-1}\left(X'D^{-1}Y\right). \tag{4.5}$$

The GLS estimator is sometimes called the Aitken estimator.

Since $Y = X\beta + e$, thus

$$\tilde{\beta} = \beta + \left(X'D^{-1}X\right)^{-1}\left(X'D^{-1}e\right).$$

Since D is a function of X, $E\left(\tilde{\beta} \mid X\right) = \beta$ and

$$\begin{aligned} Var(\tilde{\beta} \mid X) &= \left(X'D^{-1}X\right)^{-1} X'D^{-1}DD^{-1}X\left(X'D^{-1}X\right)^{-1} \\ &= \left(X'D^{-1}X\right)^{-1}. \end{aligned}$$

The class of unbiased linear estimators take the form

$$\tilde{\beta}_L = A(X)'Y, \qquad A(X)'X = I_k$$

where $A(X)$, $n \times k$, is a function only of X. This is called *linear* because it is a linear function of Y, even though it is nonlinear in X. OLS is the case $A(X) = X(X'X)^{-1}$ and GLS is the case $A(X) = D^{-1}X\left(X'D^{-1}X\right)^{-1}$. Observe that

$$E\left(\tilde{\beta}_L \mid X\right) = A(X)'X\beta = \beta$$

so $\tilde{\beta}_L$ is unbiased. Thus $\tilde{\beta}_L = \beta + A(X)'e$, and its variance is

$$Var(\tilde{\beta}_L \mid X) = A(X)'DA(X).$$

The "best" estimator within this class is the one with the smallest variance.

Theorem 4.6.1 *(Gauss-Markov). The best (minimum-variance) unbiased linear estimator is GLS.*

Proof. Let $A^* = D^{-1}X\left(X'D^{-1}X\right)^{-1}$ and A be any other $n \times k$ function of X such that $A'X = I_k$. We need to show that $A'DA \geq A^{*\prime}DA^*$.

Let $C = A - A^*$. Note that

$$\begin{aligned} C'X &= A'X - A^{*\prime}X \\ &= I_k - I_k = 0 \end{aligned}$$

and

$$\begin{aligned} C'DA^* &= C'DD^{-1}X\left(X'D^{-1}X\right)^{-1} \\ &= C'X\left(X'D^{-1}X\right)^{-1} = 0. \end{aligned}$$

Then

$$\begin{aligned} A'DA &= (C + A^*)'D(C + A^*) \\ &= C'DC + C'DA^* + A^{*\prime}DC + A^{*\prime}DA^* \\ &= C'DC + A^{*\prime}DA^* \geq A^*DA^{*\prime}. \end{aligned}$$

■

Theoretically speaking, the Gauss-Markov theorem is not a very powerful theorem, because the restriction to linear estimators is quite unnatural. That is, perhaps a "nonlinear" estimator can do even better. However, at least the theorem points to the inefficiency of OLS in regression models.

Chamberlain (1987) established the general result. He showed that in the regression model, no regular consistent estimator can have a lower asymptotic variance than the GLS estimator. This establishes that the GLS estimator is asymptotically efficient. The proof of his theorem is quite deep and we cannot cover it here.

Except in the special case of homoskedastic errors (where $D = I\sigma^2$ and GLS=OLS), these results show that in the regression model, OLS is inefficient. Unfortunately, it is not trivial to achieve the efficiency improvement, as GLS is not feasible as the matrix D is unknown. This raises the difficult issue of implementation.

Earlier, we claimed that OLS is asymptotically efficient in the class of models with $E(x_i e_i) = 0$. In this section we claimed that OLS is inefficient (but GLS is efficient) if $E(e_i \mid x_i) = 0$. The gain of efficiency (through use of GLS) comes through the exploitation of the stronger conditional mean assumption, which has the cost of reduced robustness. If $E(e_i \mid x_i) \neq 0$ then the GLS estimator will be inconsistent for the projection coefficient β, but OLS will be consistent.

4.7 Monte Carlo Simulation

It will be helpful to define some general concepts so our discussion will be at a higher level of generality than the linear regression framework. Let w_i be the data and let $F(w) = P(w_i \leq w)$ be the cumulative distribution function (CDF) of w_i. Let F denote a general CDF, and let F_0 denote the true value. Let θ be some parameter of interest and let $T_n = T_n(w_1, ..., w_n, \theta)$ be a statistic of interest. For example, in a linear regression model, $w_i = (y_i, x_i)$, $\theta = h(\beta)$ is a function of the parameter vector, and $T_n = \hat{\theta} - \theta$ where $\hat{\theta} = h(\hat{\beta})$ with $\hat{\beta}$ the OLS estimator. Or alternatively $T_n = \left(\hat{\theta} - \theta\right) / s(\hat{\theta})$.

For inference on θ, we want to know the sampling distribution of T_n. The exact CDF of T_n when the data are sampled from the distribution F is

$$G_n(x, F) = P(T_n \leq x \mid F)$$

Given the structure of the problem, G_n is a function only of F the distribution of w_i. In general, $G_n(x, F)$ will depend on F, meaning that if F as changes, G_n will change.

Ideally, inference on θ would be based on $G_n(x, F_0)$, the true value of the sampling distribution. This is (generally) impossible for two reasons. First, the function $G_n(x, F)$ is unknown. Second, F_0 is unknown.

The idea of Monte Carlo simulation is to solve the first problem through numerical simulation. The idea is that for any given F, the distribution function $G_n(x, F)$ can be calculated by numerical simulation. Since F is unknown, this does not solve the problem of inference. Instead, the method is typically used to assess the adequacy of statistical methods in practical settings.

The name Monte Carlo derives from the famous Mediterranean gambling resort, where games of chance are played.

The method of Monte Carlo is quite simple to describe. The researcher chooses F (the distribution of the data) and the sample size n. A "true" value of θ is implied by this choice, or equivalently the value of θ is selected directly by the researcher.

Then the following experiment is conducted

- n independent random vectors w_i^*, $i = 1, ..., n$, are drawn from the distribution F using the computer's random number generator.
- The statistic $T_n^* = T_n(w_1^*, ..., w_n^*, \theta)$ is calculated on this pseudo data.

For step 1, most computer packages have built-in procedures for generating $U[0, 1]$ and $N(0, 1)$ random numbers, and from these most random variables can be constructed. (For example, a chi-square can be generated by sums of squares of normals.)

For step 2, it is important that the statistic be evaluated at the "true" value of θ corresponding to the choice of F.

The above experiment creates one random draw from the distribution $G_n(x, F)$. This is one observation from an unknown distribution. Clearly, from one observation very little can be said.

So the researcher repeats the experiment B times, where B is a large number. Typically, we set $B = 1000$ or $B = 5000$, and we will discuss this choice later.

Notationally, let the $b'th$ experiment result in the draw T_{nb}^*, $b = 1, ..., B$. These results are stored. They constitute a random sample of size B from the distribution of $G_n(x, F) = P\left(T_{nb}^* \leq x\right) = P\left(T_n \leq x \mid F\right)$.

From a random sample, we can estimate any feature of interest using (typically) a method of moments estimator. For example:

Suppose we are interested in the bias, mean-squared error (mse), or variance of the distribution of $\hat{\theta} - \theta$. We then set $T_n = \hat{\theta} - \theta$, run the above experiment, and calculate

$$\begin{aligned}
\widehat{Bias(\hat{\theta})} &= \frac{1}{B}\sum_{b=1}^{B} T_{nb}^* \\
\widehat{MSE(\hat{\theta})} &= \frac{1}{B}\sum_{b=1}^{B} (T_{nb}^*)^2 \\
\widehat{Var(\hat{\theta})} &= \widehat{MSE(\hat{\theta})} - \left(\widehat{Bias(\hat{\theta})}\right)^2
\end{aligned}$$

Suppose we are interested in the Type I error associated with an asymptotic 5% two-sided t-test. We would then set $T_n = \left|\hat{\theta} - \theta\right| / s(\hat{\theta})$ and calculate

$$\hat{P} = \frac{1}{B}\sum_{b=1}^{B} 1\left(T_{nb}^* \geq 1.96\right), \tag{4.6}$$

the percentage of the simulated t-ratios which exceed the asymptotic 5% critical value.

Suppose we are interested in the 5% and 95% quantile of $\hat{\theta}$ or $\left(\hat{\theta} - \theta\right) / s(\hat{\theta})$. We then set T_n to either choice, and compute the 10% and 90% sample quantiles of the sample $\{T_{nb}^*\}$. The α% sample quantile is a number q_α such that α% of the sample are less than q_α. A simple way to compute sample quantiles is to sort the sample $\{T_{nb}^*\}$ from low to high. Then q_α is the N'th number in this ordered sequence, where $N = (B + 1)\alpha$. It is therefore convenient to pick B so that N is an integer. For example, if we set $B = 999$, then the 5% sample quantile is 50'th sorted value and the 95% sample quantile is the 950'th sorted value.

The typical purpose of a monte carlo simulation is to investigate the performance of a statistical procedure (estimator or test) in realistic settings. Generally, the performance will depend on n and F. In many cases, an estimator or test may perform wonderfully for some values, and poorly for others. It is therefore useful to conduct a variety of experiments, for a selection of choices of n and F.

As discussed above, the researcher must select the number of experiments, B. Often this is called the number of **replications**. Quite simply, a larger B results in more precise estimates of the features of interest of G_n, but requires more computational time. In practice, therefore, the

choice of B is often guided by the computational demands of the statistical procedure. However, it should be recognized that the results of a monte carlo experiment are all estimates computed from a random sample of size B, and therefore it is straightforward to calculate standard errors for any quantity of interest. If the standard error is too large to make a reliable inference, then B will have to be increased.

In particular, it is simple to make inferences about rejection probabilities from statistical tests, such as the percentage estimate reported in (4.6). The random variable $1\left(T_{nb}^{*} \geq 1.96\right)$ is iid Bernoulli, equalling 1 with probability $P = E1\left(T_{nb}^{*} \geq 1.96\right)$. The average (4.6) is therefore an unbiased estimator of P with standard error $s\left(\hat{P}\right) = \sqrt{P\left(1-P\right)/B}$. As P is unknown, this may be approximated using the estimated value $s\left(\hat{P}\right) = \sqrt{\hat{P}\left(1-\hat{P}\right)/B}$ or using a hypothesized value. For example, if we are assessing an asymptotic 5% test, then we can set $s\left(\hat{P}\right) = \sqrt{\left(.05\right)\left(.95\right)/B} \simeq .22/\sqrt{B}$. Hence the standard errors for $B = 100$, 1000, and 5000, are, respectively, $s\left(\hat{P}\right) = .022$, .007, and .003.

4.8 An Example

Here we illustrate Monte Carlo simulation to investigate estimation and tests on a nonlinear function of regression parameters.

Our model is an iid sample $\{y_i, x_{1i}, x_{2i}\}$ by the linear Gaussian regression

$$\begin{aligned} y_i &= \beta_0 + x_{1i}\beta_1 + x_{2i}\beta_2 + e_i \\ \begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} &\sim N\left(0, I_2\right) \\ e_i &\sim N\left(0, \sigma^2\right) \end{aligned}$$

We set $\sigma = 3$, $\beta_0 = 0$, $\beta_1 = 1$, $\beta_2 = .5$, and $n = 300$. In this example, the distribution of the data, F, is determined by the above choices. We don't have to be more explicit about F, as the above equations are sufficient to specify the joint distribution.

The parameter of interest is the ratio of the regression slopes

$$\theta = \frac{\beta_1}{\beta_2}.$$

The goal is to estimate θ, construct confidence intervals for θ, and test the hypothesis that $\theta = \theta_0$. Note that $\theta_0 = 2$.

We estimate the parameters of the model by OLS, and then estimate θ by

$$\hat{\theta} = \frac{\hat{\beta}_1}{\hat{\beta}_2}.$$

The standard error for $\hat{\theta}$ is calculated by the "delta method":

$$se(\hat{\theta}) = \left(\hat{H}_\beta' \hat{V} \hat{H}_\beta\right)^{1/2}, \qquad \hat{H}_\beta = \begin{pmatrix} 0 \\ \frac{1}{\hat{\beta}_2} \\ -\frac{\hat{\beta}_1}{\hat{\beta}_2^2} \end{pmatrix},$$

where $\hat{V}$ is the White covariance matrix estimate.

The asymptotic approximation to the distribution of $\hat{\theta}$ is $N(\theta, AVar(\hat{\theta}))$, where

$$AVar(\hat{\theta}) = n^{-1}\left(H_\beta' \left(Ex_i x_i'\right)^{-1} H_\beta\right) Ee_i^2 \simeq 0.6.$$

Thus the asymptotic approximation is $\hat{\theta} \sim N(2, 0.6)$.

We first investigate the distribution of $\hat{\theta}$. We set $B = 100,000$, which is extremely high, simply because the computation time was minimal. Consequently, the error due to the simulation is minimal.

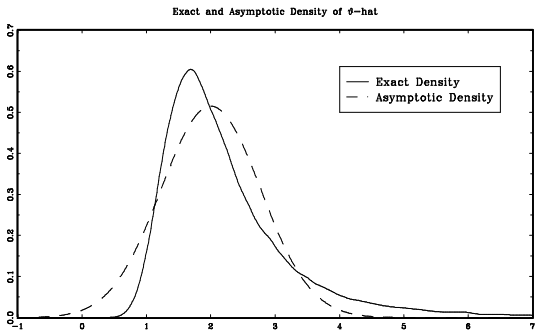
First, I nonparametrically estimated the density of $\hat{\theta}$ (using a kernel density estimator). This density, along with the asymptotic distribution, is displayed in the top section of Figure 1. The divergence between the exact and asymptotic densities is quite dramatic. The exact density is skewed and thick-tailed. Next I estimated[1] the mean and standard deviation of $\hat{\theta}$, finding $E\left(\hat{\theta}\right) = 2.32$ and $sd(\hat{\theta}) = 1.28$. Note that these are quite different from the asymptotic approximation (which are 2.0 and 0.77, respectively).

Inference is typically based on the t-ratio $t_n = (\hat{\theta} - \theta_0)/s(\hat{\theta})$. The asymptotic approximation to t_n is $N(0,1)$. Using the same simulated samples, we estimated the exact density of t_n, which is displayed along with the asymptotic distribution in the bottom section of Figure 1.

Once again we find that the divergence between the exact and asymptotic distributions to be dramatic. The exact distribution is highly skewed and non-normal. The accuracy of hypothesis testing and confidence interval coverage depends on tail probabilities. We calculate the exact probabilities $P(t_n > 1.645) = 0.00$, $P(t_n < -1.645) = .115$ and $P(|t_n| > 1.96) = .084$, meaning that both one-tailed and two-tailed tests and confidence intervals based on asymptotic approximations have significant Type I error. (The nominal, or asymptotic, Type I error for each of these tests is .050).

This example shows that asymptotic approximations may be quite poor, even in very simple regression models with reasonably large samples.

[1] Actually, these are trimmed means and standard deviations, calculated on a sample where the upper and lower 0.5% of the sample has been trimmed off. This was done to obtain a robust measure of mean and variance, due to the presence of extreme outliers.


Exact and Asymptotic Density of ϑ−hat
Exact Density
Asymptotic Density
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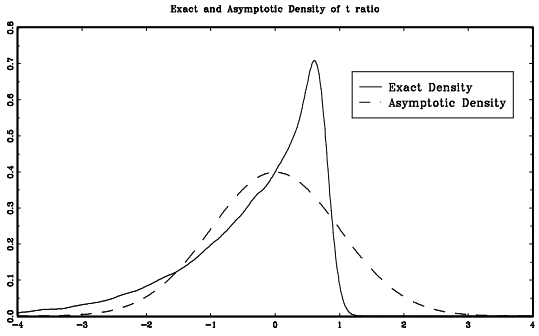
Exact and Asymptotic Density of t ratio
Exact Density
Asymptotic Density
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Chapter 5

Functional Form

5.1 Dummy Variables

In many applications, a bulk of the regressors are *binary,* variables which take on the value 0 or 1. We call these *dummy* variables. Often the regressor is binary because this is the way the data was recorded. In other cases, the binary regressor has been constructed from other variables in the dataset.

For example, a dummy variable may be used to denote the gender (male/female) of an individual. The interest is the regression $E(Wage \mid Gender)$. There are several equivalent ways to write this down. One is to define a dummy variable

$$d_{1i} = \begin{cases} 1 & \text{female} \\ 0 & \text{male} \end{cases}$$

and the model is $W_i = \beta_0 + \beta_1 d_{1i} + e_i$. In this model, $\beta_0 = E(Wage \mid Male)$ and $\beta_0 + \beta_1 = E(Wage \mid Female)$. Second, we could define the variable

$$d_{2i} = \begin{cases} 0 & \text{female} \\ 1 & \text{male} \end{cases}$$

and the model $W_i = \beta_0 + \beta_1 d_{2i} + e_i$. In this model, $\beta_0 = E(Wage \mid Female)$ and $\beta_0 + \beta_1 = E(Wage \mid Male)$. Third, we could define both d_{1i} and d_{2i} as above, and write the model as $W_i = \beta_1 d_{1i} + \beta_2 d_{2i} + e_i$. Here, $\beta_1 = E(Wage \mid Female)$ and $\beta_2 = E(Wage \mid Male)$. These three models are equivalent.

A standard easy mistake is to include an intercept, d_{1i} and d_{2i} all in the regression. These are clearly collinear so this cannot be done.

If the equation of interest is $E(Wage \mid Education, Gender)$, a typical regression model is

$$W_i = \beta_0 + \beta_1 d_{1i} + \beta_2 E_i + e_i.$$

This model specifies an *intercept effect* for gender. That is, males and females have different intercepts, but the same slope on Education.

A regression model allowing for slope differences is

$$W_i = \beta_0 + \beta_1 d_{1i} + \beta_2 E_i + \beta_3 d_{1i} E_i + e_i.$$

This allows for greater differences between groups. When there are several continuous regressors, it may be desirable to have slope effects for some, but not all, of these regressors.

From the standpoint of our regression theory, we think of d_i as a random variable, from the same sampling process which generated the other variables. The idea is that if we sample from the entire population of individuals, some random draws will be women and some will be men.

It is interesting to see how our estimators algebraically handle dummy variables. Take the simple model

$$Y_i = \beta_1 d_{1i} + \beta_2 d_{2i} + e_i.$$

We can write this in matrix notation as

$$Y = X\beta + e$$

where $\beta = (\beta_1 \ \beta_2)'$ and

$$X = [D_1 \quad D_2].$$

Thus,

$$\begin{aligned}
\hat{\beta} &= \left(X'X\right)^{-1} X'Y \\
&= \begin{bmatrix} D_1'D_1 & D_1'D_2 \\ D_2'D_1 & D_2'D_2 \end{bmatrix}^{-1} \begin{bmatrix} D_1'Y \\ D_2'Y \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{n} d_{1i}^2 & \sum_{i=1}^{n} d_{1i}d_{2i} \\ \sum_{i=1}^{n} d_{1i}d_{2i} & \sum_{i=1}^{n} d_{2i}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} d_{1i}y_i \\ \sum_{i=1}^{n} d_{1i}y_i \end{bmatrix} \\
&= \begin{bmatrix} n_1 & 0 \\ 0 & n_2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} d_{1i}y_i \\ \sum_{i=1}^{n} d_{1i}y_i \end{bmatrix} \\
&= \begin{bmatrix} \overline{y}_1 \\ \overline{y}_2 \end{bmatrix}
\end{aligned}$$

where n_1 and n_2 are the number of observations with $d_{1i} = 1$ and $d_{2i} = 1$, respectively, and $\overline{y}_1$ and $\overline{y}_2$ are the sample means among those respective observations.

Now consider the variance estimates.

$$\hat{V}^0 = \left(X'X\right)^{-1}\hat{\sigma}^2 = \begin{bmatrix} \dfrac{\hat{\sigma}^2}{n_1} & 0 \\ 0 & \dfrac{\hat{\sigma}^2}{n_2} \end{bmatrix}$$

where

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\hat{e}_i^2,$$

is the estimate of σ^2 based on the full sample. And

$$\begin{aligned} \hat{V} &= \begin{bmatrix} \left(D_1'D_1\right)^{-1}\hat{\sigma}_1^2 & 0 \\ 0 & \left(D_2'D_2\right)^{-1}\hat{\sigma}_2^2 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{\hat{\sigma}_1^2}{n_1} & 0 \\ 0 & \dfrac{\hat{\sigma}_2^2}{n_2} \end{bmatrix} \end{aligned}$$

where

$$\hat{\sigma}_1^2 = \frac{1}{n_1}\sum_{i=1}^{n} d_{1i}\hat{e}_i^2$$

and

$$\hat{\sigma}_2^2 = \frac{1}{n_2}\sum_{i=1}^{n} d_{2i}\hat{e}_i^2$$

are the estimates of σ^2 based on observations with $d_{1i} = 1$ and $d_{2i} = 1$, respectively. Thus the conventional estimate imposes the restriction that the conditional variance of e_i is the same for the two groups, and the White estimate estimates the conditional variance separately for each group.

5.2 NonLinearity in Regressors

Suppose we are interested in $E\left(y_i \mid x_i = x\right) = g(x)$, $x \in R$, and the form of g is unknown. A common approach is to consider a polynomial approximation:

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \cdots + \beta_k x_i^k + e_i.$$

Letting $\beta = (\beta_0, \beta_1, ..., \beta_k)$ and $z_i = (1, x_i, x_i^2, ..., x_i^k)$, this is $y_i = z_i'\beta + e_i$, which is a linear regression model. Typically, the polynomial order k is kept quite small.

Now suppose that $x \in R^2$. A simple quadratic approximation is

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{1i}^2 + \beta_4 x_{2i}^2 + \beta_5 x_{1i}x_{2i} + e_i.$$

As the dimensionality of x increases, such approximations can become quite non-parsimonious! In practice, therefore, most applications do appear to use more than quadratic terms. Some applications add cubics without interactions:

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{1i}^2 + \beta_4 x_{2i}^2 + \beta_5 x_{1i}^3 + \beta_6 x_{2i}^3 + \beta_7 x_{1i} x_{2i} + e_i.$$

Non-linear approximations can also be made using alternative *basis functions*, such as Fourier series (sins and cosines), splines, "neural nets", or "wavelets".

Since these non-linear models are linear in the parameters, they can be estimated by OLS, and inference is convention. However, the model is non-linear so interpretation must take this into account. For example, in the cubic model given above, the slope with respect to x_{i1} is

$$\frac{\partial}{\partial x_{1i}} E\left(y_i \mid x_i\right) = \beta_1 + 2\beta_3 x_{1i} + 3\beta_5 x_{1i}^2 + \beta_7 x_{2i},$$

which is a function of x_{1i} and x_{2i}, making reporting of the "slope" difficult. In many applications, it will be important to report the slopes for different values of the regressors, carefully chosen to illustrate the point of interest. In other applications, an average slope may be sufficient. There are two obvious candidates: the derivative evaluated at the sample averages

$$\frac{\partial}{\partial x_{1i}} E\left(y_i \mid x_i\right) \mid_{x_i = \overline{x}} = \beta_1 + 2\beta_3 \overline{x}_1 + 3\beta_5 \overline{x}_1^2 + \beta_7 \overline{x}_2$$

and the average derivative

$$\frac{1}{n}\sum_{i=1}^{n} \frac{\partial}{\partial x_{1i}} E\left(y_i \mid x_i\right) = \beta_1 + 2\beta_3 \overline{x}_1 + 3\beta_5 \frac{1}{n}\sum_{i=1}^{n} x_{1i}^2 + \beta_7 \overline{x}_2.$$

5.3 Testing for Omitted NonLinearity

If the goal is to estimate the conditional expectation $E\left(y_i \mid x_i\right)$, it is useful to know how to test a given specification. Many such tests have been proposed. Here we discuss two simple tests.

One simple test for neglected nonlinearity is to add nonlinear functions of the regressors to the regression, and test their significance using a Wald test. Thus, if the model $y_i = x_i'\hat{\beta} + \hat{e}_i$ has been fit by OLS, let $z_i = h(x_i)$ denote functions of x_i which are not linear functions of x_i (perhaps squares of non-binary regressors) and then fit $y_i = x_i'\tilde{\beta} + z_i'\tilde{\gamma} + \tilde{e}_i$ by OLS, and form a Wald statistic for $\gamma = 0$.

Ramsey (1969) introduced the RESET test. The null model is

$$y_i = x_i'\beta + e_i$$

which is estimated by OLS, yielding predicted values $\hat{y}_i = x_i'\hat{\beta}$. Now let

$$z_i = \begin{pmatrix} \hat{y}_i^2 \\ \vdots \\ \hat{y}_i^m \end{pmatrix}$$

be an $(m-1)$-vector of powers of $\hat{y}_i$. Then run the auxiliary regression

$$y_i = x_i'\tilde{\beta} + z_i'\tilde{\gamma} + \tilde{e}_i \tag{5.1}$$

by OLS, and form the Wald statistic W_n for $\gamma = 0$. It is easy (although involves a somewhat lengthy derivation) to show that under the null hypothesis, $W_n \to_d \chi^2_{m-1}$. Thus the null is rejected at the $\alpha\%$ level if W_n exceeds the upper $\alpha\%$ tail critical value of the χ^2_{m-1} distribution.

To implement the test, m must be selected in advance. Typically, small values such as $m = 2$, 3, or 4 seem to work best.

The RESET test appears to work well as a test of functional form against a wide range of smooth alternatives. It is particularly powerful at detecting *single-index* models of the form

$$y_i = G(x_i'\beta) + e_i$$

where $G(\cdot)$ is a smooth "link" function. To see why this is the case, note that (5.1) may be written as

$$y_i = x_i'\tilde{\beta} + \left(x_i'\hat{\beta}\right)^2 \tilde{\gamma}_1 + \left(x_i'\hat{\beta}\right)^3 \tilde{\gamma}_2 + \cdots \left(x_i'\hat{\beta}\right)^m \tilde{\gamma}_{m-1} + \tilde{e}_i$$

which has essentially approximated $G(\cdot)$ by a m'th order polynomial.

5.4 $\log(Y)$ versus Y as Dependent Variable

An econometrician can estimate $Y = X\hat{\beta} + \hat{e}$ or $\log(Y) = X\hat{\beta} + \hat{e}$ (or perhaps both). Which is preferable? There is a large literature on this subject, much of it quite misleading.

The plain truth is that either regression is "okay", in the sense that both $E(y_i \mid x_i)$ and $E(\log(y_i) \mid x_i)$ are well-defined (so long as $y_i > 0$). It is perfectly valid to estimate either or both regressions. They are *different* regression functions, neither is more nor less valid than the other. To *test* one specification versus the other, or select one specification over the other, requires the imposition of additional structure, such as the assumptions that the conditional expectation is linear in x_i, and $e_i \sim N(0, \sigma^2)$.

There still may be good reasons for preferring the $\log(Y)$ regression over the Y regression. First, it may be the case that $E(\log(y_i) \mid x_i)$ is roughly linear in x_i over the support of x_i, while the regression $E(y_i \mid x_i)$ is non-linear, and linear models are easier to report and interpret. Second, it may be the case that the errors in $e_i = \log(y_i) - E(\log(y_i) \mid x_i)$ may be less heteroskedastic than the errors from the linear specification (although the reverse may be true!). Finally, and this may be most important reason, if the distribution of y_i is highly skewed, the conditional mean $E(y_i \mid x_i)$ may not be a useful measure of central tendency, and estimates will be undesirably influenced by extreme observations ("outliers"). In this case, the conditional mean-log $E(\log(y_i) \mid x_i)$ may be a better measure of central tendency, and hence more interesting to estimate and report.

5.5 Multicollinearity

If $rank(X'X) < k$, then $\hat{\beta}$ is not defined. This is defined as multicollinearity. It happens if $rank(X) < k$, which holds if and only if the columns of X are linearly dependent, i.e., there is some α such that $X\alpha = 0$. Most commonly, this arises when sets of regressors are included which are identically related. For example, if X includes the logs of two prices and the log of the relative prices $\log(p_1)$, $\log(p_2)$ and $\log(p_1/p_2)$. When this happens, the applied researcher quickly discovers the error, as the statistical software will be unable to construct $(X'X)^{-1}$. Since the error is discovered quickly, this is rarely a *problem* for applied econometric practice.

The more relevant issue is *near multicollinearity*, which is often called "multicollinearity" for brevity. This is the situation when the $X'X$ matrix is *near* singular, when the columns of X are *close* to linearly dependent. Notice that this definition is not precise, because we have not said what it means for a matrix to be "near singular". This is one difficulty with the definition and interpretation of multicollinearity.

One implication of near singularity of matrices is that the numerical reliability of the calculations is reduced. It is more likely that the reported calculations will be in error due to floating-point calculation difficulties.

The other implication of near multicollinearity is that estimation precision of individual coefficients will be reduced. To see this, consider a simple example. Suppose that

$$Y = X_1\beta_1 + X_2\beta_2 + e,$$

where $E(e_i^2 \mid x_i) = \sigma^2$ and

$$X'X = \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

When $\rho = 1$, then $X'X$ is singular, and when $\rho \approx 1$, then $X'X$ is close to singular. Then

$$\begin{aligned} Var(\hat{\beta}_1) &= \left(X_1'M_2X_1\right)^{-1} \\ &= \left(X_1'X_1 - X_1'X_2\left(X_2'X_2\right)^{-1}X_2'X_1\right)^{-1} \\ &= \frac{1}{1-\rho^2}. \end{aligned}$$

As $\rho \to 1$, $Var(\hat{\beta}_1) \to \infty$. Thus the more "collinear" are the regressors, the worse the precision of $\hat{\beta}_1$. (Similarly $\hat{\beta}_2$).

Basically, what is happening is that when X_1 and X_2 are highly dependent, it is statistically difficult to disentangle the impact of β_1 from that of β_2. The precision of individual estimates are reduced.

Is there a simple solution? Basically, *No*. Fortunately, multicollinearity does not lead to errors in inference. The asymptotic distribution is still valid. Regression estimates are asymptotically normal, and estimated standard errors are consistent for the asymptotic variance. So reported confidence intervals are not inherently misleading. They will be *large*, correctly indicating the inherent uncertainty about the true parameter value.

5.6 Omitted Variables

Let x_{1i} and x_{2i} be two sets of regressors. We can define

$$g_1(x_1) = E\left(y_i \mid x_{1i} = x_1\right)$$

and

$$g_2(x_1, x_2) = E\left(y_i \mid x_{1i} = x_1, x_{2i} = x_2\right).$$

Both of these functions exist and are well defined. Given data, either can be estimated. Thus, if the function $g_1(x_1)$ is estimated by regression of y_i on x_{1i} only, there is no "bias" due to the omission of x_{2i}. In this sense, there is no such thing as "omitted variable bias".

However, the function g_1 may not be of interest. Rather, the function g_2 may be of interest. Thus if g_1 is estimated, when the true relationship of interest is g_2, then there will be estimation bias. That is, what may be of interest is the effect of x_{1i} on the conditional mean of y_i , holding x_{2i} constant, namely

$$\frac{\partial}{\partial x_1} g_2(x_1, x_2) \neq \frac{\partial}{\partial x_1} g_1(x_1).$$

In this sense, omission of x_{2i} from the regression can induce bias.

Another way to see this is by focusing on the linear regression model. Suppose that

$$\begin{aligned} y_i &= x_{1i}'\beta_1 + x_{2i}'\beta_2 + e_i \\ E\left(e_i \mid x_{1i}, x_{2i}\right) &= 0. \end{aligned}$$

Then

$$\begin{aligned} E\left(y_i \mid x_{1i}\right) &= E\left(x_{1i}'\beta_1 + x_{2i}'\beta_2 + e_i \mid x_{1i}\right) \\ &= x_{1i}'\beta_1 + E\left(x_{2i} \mid x_{1i}\right)'\beta_2 \\ &\neq x_{1i}'\beta_1. \end{aligned}$$

Thus a regression of y_i on x_{1i} does not yield the coefficient β_1, unless $E\left(x_{2i} \mid x_{1i}\right) = 0$ or $\beta_2 = 0$.

Furthermore, suppose $E\left(x_{2i} \mid x_{1i}\right) = \Gamma x_{1i}$. Then

$$\begin{aligned} E\left(y_i \mid x_{1i}\right) &= x_{1i}'\beta_1 + \left(\Gamma x_{1i}\right)'\beta_2 \\ &= x_{1i}'\left(\beta_1 + \Gamma'\beta_2\right). \end{aligned}$$

So a regression of y_i on x_{1i} will consistently estimate $\beta_1 + \Gamma'\beta_2$. β_1 cannot be uncovered from this regression, unless $\Gamma = 0$ or $\beta_2 = 0$, and thus the regression is "biased", if the parameter β_1 is of interest.

Notice that the omitted variable bias problem disappears if $\Gamma = 0$ (so x_{1i} and x_{2i} are uncorrelated) or if $\beta_2 = 0$ (so x_{2i} does not enter the joint regression). The first can be assessed by examining the correlation between x_{1i} and x_{2i}, but the second can only be assessed by computing the joint regression. Therefore the standard advice is when in doubt, to always estimate the more general model, since it is by construction free of the omitted variables problem.

5.7 Irrelevant Variables

In the model

$$\begin{aligned} y_i &= x_{1i}'\beta_1 + x_{2i}'\beta_2 + e_i \\ E\left(e_i \mid x_{1i}, x_{2i}\right) &= 0, \end{aligned}$$

x_{2i} is "irrelevant" if β_1 is the parameter of interest and $\beta_2 = 0$. That is, the truth can be written as

$$\begin{aligned} y_i &= x_{1i}'\beta_1 + e_i \\ E\left(e_i \mid x_{1i}, x_{2i}\right) &= 0. \end{aligned}$$

One estimator of β_1 is to regress y_i on x_{1i} alone, denoted $\tilde{\beta}_1$. Another is to regress y_i on x_{1i} and x_{2i}, yielding $(\hat{\beta}_1, \hat{\beta}_2)$. Under which conditions is $\hat{\beta}_1$ or $\tilde{\beta}_1$ superior?

First, it is easy to see that both are unbiased and consistent for β_1. So in comparison with the problem of omitted variables, we see that the presence (or absence) of irrelevant variables is relatively less important.

Second, we can consider the relative efficiency of $\tilde{\beta}_1$ versus $\hat{\beta}_1$. It is harder to make comparisons in the general case, so we focus on the homoskedastic case $E\left(e_i^2 \mid x_{1i}, x_{2i}\right) = \sigma^2$. Then

$$\lim_{n\to\infty} nVar(\tilde{\beta}_1 \mid X) = \left(Ex_{1i}x_{1i}'\right)^{-1}\sigma^2 = Q_{11}^{-1}\sigma^2,$$

say, and

$$\lim_{n\to\infty} nVar(\hat{\beta}_1 \mid X) = \left(Ex_{1i}x_{1i}' - Ex_{1i}x_{2i}'\left(Ex_{2i}x_{2i}'\right)Ex_{2i}x_{1i}'\right)^{-1}\sigma^2 = \left(Q_{11} - Q_{121}\right)^{-1}\sigma^2,$$

say. If $Ex_{1i}x_{2i}' = 0$ (so the variables are uncorrelated) then these two variance matrices equal, and the two estimators are have equal asymptotic efficiency.

Proposition 5.7.1 *If $Ex_{1i}x_{2i}' = 0$, $\hat{\beta}_1$ and $\tilde{\beta}_1$ are both consistent and have equal asymptotic variances.*

When $Ex_{1i}x_{2i}' \neq 0$, however, then $Q_{121} > 0$ and

$$Q_{11} = Q_{11} - Q_{121} + Q_{121} > Q_{11} - Q_{121},$$

so

$$Q_{11}^{-1} < \left(Q_{11} - Q_{121}\right)^{-1},$$

meaning that $\tilde{\beta}_1$ has a lower asymptotic variance matrix than $\hat{\beta}_1$. The inclusion of irrelevant variables reduces efficiency if these variables are correlated with the relevant variables.

Proposition 5.7.2 *If $Ex_{1i}x_{2i}' \neq 0$, $\hat{\beta}_1$ and $\tilde{\beta}_1$ are both consistent, and*

$$\lim_{n\to\infty} nVar(\tilde{\beta}_1 \mid X) < \lim_{n\to\infty} nVar(\hat{\beta}_1 \mid X).$$

5.8 Model Selection

We have discussed the costs and benefits of inclusion/exclusion of variables. How does a researcher go about selecting an econometric specification, when economic theory does not provide complete guidance? This is the question of model selection. It is important that the model selection question be well-posed. For example, the question: "What is the right model for y?" is not well posed, because it does not make clear the conditioning set. In contrast, the question, "Which subset of $(x_1, ..., x_K)$ enters the regression function $E(y_i \mid x_{1i} = x_1, ..., x_{Ki} = x_K)$?" is well posed.

In many cases the problem of model selection can be reduced to the comparison of two nested models, as the larger problem can be written as a sequence of such comparisons. We thus consider the question of the inclusion of X_2 in the linear regression

$$Y = X_1\beta_1 + X_2\beta_2 + e,$$

where X_1 is $n \times k_1$ and X_2 is $n \times k_2$. This is equivalent to the comparison of the two models

$$\begin{aligned} \mathcal{M}_1 &: \quad Y = X_1\beta_1 + e, \qquad E(e \mid X_1, X_2) = 0 \\ \mathcal{M}_2 &: \quad Y = X_1\beta_1 + X_2\beta_2 + e, \qquad E(e \mid X_1, X_2) = 0. \end{aligned}$$

Note that $\mathcal{M}_1 \subset \mathcal{M}_2$. To be concrete, we say that $\mathcal{M}_2$ is true if $\beta_2 \neq 0$.

To fix notation, models 1 and 2 are estimated by OLS, with residual vectors $\hat{e}_1$ and $\hat{e}_2$, estimated variances $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$, etc., respectively. To simplify some of the statistical discussion, we will on occasion use the homoskedasticity assumption $E\left(e_i^2 \mid x_{1i}, x_{2i}\right) = \sigma^2$.

A model selection procedure is a data-dependent rule which selects one of the true models. We can write this as $\widehat{\mathcal{M}}$. There are many possible desirable properties for a model selection procedure. One useful property is consistency, that it selects the true model with probability one if the sample is sufficiently large. A model selection procedure is consistent if

$$\begin{aligned} P\left(\widehat{\mathcal{M}} = \mathcal{M}_1 \mid \mathcal{M}_1\right) &\to 1 \\ P\left(\widehat{\mathcal{M}} = \mathcal{M}_2 \mid \mathcal{M}_2\right) &\to 1 \end{aligned}$$

We now discuss a number of possible model selection methods.

Selection Based on Fit

Natural measures of fit of a regression are the residual sum of squares $\hat{e}'\hat{e}$, $R^2 = 1 - (\hat{e}'\hat{e})/\hat{\sigma}_y^2$ or Gaussian log-likelihood $l = -(n/2)\log\hat{\sigma}^2$. It might therefore be thought attractive to base a model selection procedure on one of these measures of fit. The problem is that each of these measures are necessarily monotonic between nested models, namely $\hat{e}_1'\hat{e}_1 \geq \hat{e}_2'\hat{e}_2$, $R_1^2 \leq R_2^2$, and $l_1 \leq l_2$, so model $\mathcal{M}_2$ would always be selected, regardless of the actual data and probability structure. This is clearly an inappropriate decision rule!

Selection based on Testing

A common approach to model selection is to base the decision on a statistical test such as the Wald W_n or Gaussian Likelihood Ratio

$$\begin{aligned} LR_n &= 2(l_2 - l_1) = n\left(\log\hat{\sigma}_1^2 - \log\hat{\sigma}_2^2\right) = n\left(\log\left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}\right)\right) \\ &\simeq n\left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} - 1\right) = n\left(\frac{\hat{\sigma}_1^2 - \hat{\sigma}_2^2}{\hat{\sigma}_2^2}\right) = W_n. \end{aligned}$$

The model selection rule is as follows, exposited for the Wald test. For some critical level α, let c_α satisfy $P\left(\chi_{k_2}^2 > c_\alpha\right)$. Then select $\mathcal{M}_1$ if $W_n \leq c_\alpha$, else select $\mathcal{M}_2$.

The major problem with this approach is that the critical level α is indeterminate. The reasoning which helps guide the choice of α in hypothesis testing (controlling Type I error) is not relevant for model selection. That is, if α is set to be a small number, then $P\left(\widehat{\mathcal{M}} = \mathcal{M}_1 \mid \mathcal{M}_1\right) \approx 1 - \alpha$ but $P\left(\widehat{\mathcal{M}} = \mathcal{M}_2 \mid \mathcal{M}_2\right)$ could vary dramatically, depending on the sample size, etc. Another problem is that if α is held fixed, then this model selection procedure is inconsistent, as $P\left(\widehat{\mathcal{M}} = \mathcal{M}_1 \mid \mathcal{M}_1\right) \to 1 - \alpha < 1$.

Adjusted R-squared

Since R^2 is not a useful model selection rule, as it always "prefers" the larger model, Theil proposed an adjusted coefficient of determination

$$\begin{aligned} \overline{R}^2 &= 1 - \frac{\left(\hat{e}'\hat{e}\right)/(n-k)}{\hat{\sigma}_y^2} \\ &= 1 - \frac{s^2}{\hat{\sigma}_y^2}. \end{aligned}$$

At one time, it was popular to pick between models based on $\overline{R}^2$. This rule is to select $\mathcal{M}_1$ if $\overline{R}_1^2 > \overline{R}_2^2$, else select $\mathcal{M}_2$. Since $\overline{R}^2$ is a monotonically decreasing function of s^2, this rule is the same as selecting the model with the smaller s^2, or equivalently, the smaller $\log(s^2)$. It is helpful to observe that

$$\begin{aligned} \log(s^2) &= \log\left(\hat{\sigma}^2\frac{n}{n-k}\right) \\ &= \log\left(\hat{\sigma}^2\right) + \log\left(1 + \frac{k}{n-k}\right) \\ &\simeq \log\left(\hat{\sigma}^2\right) + \frac{k}{n-k} \\ &\simeq \log\left(\hat{\sigma}^2\right) + \frac{k}{n}, \end{aligned}$$

(the first approximation is $\log(1+x) \simeq x$ for small x). Thus selecting based on $\overline{R}^2$ is the same as selecting based on $\log\left(\hat{\sigma}^2\right) + \frac{k}{n}$, which is a particular choice of penalized likelihood criteria. It

turns out that model selection based on any criterion of the form

$$\log\left(\hat{\sigma}^2\right) + c\frac{k}{n}, \qquad c > 0, \tag{5.2}$$

is inconsistent, as the rule tends to overfit. Indeed, since under $\mathcal{M}_1$,

$$LR_n = n\left(\log\hat{\sigma}_1^2 - \log\hat{\sigma}_2^2\right) \simeq W_n \to_d \chi^2_{k_2}, \tag{5.3}$$

$$\begin{aligned} P\left(\widehat{\mathcal{M}} = \mathcal{M}_1 \mid \mathcal{M}_1\right) &= P\left(\overline{R}_1^2 > \overline{R}_2^2 \mid \mathcal{M}_1\right) \\ &\simeq P\left(n\log(s_1^2) < n\log(s_2^2) \mid \mathcal{M}_1\right) \\ &\simeq P\left(n\log\left(\hat{\sigma}_1^2\right) + ck_1 < n\log\left(\hat{\sigma}_2^2\right) + c\left(k_1 + k_2\right) \mid \mathcal{M}_1\right) \\ &= P\left(LR_n < ck_2 \mid \mathcal{M}_1\right) \\ &\to P\left(\chi^2_{k_2} < ck_2\right) < 1. \end{aligned}$$

Akaike Information Criterion

Akaike proposed an information criterion which takes the form (5.2) with $c = 2$:

$$AIC = \log\left(\hat{\sigma}^2\right) + 2\frac{k}{n}. \tag{5.4}$$

This imposes a larger penalty on overparameterization than does $\overline{R}^2$. Akaike's motivation for this criterion is that a good measure of the fit of a model density $f(Y \mid X, \mathcal{M})$ to the true density $f(Y \mid X)$ is the Kullback distance $K(\mathcal{M}) = E\left(\log f(Y \mid X) - \log f(Y \mid X, \mathcal{M})\right)$. The log-likelihood function provides a decent estimate of this distance, but it is biased, and a better, less-biased estimate can be obtained by introducing the penalty $2k$. The actual derivation is not very enlightening, and the motivation for the argument is not fully satisfactory, so we omit the details. Despite these concerns, the AIC is a popular method of model selection. The rule is to select $\mathcal{M}_1$ if $AIC_1 < AIC_2$, else select $\mathcal{M}_2$.

Since the AIC criterion (5.4) takes the form (5.2), it is an inconsistent model selection criterion, and tends to overfit.

Schwarz Criterion

While many modifications of the AIC have been proposed, the most popular appears to be one proposed by Schwarz, based on Bayesian arguments. His criterion, known as the BIC, is

$$BIC = \log\left(\hat{\sigma}^2\right) + \log(n)\frac{k}{n}. \tag{5.5}$$

Since $\log(n) > 2$ (if $n > 8$), the BIC places a larger penalty than the AIC on the number of estimated parameters and is more parsimonious.

In contrast to the other methods studied above, BIC model selection is consistent. Indeed, since (5.3) holds under $\mathcal{M}_1$,

$$\frac{LR_n}{\log(n)} \to_p 0,$$

so

$$\begin{aligned} P\left(\widehat{\mathcal{M}} = \mathcal{M}_1 \mid \mathcal{M}_1\right) &= P\left(BIC_1 < BIC_2 \mid \mathcal{M}_1\right) \\ &= P\left(LR_n < \log(n) k_2 \mid \mathcal{M}_1\right) \\ &= P\left(\frac{LR_n}{\log(n)} < k_2 \mid \mathcal{M}_1\right) \\ &\to P\left(0 < k_2\right) = 1. \end{aligned}$$

Also under $\mathcal{M}_2$, one can show that

$$\frac{LR_n}{\log(n)} \to_p \infty,$$

thus

$$\begin{aligned} P\left(\widehat{\mathcal{M}} = \mathcal{M}_2 \mid \mathcal{M}_2\right) &= P\left(\frac{LR_n}{\log(n)} > k_2 \mid \mathcal{M}_2\right) \\ &\to 1. \end{aligned}$$

Selection Among Multiple Regressors

We have discussed model selection between two models. The methods extend readily to the issue of selection among multiple regressors. The general problem is the model

$$y_i = \beta_1 x_{1i} + \beta_2 x_{1i} + \cdots + \beta_K x_{Ki} + e_i, \qquad E\left(e_i \mid x_i\right) = 0$$

and the question is which subset of the coefficients are non-zero (equivalently, which regressors enter the regression).

There are two leading cases: ordered regressors and unordered.

In the ordered case, the models are

$$\begin{aligned} \mathcal{M}_1 &: \beta_1 \neq 0, \beta_2 = \beta_3 = \cdots = \beta_K = 0 \\ \mathcal{M}_2 &: \beta_1 \neq 0, \beta_2 \neq 0, \beta_3 = \cdots = \beta_K = 0 \\ &\vdots \\ \mathcal{M}_K &: \beta_1 \neq 0, \beta_2 \neq 0, \ldots, \beta_K \neq 0. \end{aligned}$$

which are nested. The AIC selection criteria estimates the K models by OLS, stores the residual variance $\hat{\sigma}^2$ for each model, and then selects the model with the lowest AIC (5.4). Similarly for the BIC, selecting based on (5.5).

In the unordered case, a model consists of any possible subset of the regressors $\{x_{1i}, ..., x_{Ki}\}$, and the AIC or BIC in principle can be implemented by estimating all possible subset models. However, there are 2^K such models, which can be a very large number. For example, $2^{10} = 1024$, and $2^{20} = 1,048,576$. In the latter case, a full-blown implementation of the BIC selection criterion would seem computationally prohibitive.

Chapter 6

NonLinear Regression

6.1 NonLinear Regression Models

We say that the regression function $g(x,\theta) = E(y_i \mid x_i = x)$ is nonlinear in the parameters if it cannot be written as $g(x,\theta) = z(x)'\theta$ for some function $z(x)$. Examples of nonlinear regression functions include

$$\begin{aligned}
g(x,\theta) &= \theta_1 + \theta_2 \frac{x}{1+\theta_3 x} \\
g(x,\theta) &= \theta_1 + \theta_2 x^{\theta_3} \\
g(x,\theta) &= \theta_1 + \theta_2 \exp(\theta_3 x) \\
g(x,\theta) &= G(x'\theta),\ G \text{ known} \\
g(x,\theta) &= \theta_1 + \theta_2 x_1 + (\theta_3 + \theta_4 x_1)\, \Phi\left(\frac{x_2 - \theta_5}{\theta_6}\right) \\
g(x,\theta) &= \theta_1 + \theta_2 x + \theta_4 (x - \theta_3)\, 1(x > \theta_3) \\
g(x,\theta) &= (\theta_1 + \theta_2 x_1)\, 1(x_2 < \theta_3) + (\theta_4 + \theta_5 x_1)\, 1(x_2 > \theta_3)
\end{aligned}$$

In the first five examples, $g(x,\theta)$ is (generically) differentiable in the parameters θ. In the final two examples, g is not differentiable with respect to θ_3, which alters some of the analysis. When it exists, let

$$g_\theta(x,\theta) = \frac{\partial}{\partial \theta} g(x,\theta).$$

Nonlinear regression is frequently adopted because the functional form $g(x,\theta)$ is suggested by an economic model. In other cases, it is adopted as a flexible approximation to an unknown regression function.

6.2 NLLS Estimation

The least squares estimator $\hat{\theta}$ minimizes the sum-of-squared-errors

$$S_n(\theta) = \sum_{i=1}^{n} (y_i - g(x_i, \theta))^2 .$$

When the regression function is nonlinear, we call this the *nonlinear least squares* (NLLS) estimator.

The NLLS residuals are $\hat{e}_i = y_i - g(x_i, \hat{\theta})$.

Since the problem is non-linear, $\hat{\theta}$ must be found by numerical methods. A common method to minimize the function $S_n(\theta)$ is the Gauss-Newton method or one of its variants. When $g(x, \theta)$ is differentiable, then the FOC for minimization are

$$0 = \sum_{i=1}^{n} g_\theta(x_i, \hat{\theta}) \hat{e}_i. \tag{6.1}$$

6.3 Concentration

A major simplification can be achieved through concentration. This can be done when we partition $\theta = (\beta, \gamma)$ so that

$$g(x_i, \theta) = \beta' x_i(\gamma)$$

where $x_i(\gamma)$ is a $k \times 1$ function of x_i and γ. In all the examples, this can be done with γ of much smaller dimension than β. In many cases, γ is scalar.

The SSE function is $S_n(\theta) = S_n(\beta, \gamma)$ and thus

$$\min_\theta S_n(\theta) = \min_\gamma \min_\beta S_n(\beta, \gamma).$$

Since β enters the model linearly, we see that

$$\begin{aligned} \hat{\beta}(\gamma) &= \arg\min_\beta S_n(\beta, \gamma) \\ &= \left(X(\gamma)' X(\gamma) \right)^{-1} X(\gamma)' Y \end{aligned} \tag{6.2}$$

where $X(\gamma)$ is the $n \times k$ matrix of the stacked $x_i(\gamma)'$.

Now set

$$S_n(\gamma) = S_n(\hat{\beta}(\gamma), \gamma)$$

which is the concentrated sum of squared errors. We have $\hat{\gamma} = \arg\min S_n(\gamma)$ and $\hat{\beta} = \hat{\beta}(\hat{\gamma})$. The pair $(\hat{\beta}, \hat{\gamma})$ are the joint NLLS estimates of (β, γ).

The main benefit of concentration is that the dimension of the numerical optimization is typically reduced dramatically. When γ is scalar, the final minimization over γ can be done by a grid search, for example.

6.4 Computation Using Linearization

A linearized regression can be used to find the NLLS estimator $\hat{\theta}$. It is an iterative technique, meaning that we start with an initial guess $\hat{\theta}_1$, and then define an iteration rule $\hat{\theta}_j \to \hat{\theta}_{j+1}$, stopping when the iteration "converges", meaning in practice that the difference $\left|\hat{\theta}_{j+1} - \hat{\theta}_j\right|$ is smaller than some pre-specified level.

We now define the iteration rule

$$\hat{\theta}_{j+1} = \hat{\theta}_j + d_j. \tag{6.3}$$

where the "direction" d_j is a function of $\hat{\theta}_j$. Let

$$\begin{aligned} g_{\theta i}(j) &= g_\theta(x_i, \hat{\theta}_j), \\ \hat{e}_i(j) &= y_i - g(x_i, \hat{\theta}_j), \end{aligned}$$

and

$$d_j = \left(\sum_{i=1}^n g_{\theta i}(j) g_{\theta i}(j)'\right)^{-1} \left(\sum_{i=1}^n g_{\theta i}(j)\hat{e}_i(j)\right).$$

Convergence requires $d_j = 0$, which requires $\sum_{i=1}^n g_{\theta i}\hat{e}_i = 0$, which is the same as the first-order condition for NLLS minimization (6.1). Thus if (6.3) converges, it yields the NLLS estimator.

One problem is that the updating rule (6.3) may tend to overshoot and thus fail to converge. The algorithm can be easily modified to correct for this, by substituting for (6.3) the rule

$$\hat{\theta}_{j+1} = \hat{\theta}_j + \lambda d_j,$$

where $\lambda > 0$ is a scalar "step length". Rules for determining the step length are discussed in the numerical optimization literature. The goal is to find λ so that $S_n^*(\lambda) = S_n\left(\hat{\theta}_j + \lambda d_j\right)$ is minimized. One simple rule is the "half" rule. Essentially, try the sequence $\lambda = 1, \frac{1}{2}, \frac{1}{4}, ...$, until a value of λ is found which reduces the criterion $S_n^*(\lambda)$. Specifically, first compute $S_n^*(1)$. If $S_n^*(1) < S_n^*(0) = S_n\left(\hat{\theta}_j\right)$, then set $\hat{\theta}_{j+1} = \hat{\theta}_j + d_j$. If not, compute $S_n^*(1/2)$. If $S_n^*(1/2) < S_n^*(0)$, then set $\hat{\theta}_{j+1} = \hat{\theta}_j + \frac{1}{2}d_j$. This is continued until a value of λ yields an "improvement" in the criterion.

6.5 Asymptotic Distribution

Let $g_{\theta i} = g_\theta(x_i, \theta_0)$.

Theorem 6.5.1 *If the model is identified and $g(x, \theta)$ is differentiable with respect to θ,*

$$\sqrt{n}\left(\hat{\theta} - \theta_0\right) \to_d N(0, V)$$

$$V = \left(E\left(g_{\theta i} g_{\theta i}'\right)\right)^{-1} \left(E\left(g_{\theta i} g_{\theta i}' e_i^2\right)\right) \left(E\left(g_{\theta i} g_{\theta i}'\right)\right)^{-1}.$$

"Proof". First, it must be shown that $\hat{\theta} \to_p \theta_0$. This can be done using arguments for optimization estimators, but we won't cover that argument here.

Since $\hat{\theta} \to_p \theta_0$, $\hat{\theta}$ is close to θ_0 for n large, so the minimization of $S_n(\theta)$ only needs to be examined for θ close to θ_0.

Let

$$y_i^0 = e_i + g'_{\theta i}\theta_0.$$

For θ close to the true value θ_0, by a first-order Taylor series approximation,

$$g(x_i, \theta) \simeq g(x_i, \theta_0) + g'_{\theta i}(\theta - \theta_0).$$

Thus

$$\begin{aligned} y_i - g(x_i, \theta) &\simeq (e_i + g(x_i, \theta_0)) - \left(g(x_i, \theta_0) + g'_{\theta i}(\theta - \theta_0)\right) \\ &= e_i - g'_{\theta i}(\theta - \theta_0) \\ &= y_i^0 - g'_{\theta i}\theta. \end{aligned}$$

Hence the sum of squared errors function is

$$S_n(\theta) = \sum_{i=1}^n (y_i - g(x_i, \theta))^2 \simeq \sum_{i=1}^n \left(y_i^0 - g'_{\theta i}\theta\right)^2$$

and the right-hand-side is the SSE function for a linear regression of y_i^0 on $g_{\theta i}$. Thus the NLLS estimator $\hat{\theta}$ has the same asymptotic distribution as the (infeasible) OLS regression of y_i^0 on $g_{\theta i}$, which is that stated in the theorem. ■

Based on Theorem 6.5.1, an estimate of the variance matrix for $\hat{\theta}$ is

$$\hat{V} = \left(\sum_{i=1}^n \hat{g}_{\theta i}\hat{g}'_{\theta i}\right)^{-1} \left(\sum_{i=1}^n \hat{g}_{\theta i}\hat{g}'_{\theta i}\hat{e}_i^2\right) \left(\sum_{i=1}^n \hat{g}_{\theta i}\hat{g}'_{\theta i}\right)^{-1}$$

where $\hat{g}_{\theta i} = g_\theta(x_i, \hat{\theta})$ and $\hat{e}_i = y_i - g(x_i, \hat{\theta})$.

6.6 Identification

Identification is often tricky in nonlinear regression models. Suppose that

$$g(x_i, \theta) = \beta_1' z_i + \beta_2' x_i(\gamma).$$

The model is linear when $\beta_2 = 0$, and this is often a useful hypothesis (sub-model) to consider. Thus we want to test

$$H_0 : \beta_2 = 0.$$

However, under H_0, the model is

$$y_i = \beta_1' z_i + e_i$$

and both β_2 and γ have dropped out. This means that under H_0, γ is not identified. This renders the distribution theory presented in the previous section invalid. Thus when the truth is that $\beta_2 = 0$, the parameter estimates are not asymptotically normally distributed. Furthermore, tests of H_0 do not have asymptotic normal or chi-square distributions.

The asymptotic theory of such tests have been worked out by Andrews and Ploberger (1994) and B. Hansen (1996). In particular, Hansen shows how to use simulation (similar to the bootstrap) to construct the asymptotic critical values (or p-values) in a given application.

Chapter 7

Feasible GLS

The Gauss-Markov Theorem (Theorem 4.6.1) shows that in regression models OLS is inefficient relative to infeasible GLS. Now we discuss feasible approximate GLS estimation.

7.1 Skedastic Regression

Suppose that the conditional variance takes the parametric form

$$\begin{aligned} \sigma_i^2 &= \alpha_0 + z_{1i}'\alpha_1 \\ &= \alpha' z_i, \end{aligned}$$

where z_{1i} is some $q \times 1$ function of x_i. Typically, z_{1i} are squares (and perhaps levels) of some (or all) elements of x_i. Often the functional form is kept simple for parsimony.

Let $\eta_i = e_i^2$. Then

$$E(\eta_i \mid x_i) = \alpha_0 + z_{1i}'\alpha_1$$

and we have the regression equation

$$\begin{aligned} \eta_i &= \alpha_0 + z_{1i}'\alpha_1 + \xi_i \\ E(\xi_i \mid x_i) &= 0. \end{aligned} \tag{7.1}$$

It is helpful to think about the regression error ξ_i. It has conditional variance

$$\begin{aligned} Var(\xi_i \mid x_i) &= Var\left(e_i^2 \mid x_i\right) \\ &= E\left(\left(e_i^2 - E\left(e_i^2 \mid x_i\right)\right)^2 \mid x_i\right) \\ &= E\left(e_i^4 \mid x_i\right) - \left(E\left(e_i^2 \mid x_i\right)\right)^2. \end{aligned}$$

If e_i is heteroskedastic, then $Var(\xi_i \mid x_i)$ will depend on x_i. In contrast, when e_i is independent of x_i then it is a constant

$$Var(\xi_i \mid x_i) = E\left(e_i^4\right) - \sigma^4$$

and under normality it simplifies to

$$Var\left(\xi_i \mid x_i\right) = 2\sigma^4. \tag{7.2}$$

7.2 Estimation of Skedastic Regression

Suppose e_i (and thus η_i) were observed. Then we could estimate α by OLS:

$$\hat{\alpha} = \left(Z'Z\right)^{-1} Z'\eta \to_p \alpha$$

and

$$\sqrt{n}\left(\hat{\alpha} - \alpha\right) \to_d N\left(0, V_\alpha\right)$$

where

$$V_\alpha = \left(E\left(z_i z_i'\right)\right)^{-1} E\left(z_i z_i' \xi_i^2\right) \left(E\left(z_i z_i'\right)\right)^{-1}. \tag{7.3}$$

While e_i is not observed, we have the OLS residual $\hat{e}_i = y_i - x_i'\hat{\beta} = e_i - x_i'(\hat{\beta} - \beta)$. Thus

$$\begin{aligned} \hat{\eta} - \eta_i &= \hat{e}_i^2 - e_i^2 \\ &= -2e_i x_i' \left(\hat{\beta} - \beta\right) + (\hat{\beta} - \beta)' x_i x_i' (\hat{\beta} - \beta) \\ &= \phi_i, \end{aligned}$$

say. Note that

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \phi_i &= \frac{-2}{n} \sum_{i=1}^n z_i e_i x_i' \sqrt{n}\left(\hat{\beta} - \beta\right) + \frac{1}{n} \sum_{i=1}^n z_i (\hat{\beta} - \beta)' x_i x_i' (\hat{\beta} - \beta)\sqrt{n} \\ &\to_p 0 \end{aligned}$$

Let

$$\tilde{\alpha} = \left(Z'Z\right)^{-1} Z'\hat{\eta} \tag{7.4}$$

be from OLS regression of $\hat{\eta}_i$ on z_i. Then

$$\begin{aligned} \sqrt{n}\left(\tilde{\alpha} - \alpha\right) &= \sqrt{n}\left(\hat{\alpha} - \alpha\right) + \left(n^{-1} Z'Z\right)^{-1} n^{-1/2} Z'\phi \\ &\to_d N\left(0, V_\alpha\right) \end{aligned} \tag{7.5}$$

Thus the fact that η_i is replaced with $\hat{\eta}_i$ is asymptotically irrelevant. We may call (7.4) the *skedastic* regression, as it is estimating the conditional variance of the regression of y_i on x_i.

We have shown that α is consistently estimated by a simple procedure, and hence we can estimate $\sigma_i^2 = z_i'\alpha$ by $\tilde{\sigma}_i^2 = z_i'\tilde{\alpha}$. We now discuss how to use these results to test hypotheses on α, and construct a FGLS estimator for β.

7.3 Testing for Heteroskedasticity

The hypothesis of homoskedasticity is that $E\left(e_i^2 \mid x_i\right) = \sigma^2$, or equivalently that

$$H_0 : \alpha_1 = 0$$

in the regression (7.1). We may therefore test this hypothesis by the estimation (7.4) and constructing a Wald statistic.

This hypothesis does not imply that ξ_i is independent of x_i. Typically, however, we impose the stronger hypothesis and test the hypothesis that e_i is independent of x_i, in which case ξ_i is independent of x_i and the asymptotic variance (7.3) for $\tilde{\alpha}$ simplifies to

$$V_\alpha = \left(E\left(z_i z_i'\right)\right)^{-1} E\left(\xi_i^2\right). \tag{7.6}$$

Hence the standard test of H_0 is a classic F (or Wald) test for exclusion of all regressors from the skedastic regression (7.4). The asymptotic distribution (7.5) and the asymptotic variance (7.6) under independence show that this test has an asymptotic chi-square distribution.

Theorem 7.3.1 *Under H_0, and e_i independent of x_i, the Wald test of H_0 is asymptotically χ_q^2.*

Most tests for heteroskedasticity take this basic form. The main differences between popular "tests" is which transformations of x_i enter z_i. Motivated by the form of the asymptotic variance of the OLS estimator $\hat{\beta}$, White (1980) proposed that the test for heteroskedasticity be based on setting z_i to equal all non-redundant elements of x_i, its squares, and all cross-products. Breusch-Pagan (1979) proposed what might appear to be a distinct test, but the only difference is that they allowed for general choice of z_i, and used an assumption of normality to use the simplification (7.2) for their test. If this simplification is replaced by the standard formula (under independence of the error), the two tests coincide.

7.4 Feasible GLS Estimation

Let

$$\tilde{\sigma}_i^2 = \tilde{\alpha}' z_i.$$

Suppose that $\tilde{\sigma}_i^2 > 0$ for all i. Then set

$$\tilde{D} = diag\{\tilde{\sigma}_1^2, ..., \tilde{\sigma}_n^2\}$$

and

$$\tilde{\beta} = \left(X'\tilde{D}^{-1}X\right)^{-1} X'\tilde{D}^{-1}Y.$$

This is the feasible GLS, or FGLS, estimator of β.

Since there is not a unique specification for the conditional variance the FGLS estimator is not unique, and will depend on the model (and estimation method) for the skedastic regression.

One typical problem with implementation of FGLS estimation is that in a linear regression specification, there is no guarantee that $\tilde{\sigma}_i^2 > 0$ for all i. If $\tilde{\sigma}_i^2 < 0$ for some i, then the FGLS estimator is not well defined. Furthermore, if $\tilde{\sigma}_i^2 \approx 0$ for some i, then the FGLS estimator will force the regression equation through the point (y_i, x_i), which is typically undesirable. This suggests that there is a need to bound the estimated variances away from zero. A trimming rule might make sense:

$$\overline{\sigma}_i^2 = \max[\tilde{\sigma}_i^2, \underline{\sigma}^2]$$

for some $\underline{\sigma}^2 > 0$.

It is possible to show that if the skedastic regression is correctly specified, then FGLS is asymptotically equivalent to GLS, but the proof of this is tricky in our notational structure. We just state the result without proof.

Theorem 7.4.1 *If the skedastic regression is correctly specified,*

$$\sqrt{n}\left(\tilde{\beta}_{GLS} - \tilde{\beta}_{FGLS}\right) \to_p 0,$$

and thus

$$\sqrt{n}\left(\tilde{\beta}_{FGLS} - \beta\right) \to_d N(0, V),$$

where

$$V = \left(E\left(\sigma_i^{-2} x_i x_i'\right)\right)^{-1}.$$

7.5 Covariance Matrix Estimation

Examining the asymptotic distribution of Theorem 7.4.1, the natural estimator of $Var(\tilde{\beta})$ is

$$\tilde{V}^0 = \left(\sum_{i=1}^n \tilde{\sigma}_i^{-2} x_i x_i'\right)^{-1} = \left(X'\tilde{D}^{-1}X\right)^{-1}.$$

It is consistent for V in the sense that $n\tilde{V}^0 \to_p V$ as $n \to \infty$.

$\tilde{V}^0$ is appropriate when the skedastic regression (7.1) is correctly specified.

It may be the case that $\alpha' z_i$ is only an approximation to the true conditional variance $\sigma_i^2 = E(e_i^2 \mid x_i)$. In this case we interpret $\alpha' z_i$ as a linear projection of e_i^2 on z_i. $\tilde{\beta}$ should perhaps be called a quasi-FGLS estimator of β. Its asymptotic variance is not that given in Theorem 7.4.1. Instead,

$$V = \left(E\left(\left(\alpha' z_i\right)^{-1} x_i x_i'\right)\right)^{-1} \left(E\left(\left(\alpha' z_i\right)^{-2} \sigma_i^2 x_i x_i'\right)\right) \left(E\left(\left(\alpha' z_i\right)^{-1} x_i x_i'\right)\right)^{-1}.$$

V takes a sandwich form, similar to the covariance matrix of the OLS estimator. Unless $\sigma_i^2 = \alpha' z_i$, $\tilde{V}^0$ is inconsistent for V.

An appropriate solution is to use a White-type estimator in place of $\tilde{V}^0$. This may be written as

$$\begin{aligned}\tilde{V} &= \left(\sum_{i=1}^{n} \tilde{\sigma}_i^{-2} x_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} \tilde{\sigma}_i^{-4} \hat{e}_i^2 x_i x_i'\right) \left(\sum_{i=1}^{n} \tilde{\sigma}_i^{-2} x_i x_i'\right)^{-1} \\ &= \left(X'\tilde{D}^{-1}X\right)^{-1} \left(X'\tilde{D}^{-1}\hat{D}\tilde{D}^{-1}X\right) \left(X'\tilde{D}^{-1}X\right)^{-1}\end{aligned}$$

where $\hat{D} = diag\{\hat{e}_1^2, ..., \hat{e}_n^2\}$. This is an estimator which is robust to misspecification of the conditional variance, and was proposed by Cragg (*Journal of Econometrics*, 1992).

7.6 Commentary: FGLS versus OLS

In a regression model, FGLS is asymptotically superior to OLS. Why then do we not exclusively estimate regression models by FGLS? This is a good question. There are three reasons.

First, FGLS estimation depends on specification and estimation of the skedastic regression. Since the form of the skedastic regression is unknown, and it may be estimated with considerable error, the estimated conditional variances may contain more noise than information about the true conditional variances. In this case, FGLS will do worse than OLS in practice.

Second, individual estimated conditional variances may be negative, and this requires trimming to solve. This introduces an element of arbitrariness which is unsettling to empirical researchers.

Third, OLS is a more robust estimator of the parameter vector. It is consistent not only in the regression model, but also under the assumptions of linear projection. The GLS and FGLS estimators, on the other hand, require the assumption of a correct conditional mean. If the equation of interest is a linear projection, and not a conditional mean, then the OLS and FGLS estimators will converge in probability to different limits, as they will be estimating two different projections. And the FGLS probability limit will depend on the particular function selected for the skedastic regression. The point is that the efficiency gains from FGLS are built on the stronger assumption of a correct conditional mean, and the cost is a reduction of robustness to misspecification.

Chapter 8

Generalized Method of Moments

8.1 Overidentified Linear Model

The linear model

$$\begin{aligned} y_i &= x_i'\beta + e_i \\ E(x_i e_i) &= 0 \end{aligned}$$

with $x_i \in R^\ell$ and $\beta \in R^\ell$ gives rise to the ℓ moment conditions

$$0 = E(g(w_i, \beta))$$

where $w_i = (y_i, x_i')'$. If β is known to satisfy a non-trivial restriction (e.g. $h(\beta) = 0$ for known $h : R^\ell \to R^q$ with $q > 0$) then we can rewrite $g(w_i, \beta) = g^*(w_i, \beta^*)$ where $\beta^* \in R^k$ with $k = \ell - q$. For example, if we partition $\beta = (\beta_1\ \beta_2)$ and conformably x_i, then the exclusion restriction $\beta_2 = 0$ is equivalent to setting $g^*(w_i, \beta^*) = x_i(y_i - x_{1i}'\beta_1)$. This is an **overidentified** model, since $\ell > k$ – the number of moment conditions is strictly larger than the number of parameters. How should we estimate β efficiently given this information? And how should we efficiently test the hypothesis?

The general setting are **moment condition models** of the form

$$Eg(w_i, \beta_0) = 0 \tag{8.1}$$

where $\beta \in R^k$ and $g \in R^\ell$ with $\ell > k$. In the statistics literature, these are known as **estimating equations**.

We will write the class of linear models using the notation $w_i = (y_i, z_i, x_i)$, setting

$$\begin{aligned} y_i &= z_i'\theta + e_i \\ E(x_i e_i) &= 0. \end{aligned}$$

In the linear model

$$g(w_i, \beta_0) = x_i\left(y_i - z_i'\beta\right). \tag{8.2}$$

For the present we may assume that z_i are functions (components of) the regressor x_i, but this does not need to be the case, and in Chapter 10 we will examine models where z_i and x_i will be distinct.

8.2 GMM Estimator

Define the sample analog of (8.2)

$$\overline{g}_n(\beta) = \frac{1}{n}\sum_{i=1}^{n} g_i(\beta) = \frac{1}{n}\sum_{i=1}^{n} x_i\left(y_i - z_i'\beta\right) = \frac{1}{n}\left(X'Y - X'Z\beta\right). \tag{8.3}$$

The method of moments estimator for β is defined as the parameter value which sets $\overline{g}_n(\beta) = 0$, but this is generally not possible when $\ell > k$. The idea of the generalized method of moments (GMM) is to define an estimator which sets $\overline{g}_n(\beta)$ "close" to zero.

For some $\ell \times \ell$ weight matrix $W_n > 0$, let

$$J_n(\beta) = n \cdot \overline{g}_n(\beta)' W_n \overline{g}_n(\beta).$$

This is a non-negative measure of the "length" of the vector $\overline{g}_n(\beta)$. For example, if $W_n = I$, then, $J_n(\beta) = n \cdot \overline{g}_n(\beta)'\overline{g}_n(\beta) = n \cdot |\overline{g}_n(\beta)|^2$, the square of the Euclidean length. The GMM estimator minimizes $J_n(\beta)$.

Definition 8.2.1 $\hat{\beta}_{GMM} = \underset{\beta}{\operatorname{argmin}}\, J_n(\beta)$.

Note that if $k = \ell$, then $\overline{g}_n(\hat{\beta}) = 0$, and the GMM estimator is the MME.

The first order conditions for the GMM estimator are

$$\begin{aligned} 0 &= \frac{\partial}{\partial \beta} J_n(\hat{\beta}) \\ &= 2\frac{\partial}{\partial \beta}\overline{g}_n(\hat{\beta})' W_n \overline{g}_n(\hat{\beta}) \\ &= -2\frac{1}{n}Z'XW_n\frac{1}{n}X'\left(Y - Z\hat{\beta}\right) \end{aligned}$$

so

$$2Z'XW_nX'Z\hat{\beta} = 2Z'XW_nX'Y$$

which establishes the following.

Proposition 8.2.1

$$\hat{\beta}_{GMM} = \left(Z'XW_nX'Z\right)^{-1} Z'XW_nX'Y.$$

While the estimator depends on W_n, the dependence is only up to scale, for if W_n is replaced by cW_n for some $c > 0$, $\hat{\beta}_{GMM}$ does not change.

8.3 Distribution of GMM Estimator

Assume that $W_n \to_p W > 0$. Let

$$Q = E\left(x_i z_i'\right)$$

and

$$\Omega = E\left(x_i x_i' e_i^2\right) = E\left(g_i g_i'\right),$$

where $g_i = x_i e_i$. Then

$$\left(\frac{1}{n} Z'X\right) W_n \left(\frac{1}{n} X'Z\right) \to_p Q'WQ$$

and

$$\left(\frac{1}{n} Z'X\right) W_n \left(\frac{1}{n} X'e\right) \to_p Q'WN\left(0, \Omega\right).$$

We conclude:

Theorem 8.3.1 $\sqrt{n}\left(\hat{\beta} - \beta\right) \to_d N\left(0, V\right)$, *where*

$$V = \left(Q'WQ\right)^{-1}\left(Q'W\Omega WQ\right)\left(Q'WQ\right)^{-1}.$$

In general, GMM estimators are asymptotically normal with "sandwich form" asymptotic variances.

The optimal weight matrix W_0 is one which minimizes V. This turns out to be $W_0 = \Omega^{-1}$. The proof is left as an exercise. This yields the *efficient GMM* estimator:

$$\hat{\beta} = \left(Z'X\Omega^{-1}X'Z\right)^{-1} Z'X\Omega^{-1}X'Y.$$

Thus we have

Theorem 8.3.2 *For the efficient GMM estimator,* $\sqrt{n}\left(\hat{\beta} - \beta\right) \to_d N\left(0, \left(Q'\Omega^{-1}Q\right)^{-1}\right)$.

This estimator is efficient only in the sense that it is the best (asymptotically) in the class of GMM estimators with this set of moment conditions.

$W_0 = \Omega^{-1}$ is not known in practice, but it can be estimated consistently. For any $W_n \to_p W_0$, we still call $\hat{\beta}$ the efficient GMM estimator, as it has the same asymptotic distribution.

8.4 Estimation of the Efficient Weight Matrix

Given any weight matrix $W_n > 0$, the GMM estimator $\hat{\beta}$ is consistent yet inefficient. For example, we can set $W_n = I_\ell$. In the linear model, a better choice is $W_n = \left(X'X\right)^{-1}$. Given any such first-step estimator, we can define the residuals $\hat{e}_i = y_i - z_i'\hat{\beta}$ and moment equations $\hat{g}_i = x_i\hat{e}_i = g(w_i, \hat{\beta})$. Construct their

$$\overline{g}_n = \overline{g}_n(\hat{\beta}) = \frac{1}{n}\sum_{i=1}^{n} \hat{g}_i,$$

subtract

$$\hat{g}_i^* = \hat{g}_i - \overline{g}_n,$$

and define

$$W_n = \left(\frac{1}{n}\sum_{i=1}^n \hat{g}_i^* \hat{g}_i^{*\prime}\right)^{-1} = \left(\frac{1}{n}\sum_{i=1}^n \hat{g}_i \hat{g}_i' - \overline{g}_n \overline{g}_n'\right)^{-1}. \tag{8.4}$$

Then $W_n \to_p \Omega^{-1} = W_0$, and GMM using W_n as the weight matrix is asymptotically efficient.

A common alternative choice is to set

$$W_n = \left(\frac{1}{n}\sum_{i=1}^n \hat{g}_i \hat{g}_i'\right)^{-1}$$

which uses the uncentered moment conditions. Since $Eg_i = 0$, these two estimators are asymptotically equivalent under the hypothesis of correct specification. However, Alastair Hall (2000) has shown that the uncentered estimator is a poor choice. When constructing hypothesis tests, under the alternative hypothesis the moment conditions are violated, i.e. $Eg_i \neq 0$, so the uncentered estimator will contain an undesirable bias term and the power of the test will be adversely affected. A simple solution is to use the centered moment conditions to construct the weight matrix, as in (8.4) above.

Here is a simple way to compute the efficient GMM estimator.

First, set $W_n = (X'X)^{-1}$, estimate $\hat{\beta}$ using this weight matrix, and construct the residual $\hat{e}_i = y_i - z_i'\hat{\beta}$. Then set $\hat{g}_i = x_i\hat{e}_i$, and let $\hat{g}$ be the associated $n \times \ell$ matrix. Then the efficient GMM estimator is

$$\hat{\beta} = \left(Z'X\left(\hat{g}'\hat{g} - n\overline{g}_n\overline{g}_n'\right)^{-1}X'Z\right)^{-1} Z'X\left(\hat{g}'\hat{g} - n\overline{g}_n\overline{g}_n'\right)^{-1}X'Y.$$

In most cases, when we say "GMM", we actually mean "efficient GMM". There is little point in using an inefficient GMM estimator as it is easy to compute.

An estimator of $Var(\hat{\beta})$ can be seen from the above formula. Set

$$\hat{V} = \left(Z'X\left(\hat{g}'\hat{g} - n\overline{g}_n\overline{g}_n'\right)^{-1}X'Z\right)^{-1}.$$

Asymptotic standard errors are given by the square roots of the diagonal elements of $\hat{V}$.

8.5 Over-Identification Test

Overidentified models ($\ell > k$) are special in the sense that there need not be a parameter value β such that the moment condition

$$Eg(w_i, \beta) = 0$$

holds. Thus the model – the overidentifying restrictions – are testable.

For example, take the linear model $y_i = \beta_1' x_{1i} + \beta_2' x_{2i} + e_i$ with $E(x_{1i}e_i) = 0$ and $E(x_{2i}e_i) = 0$. It is possible that $\beta_2 = 0$, so that the linear equation may be written as $y_i = \beta_1' x_{1i} + e_i$. However, it is possible that $\beta_2 \neq 0$, and in this case it would be impossible to find a value of β_1 so that both $E(x_{1i}(y_i - x_{1i}'\beta_1)) = 0$ and $E(x_{2i}(y_i - x_{1i}'\beta_1)) = 0$ hold simultaneously. In this sense an exclusion restriction can be seen as an overidentifying restriction.

Note that $\overline{g}_n \to_p Eg_i$, and thus $\overline{g}_n$ can be used to assess whether or not the hypothesis that $Eg_i = 0$ is true or not. The criterion function at the parameter estimates is

$$\begin{aligned} J &= n\,\overline{g}_n' W_n \overline{g}_n \\ &= n^2 \overline{g}_n' \left(\hat{g}'\hat{g} - n\overline{g}_n\overline{g}_n'\right)^{-1} \overline{g}_n. \end{aligned}$$

is a quadratic form in $\overline{g}_n$, and is thus a natural test statistic for $H_0 : Eg_i = 0$.

Theorem 8.5.1 *(Sargon-Hansen). Under the hypothesis of correct specification, and if the weight matrix is asymptotically efficient,*

$$J = J(\hat{\beta}) \to_d \chi^2_{\ell-k}.$$

The proof of the theorem is left as an exercise. This result was established by Sargon (1958) for a specialized case, and by L. Hansen (1982) for the general case.

The degrees of freedom of the asymptotic distribution are the number of overidentifying restrictions. If the statistic J exceeds the chi-square critical value, we can reject the model. Based on this information alone, it is unclear what is wrong, but it is typically cause for concern. The GMM overidentification test is a very useful by-product of the GMM methodology, and it is advisable to report the statistic J whenever GMM is the estimation method.

When over-identified models are estimated by GMM, it is customary to report the J statistic as a general test of model adequacy.

8.6 GMM: The General Case

In its most general form, GMM applies whenever an economic or statistical model implies the $\ell \times 1$ moment condition

$$E(g_i(\beta)) = 0.$$

Often, this is *all* that is known. Identification requires $l \geq k = \dim(\beta)$. The GMM estimator minimizes

$$J(\beta) = n \cdot \overline{g}_n(\beta)' W_n \overline{g}_n(\beta)$$

where

$$\overline{g}_n(\beta) = \frac{1}{n}\sum_{i=1}^{n} g_i(\beta)$$

and

$$W_n = \left(\frac{1}{n} \sum_{i=1}^{n} \hat{g}_i \hat{g}_i' - \overline{g}_n \overline{g}_n' \right)^{-1},$$

with $\hat{g}_i = g_i(\tilde{\beta})$ constructed using a preliminary consistent estimator $\tilde{\beta}$, perhaps obtained by first setting $W_n = I$. Since the GMM estimator depends upon the first-stage estimator, often the weight matrix W_n is updated, and then $\hat{\beta}$ recomputed. This estimator can be iterated if needed.

Theorem 8.6.1 *Under general regularity conditions,* $\sqrt{n}\left(\hat{\beta} - \beta\right) \to_d N\left(0, \left(G'\Omega^{-1}G\right)^{-1}\right)$, *where* $\Omega = \left(E\left(g_i g_i'\right)\right)^{-1}$ *and* $G = E\frac{\partial}{\partial \beta'} g_i(\beta)$. *The variance of* $\hat{\beta}$ *may be estimated by* $\left(\hat{G}'\hat{\Omega}^{-1}\hat{G}\right)^{-1}$ *where* $\hat{\Omega} = n^{-1}\sum_i \hat{g}_i^* \hat{g}_i^{*\prime}$ *and* $\hat{G} = n^{-1}\sum_i \frac{\partial}{\partial \beta'} g_i(\hat{\beta})$.

The general theory of GMM estimation and testing was exposited by L. Hansen (1982).

8.7 Hypothesis Testing: The Distance Statistic

We described before how to construct estimates of the asymptotic covariance matrix of the GMM estimates. These may be used to construct Wald tests of statistical hypotheses.

If the hypothesis is non-linear, a better approach is to directly use the GMM criterion function. This is sometimes called the GMM Distance statistic, and sometimes called a LR-like statistic (the LR is for likelihood-ratio). The idea was first put forward by Newey and West (1987).

For a given weight matrix W_n, the GMM criterion function is

$$J(\beta) = n \cdot \overline{g}_n(\beta)' W_n \overline{g}_n(\beta)$$

For $h : R^k \to R^r$, the hypothesis is

$$H_0 : h(\beta) = 0.$$

The estimates under H_1 are

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}}\, J(\beta)$$

and those under H_0 are

$$\tilde{\beta} = \underset{h(\beta)=0}{\operatorname{argmin}}\, J(\beta).$$

The two minimizing criterion functions are $J(\hat{\beta})$ and $J(\tilde{\beta})$. The GMM distance statistic is the difference

$$D = J(\tilde{\beta}) - J(\hat{\beta}).$$

Proposition 8.7.1 *If the same weight matrix* W_n *is used for both null and alternative,*

1. $D \geq 0$

2. $D \to_d \chi^2_r$

3. *If h is linear in β, then D equals the Wald statistic.*

If h is non-linear, the Wald statistic can work quite poorly. In contrast, current evidence suggests that the D statistic appears to have quite good sampling properties, and is the preferred test statistic.

Newey and West (1987) suggested to use the same weight matrix W_n for both null and alternative, as this ensures that $D \geq 0$. This reasoning is not compelling, however, and some current research suggests that this restriction is not necessary for good performance of the test.

This test shares the useful feature of LR tests is that it is a natural by-product of the computation of alternative models.

8.8 GMM as Semiparametrically Efficient

We have described the estimator $\hat{\beta}$ as "efficient GMM" if the optimal (variance minimizing) weight matrix is selected. This is a weak concept of optimality, as we are only considering alternative weight matrices W_n. However, it turns out that the GMM estimator is semiparametrically efficient, as shown by Gary Chamberlain (1987).

If it is known that $E(g_i(\beta)) = 0$, and this is all that is known, this is a semi-parametric problem, as the distribution of the data is unknown. Chamberlain showed that in this context, no semiparametric estimator (one which is consistent globally for the class of models considered) can have a smaller asymptotic variance than $\left(G'\Omega^{-1}G\right)^{-1}$. Since the GMM estimator has this asymptotic variance, it is semiparametrically efficient.

This results shows that in the linear model, no estimator has greater asymptotic efficiency than the efficient linear GMM estimator. No estimator can do better (in this first-order asymptotic sense), without imposing additional assumptions.

8.9 Conditional Moment Restrictions

In many contexts, the model implies more than an unconditional moment restriction of the form $Eg_i(\beta) = 0$. It implies a conditional moment restriction of the form

$$E(e_i(\beta) \mid x_i) = 0$$

where $e_i(\beta)$ is some $s \times 1$ function of the observation and the parameters. In many cases, $s = 1$.

It turns out that this conditional moment restriction is much more powerful, and restrictive, than the unconditional moment restriction discussed above.

Our linear model $y_i = z_i'\beta + e_i$ with instruments x_i falls into this class under the stronger assumption $E(e_i \mid x_i) = 0$. Then $e_i(\beta) = y_i - z_i'\beta$.

It is also helpful to realize that conventional regression models also fall into this class, except that in this case $z_i = x_i$. For example, in linear regression, $e_i(\beta) = y_i - x_i'\beta$, while in a nonlinear regression model $e_i(\beta) = y_i - g(x_i, \beta)$. In a joint model of the conditional mean and variance

$$e_i(\beta, \gamma) = \begin{cases} y_i - x_i'\beta \\ \\ (y_i - x_i'\beta)^2 - f(x_i)'\gamma \end{cases}.$$

Here $s = 2$.

Given a conditional moment restriction, an unconditional moment restriction can always be constructed. That is for any $\ell \times 1$ function $\phi(x_i, \beta)$, we can set $g_i(\beta) = \phi(x_i, \beta)e_i(\beta)$ which satisfies $Eg_i(\beta) = 0$ and hence defines a GMM estimator. The obvious problem is that the class of functions ϕ is infinite. Which should be selected?

This is equivalent to the problem of selection of the best instruments. If x_i is a valid instrument satisfying $E(e_i \mid x_i) = 0$, then x_i, x_i^2, x_i^3, ..., etc., are all valid instruments. Which both be used?

One solution is to construct an infinite list of potent instruments, and then use the first k instruments. How is k to be determined? This is an area of theory still under development. A recent study of this problem is Donald and Newey (2001).

Another approach is to construct the *optimal instrument*. The form was uncovered by Chamberlain (1987). Take the case $s = 1$. Let

$$R_i = E\left(\frac{\partial}{\partial \beta} e_i(\beta) \mid x_i\right)$$

and

$$\sigma_i^2 = E\left(e_i(\beta)^2 \mid x_i\right).$$

Then the "optimal instrument" is

$$A_i = \sigma_i^{-2} R_i$$

so the optimal moment is

$$g_i(\beta) = A_i e_i(\beta).$$

Setting $g_i(\beta)$ to be this choice (which is $k \times 1$, so is just-identified) yields the best GMM estimator possible.

In practice, A_i is unknown, but its form does help us think about construction of optimal instruments.

In the linear model $e_i(\beta) = y_i - z_i'\beta$, note that

$$R_i = -E(z_i \mid x_i)$$

and

$$\sigma_i^2 = E\left(e_i^2 \mid x_i\right),$$

so

$$A_i = \sigma_i^{-2} E\left(z_i \mid x_i\right).$$

In the case of linear regression, $z_i = x_i$, so $A_i = \sigma_i^{-2} x_i$. Hence efficient GMM is GLS, as we discussed earlier in the course.

In the case of endogenous variables, note that the efficient instrument A_i involves the estimation of the conditional mean of z_i given x_i. In other words, to get the best instrument for z_i, we need the best conditional mean model for z_i given x_i, not just an arbitrary linear projection. The efficient instrument is also inversely proportional to the conditional variance of e_i. This is the same as the GLS estimator; namely that improved efficiency can be obtained if the observations are weighted inversely to the conditional variance of the errors.

8.10 Continuously-Updated GMM

Let the weight matrix be considered as a function of β. Define the criterion function

$$J(\beta) = n \cdot \overline{g}_n(\beta)' \left(\frac{1}{n} \sum_{i=1}^{n} g_i^*(\beta) g_i^*(\beta)' \right)^{-1} \overline{g}_n(\beta).$$

where

$$g_i^*(\beta) = g_i(\beta) - \overline{g}_n(\beta)$$

The $\hat{\beta}$ which minimizes this function is called the "continuously-updated GMM estimator", and was introduced by L. Hansen, Heaton and Yaron (1996).

The estimator appears to have some better properties than traditional GMM, but can be numerically tricky to obtain in some cases. This is a current area of research in econometrics.

Chapter 9

Empirical Likelihood

9.1 Non-Parametric Likelihood

An alternative to GMM is empirical likelihood. The idea is due to Art Owen (1988, 2001) and has been extended to moment condition models by Qin and Lawless (1994). It is a non-parametric analog of likelihood estimation.

In Section 2.3, we defined the non-parametric likelihood as the multinomial probability distribution which puts probability mass p_i at each observed observation w_i, constrained so that the assumptions of the model hold. The likelihood is maximized over all the parameters, including the probability parameters p_i, yielding the EL estimates and likelihood.

The log empirical likelihood is

$$\mathcal{L}_n(p_1, ..., p_n) = \sum_{i=1}^{n} \ln(p_i).$$

The Lagrangian is

$$\mathcal{L}_n^* (\beta, p_1, ..., p_n, \lambda, \mu) = \sum_{i=1}^{n} \ln(p_i) - \mu \left(\sum_{i=1}^{n} p_i - 1 \right) - n\lambda' \sum_{i=1}^{n} p_i g (w_i, \beta)$$

where λ and μ are Lagrange multipliers. The first-order-conditions of $\mathcal{L}_n^*$ with respect to p_i, μ, and λ are

$$\begin{aligned} \frac{1}{p_i} &= \mu + n\lambda' g (w_i, \beta) \\ \sum_{i=1}^{n} p_i &= 1 \\ \sum_{i=1}^{n} p_i g (w_i, \beta) &= 0. \end{aligned}$$

Multiplying the first equation by p_i, summing over i, and using the second and third equations, we find $\mu = n$ and

$$p_i = \frac{1}{n\left(1 + \lambda' g\left(w_i, \beta\right)\right)}.$$

Substituting into $\mathcal{L}_n^*$ we find

$$R_n\left(\beta, \lambda\right) = -n\ln\left(n\right) - \sum_{i=1}^{n} \ln\left(1 + \lambda' g\left(w_i, \beta\right)\right). \tag{9.1}$$

For given β, the Lagrange multiplier $\lambda(\beta)$ minimizes $R_n\left(\beta, \lambda\right)$:

$$\lambda(\beta) = \underset{\lambda}{\operatorname{argmin}}\, R_n(\beta, \lambda). \tag{9.2}$$

This minimization problem is the dual of the constrained maximization problem. The solution (when it exists) is well defined since $R_n(\beta, \lambda)$ is a convex function of λ. The solution cannot be obtained explicitly, but must be obtained numerically (see section 9.5.1). This yields the (profile) empirical log-likelihood function for β.

$$\begin{aligned} \mathcal{L}_n(\beta) &= R_n(\beta, \lambda(\beta)) \\ &= -n\ln\left(n\right) - \sum_{i=1}^{n} \ln\left(1 + \lambda(\beta)' g\left(w_i, \beta\right)\right) \end{aligned}$$

The EL estimate $\hat{\beta}$ is the value which maximizes $\mathcal{L}_n(\beta)$, or equivalently minimizes its negative

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}}\left[-\mathcal{L}_n(\beta)\right] \tag{9.3}$$

Numerical methods are required for calculation of $\hat{\beta}$. (see section 9.5.2)

As a by-product of estimation, we also obtain the Lagrange multiplier $\hat{\lambda} = \lambda(\hat{\beta})$, probabilities

$$\hat{p}_i = \frac{1}{n\left(1 + \hat{\lambda}' g\left(w_i, \hat{\beta}\right)\right)}.$$

and maximized empirical likelihood

$$\hat{\mathcal{L}}_n = \sum_{i=1}^{n} \ln\left(\hat{p}_i\right). \tag{9.4}$$

9.2 Asymptotic Distribution of EL Estimator

Define

$$G\left(w_i, \beta\right) = \frac{\partial}{\partial \beta'} g\left(w_i, \beta\right) \tag{9.5}$$

$$G = EG\left(w_i, \beta_0\right)$$
$$\Omega = E\left(g\left(w_i, \beta_0\right) g\left(w_i, \beta_0\right)'\right)$$

and

$$V = \left(G'\Omega^{-1}G\right)^{-1} \tag{9.6}$$

$$V_\lambda = \Omega - G\left(G'\Omega^{-1}G\right)^{-1} G' \tag{9.7}$$

For example, in the linear model, $G\left(w_i, \beta\right) = -x_i z_i'$, $G = -E\left(x_i z_i'\right)$, and $\Omega = E\left(x_i x_i' e_i^2\right)$.

Theorem 9.2.1 *Under regularity conditions,*

$$\begin{aligned} \sqrt{n}\left(\hat{\beta} - \beta_0\right) &\to^d \quad N\left(0, V\right) \\ \sqrt{n}\hat{\lambda} &\to^d \quad \Omega^{-1} N\left(0, V_\lambda\right) \end{aligned}$$

where V and V_λ are defined in (9.6) and (9.7), and $\sqrt{n}\left(\hat{\beta} - \beta_0\right)$ and $\sqrt{n}\hat{\lambda}$ are asymptotically independent.

The asymptotic variance V for $\hat{\beta}$ is the same as for efficient GMM. Thus the EL estimator is asymptotically efficient.

Proof. $(\hat{\beta}, \hat{\lambda})$ jointly solve

$$0 = \frac{\partial}{\partial \lambda} R_n(\beta, \lambda) = -\sum_{i=1}^n \frac{g\left(w_i, \hat{\beta}\right)}{\left(1 + \hat{\lambda}' g\left(w_i, \hat{\beta}\right)\right)} \tag{9.8}$$

$$0 = \frac{\partial}{\partial \beta} R_n(\beta, \lambda) = -\sum_{i=1}^n \frac{G\left(w_i, \hat{\beta}\right)' \lambda}{1 + \hat{\lambda}' g\left(w_i, \hat{\beta}\right)}. \tag{9.9}$$

Let $G_n = \frac{1}{n}\sum_{i=1}^n G\left(w_i, \beta_0\right)$, $\overline{g}_n = \frac{1}{n}\sum_{i=1}^n g\left(w_i, \beta_0\right)$ and $\Omega_n = \frac{1}{n}\sum_{i=1}^n g\left(w_i, \beta_0\right) g\left(w_i, \beta_0\right)'$.

Expanding (9.9) around $\beta = \beta_0$ and $\lambda = \lambda_0 = 0$ yields

$$0 \simeq G_n'\left(\hat{\lambda} - \lambda_0\right). \tag{9.10}$$

Expanding (9.8) around $\beta = \beta_0$ and $\lambda = \lambda_0 = 0$ yields

$$0 \simeq -\overline{g}_n - G_n\left(\hat{\beta} - \beta_0\right) + \Omega_n\left(\hat{\lambda} - \lambda_0\right) \tag{9.11}$$

Premultiplying by $G_n'\Omega_n^{-1}$ and using (9.10) yields

$$\begin{aligned} 0 &\simeq -G_n'\Omega_n^{-1}\overline{g}_n - G_n'\Omega_n^{-1}G_n\left(\hat{\beta} - \beta_0\right) + G_n'\Omega_n^{-1}\Omega_n\hat{\lambda} \\ &= -G_n'\Omega_n^{-1}\overline{g}_n - G_n'\Omega_n^{-1}G_n\left(\hat{\beta} - \beta_0\right) \end{aligned}$$

Solving for $\hat{\beta}$ and using the WLLN and CLT yields

$$\begin{aligned} \sqrt{n}\left(\hat{\beta}-\beta_0\right) &\simeq -\left(G_n'\Omega_n^{-1}G_n\right)^{-1}G_n'\Omega_n^{-1}\sqrt{n}\overline{g}_n \\ &\to^d \left(G'\Omega^{-1}G\right)^{-1}G'\Omega^{-1}N\left(0,\Omega\right) \\ &= N\left(0,V\right) \end{aligned} \tag{9.12}$$

Solving (9.11) for $\hat{\lambda}$ and using (9.12) yields

$$\begin{aligned} \sqrt{n}\hat{\lambda} &\simeq \Omega_n^{-1}\left(I-G_n\left(G_n'\Omega_n^{-1}G_n\right)^{-1}G_n'\Omega_n^{-1}\right)\sqrt{n}\overline{g}_n \\ \to^d & \quad \Omega^{-1}\left(I-G\left(G'\Omega^{-1}G\right)^{-1}G'\Omega^{-1}\right)N\left(0,\Omega\right) \\ &= \Omega^{-1}N\left(0,V_\lambda\right) \end{aligned} \tag{9.13}$$

Furthermore, since

$$G'\left(I-\Omega^{-1}G\left(G'\Omega^{-1}G\right)^{-1}G'\right)=0$$

$\sqrt{n}\left(\hat{\beta}-\beta_0\right)$ and $\sqrt{n}\hat{\lambda}$ are asymptotically uncorrelated and hence independent. ■

Chamberlain (1987) showed that V is the semiparametric efficiency bound for β in the overidentified moment condition model. This means that no consistent estimator for this class of models can have a lower asymptotic variance than V. Since the EL estimator achieves this bound, it is an asymptotically efficient estimator for β.

9.3 Overidentifying Restrictions

In a parametric likelihood context, tests are based on the difference in the log likelihood functions. The same statistic can be constructed for empirical likelihood. Twice the difference between the unrestricted empirical likelihood $-n\log\left(n\right)$ and the maximized empirical likelihood for the model (9.4) is

$$LR_n=\sum_{i=1}^{n}2\ln\left(1+\hat{\lambda}'g\left(w_i,\hat{\beta}\right)\right). \tag{9.14}$$

Theorem 9.3.1 *If* $Eg(w_i,\beta_0)=0$ *then* $LR_n\to_d\chi^2_{\ell-k}$.

The EL overidentification test is similar to the GMM overidentification test. They are asymptotically first-order equivalent, and have the same interpretation. The overidentification test is a very useful by-product of EL estimation, and it is advisable to report the statistic LR_n whenever EL is the estimation method.

Proof. First, by a Taylor expansion, (9.12), and (9.13),

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} g\left(w_i,\hat{\beta}\right) &\simeq \sqrt{n}\left(\overline{g}_n + G_n\left(\hat{\beta}-\beta_0\right)\right)\\ &\simeq \left(I - G_n\left(G_n'\Omega_n^{-1}G_n\right)^{-1}G_n'\Omega_n^{-1}\right)\sqrt{n}\overline{g}_n\\ &\simeq \Omega_n\sqrt{n}\hat{\lambda}.\end{aligned}$$

Second, since $\ln(1+x) \simeq x - x^2/2$ for x small,

$$\begin{aligned}LR_n &= \sum_{i=1}^{n} 2\ln\left(1+\hat{\lambda}'g\left(w_i,\hat{\beta}\right)\right)\\ &\simeq 2\hat{\lambda}'\sum_{i=1}^{n} g\left(w_i,\hat{\beta}\right) - \hat{\lambda}'\sum_{i=1}^{n} g\left(w_i,\hat{\beta}\right)g\left(w_i,\hat{\beta}\right)'\hat{\lambda}\\ &\simeq n\hat{\lambda}'\Omega_n\hat{\lambda}\\ \to_d &\quad N\left(0,V_\lambda\right)'\Omega^{-1}N\left(0,V_\lambda\right)\\ &= \chi^2_{\ell-k}\end{aligned}$$

where the proof of the final equality is left as an exercise. ■

9.4 Testing

Let the maintained model be

$$Eg(w_i,\beta) = 0 \tag{9.15}$$

where g is $\ell \times 1$ and β is $k \times 1$. By "maintained" we mean that the overidentfying restrictions contained in (9.15) are assumed to hold and are not being challenged (at least for the test discussed in this section). The hypothesis of interest is

$$h(\beta) = 0.$$

where $h : R^k \to R^a$. The restricted EL estimator and likelihood are the values which solve

$$\begin{aligned}\tilde{\beta} &= \underset{h(\beta)=0}{\operatorname{argmax}}\,\mathcal{L}_n(\beta)\\ \tilde{\mathcal{L}}_n &= \mathcal{L}_n(\tilde{\beta}) = \max_{h(\beta)=0}\mathcal{L}_n(\beta).\end{aligned}$$

Fundamentally, the restricted EL estimator $\tilde{\beta}$ is simply an EL estimator with $\ell - k + a$ overidentifying restrictions, so there is no fundamental change in the distribution theory for $\tilde{\beta}$ relative to $\hat{\beta}$.

To test the hypothesis $h(\beta)$ while maintaining (9.15), the simple overidentifying restrictions test (9.14) is not appropriate. Instead we use the difference in log-likelihoods:

$$LR_n = 2\left(\hat{\mathcal{L}}_n - \tilde{\mathcal{L}}_n\right).$$

This test statistic is a natural analog of the GMM distance statistic.

Theorem 9.4.1 *Under (9.15) and* $H_0 : h(\beta) = 0$, $LR_n \to_d \chi^2_a$.

The proof of this result is more challenging and is omitted.

9.5 Numerical Computation

Gauss code which implements the methods discussed below can be found at

http://www.ssc.wisc.edu/~bhansen/progs/elike.prc

9.5.1 Derivatives

The numerical calculations depend on derivatives of the dual likelihood function (9.1). Define

$$\begin{aligned} g_i^*(\beta,\lambda) &= \frac{g(w_i,\beta)}{\left(1+\lambda' g(w_i,\beta)\right)} \\ G_i^*(\beta,\lambda) &= \frac{G(w_i,\beta)'\lambda}{1+\lambda' g(w_i,\beta)} \end{aligned}$$

The first derivatives of (9.1) are

$$\begin{aligned} R_\lambda &= \frac{\partial}{\partial\lambda} R_n(\beta,\lambda) = -\sum_{i=1}^n g_i^*(\beta,\lambda) \\ R_\beta &= \frac{\partial}{\partial\beta} R_n(\beta,\lambda) = -\sum_{i=1}^n G_i^*(\beta,\lambda). \end{aligned}$$

The second derivatives are

$$\begin{aligned} R_{\lambda\lambda} &= \frac{\partial^2}{\partial\lambda\partial\lambda'} R_n(\beta,\lambda) = \sum_{i=1}^n g_i^*(\beta,\lambda)\, g_i^*(\beta,\lambda)' \\ R_{\lambda\beta} &= \frac{\partial^2}{\partial\lambda\partial\beta'} R_n(\beta,\lambda) = \sum_{i=1}^n \left(G_i^*(\beta,\lambda)\, g_i^*(\beta,\lambda)' - G_i^*(\beta,\lambda)\right) \\ R_{\beta\beta} &= \frac{\partial^2}{\partial\beta\partial\beta'} R_n(\beta,\lambda) = \sum_{i=1}^n \left(G_i^*(\beta,\lambda)\, G_i^*(\beta,\lambda)' - \frac{\frac{\partial^2}{\partial\beta\partial\beta'}\left(g(w_i,\beta)'\lambda\right)}{1+\lambda' g(w_i,\beta)} \right) \end{aligned}$$

9.5.2 Inner Loop

The so-called "inner loop" solves (9.2) for given β. The modified Newton method takes a quadratic approximation to $R_n(\beta, \lambda)$ yielding the iteration rule

$$\lambda_{j+1} = \lambda_j - \delta \left(R_{\lambda\lambda}(\beta, \lambda_j)\right)^{-1} R_\lambda(\beta, \lambda_j). \tag{9.16}$$

where $\delta > 0$ is a scalar steplength (to be discussed next). The starting value λ_1 can be set to the zero vector. The iteration (9.16) is continued until the gradient $R_\lambda(\beta, \lambda_j)$ is smaller than some prespecified tolerance.

Efficient convergence requires a good choice of steplength δ. A good choice is based on a a quadratic approximation, which works as follows. Set $\delta_0 = 0$, $\delta_1 = \frac{1}{2}$ and $\delta_2 = 1$. For $p = 0, 1, 2$, set

$$\begin{aligned} \lambda_p &= \lambda_j - \delta_p \left(R_{\lambda\lambda}(\beta, \lambda_j)\right)^{-1} R_\lambda(\beta, \lambda_j)) \\ R_p &= R_n(\beta, \lambda_p) \end{aligned}$$

Fitting a quadratic function to these three points, the minimizing value of δ is

$$\hat{\delta} = \frac{R_2 + 3R_0 - 4R_1}{4R_2 + 4R_0 - 8R_1}.$$

A complication is that λ must be constrained so that $0 \le p_i \le 1$ which holds if

$$n\left(1 + \lambda' g(w_i, \beta)\right) \ge 1 \tag{9.17}$$

for all i. If (9.17) fails, the stepsize δ needs to be decreased.

9.5.3 Outer Loop

The outer loop is the minimization (9.3). This can be done by the modified Newton method described in the previous section. The gradient for (9.3) is

$$\mathcal{L}_\beta = \frac{\partial}{\partial \beta}\mathcal{L}_n(\beta) = \frac{\partial}{\partial \beta} R_n(\beta, \lambda) = R_\beta + \lambda_\beta' R_\lambda = R_\beta$$

since $R_\lambda(\beta, \lambda) = 0$ at $\lambda = \lambda(\beta)$, where

$$\lambda_\beta = \frac{\partial}{\partial \beta'}\lambda(\beta) = -R_{\lambda\lambda}^{-1} R_{\lambda\beta},$$

the second equality following from the implicit function theorem applied to $R_\lambda(\beta, \lambda(\beta)) = 0$.

The Hessian for (9.3) is

$$\begin{aligned}
\mathcal{L}_{\beta\beta} &= -\frac{\partial}{\partial\beta\partial\beta'}\mathcal{L}_n(\beta) \\
&= -\frac{\partial}{\partial\beta'}\left[R_\beta\left(\beta,\lambda(\beta)\right) + \lambda'_\beta R_\lambda\left(\beta,\lambda(\beta)\right)\right] \\
&= -\left(R_{\beta\beta}\left(\beta,\lambda(\beta)\right) + R'_{\lambda\beta}\lambda_\beta + \lambda'_\beta R_{\lambda\beta} + \lambda'_\beta R_{\lambda\lambda}\lambda_\beta\right) \\
&= R'_{\lambda\beta}R_{\lambda\lambda}^{-1}R_{\lambda\beta} - R_{\beta\beta}.
\end{aligned}$$

It is not guaranteed that $\mathcal{L}_{\beta\beta} > 0$. If not, the eigenvalues of $\mathcal{L}_{\beta\beta}$ should be adjusted so that all are positive. The Newton iteration rule is

$$\beta_{j+1} = \beta_j - \delta\mathcal{L}_{\beta\beta}^{-1}\mathcal{L}_\beta$$

where δ is a scalar stepsize, and the rule is iterated until convergence.

Chapter 10

Endogeneity

We say that there is endogeneity in the linear model $y = z_i'\beta + e_i$ if β is the parameter of interest and $E(z_i e_i) \neq 0$. This cannot happen if β is defined by linear projection, so requires a structural interpretation. The coefficient β must have meaning separately from the definition of a conditional mean or linear projection.

Example: Measurement error in the regressor. Suppose that (y_i, x_i^*) are joint random variables, $E(y_i \mid x_i^*) = x_i^{*\prime}\beta$ is linear, β is the parameter of interest, and x_i^* is not observed. Instead we observe $x_i = x_i^* + u_i$ where u_i is an $k \times 1$ measurement error, independent of y_i and x_i^*. Then

$$\begin{aligned} y_i &= x_i^{*\prime}\beta + e_i \\ &= (x_i - u_i)'\beta + e_i \\ &= x_i'\beta + v_i \end{aligned}$$

where

$$v_i = e_i - u_i'\beta.$$

The problem is that

$$E(x_i v_i) = E\left[(x_i^* + u_i)\left(e_i - u_i'\beta\right)\right] = -E\left(u_i u_i'\right)\beta \neq 0$$

if $\beta \neq 0$ and $E(u_i u_i') \neq 0$. It follows that if $\hat{\beta}$ is the OLS estimator, then

$$\hat{\beta} \to_p \beta^* = \beta - \left(E\left(x_i x_i'\right)\right)^{-1} E\left(u_i u_i'\right)\beta \neq \beta.$$

This is called **measurement error bias**.

Example: Supply and Demand. The variables q_i and p_i (quantity and price) are determined jointly by the demand equation

$$q_i = -\beta_1 p_i + e_{1i}$$

and the supply equation

$$q_i = \beta_2 p_i + e_{2i}.$$

Assume that $e_i = \begin{pmatrix} e_{1i} \\ e_{2i} \end{pmatrix}$ is iid, $Ee_i = 0$, $\beta_1 + \beta_2 = 1$ and $Ee_i e_i' = I_2$ (the latter for simplicity). The question is, if we regress q_i on p_i, what happens?

It is helpful to solve for q_i and p_i in terms of the errors. In matrix notation,

$$\begin{bmatrix} 1 & \beta_1 \\ 1 & -\beta_2 \end{bmatrix} \begin{pmatrix} q_i \\ p_i \end{pmatrix} = \begin{pmatrix} e_{1i} \\ e_{2i} \end{pmatrix}$$

so

$$\begin{aligned} \begin{pmatrix} q_i \\ p_i \end{pmatrix} &= \begin{bmatrix} 1 & \beta_1 \\ 1 & -\beta_2 \end{bmatrix}^{-1} \begin{pmatrix} e_{1i} \\ e_{2i} \end{pmatrix} \\ &= \begin{bmatrix} \beta_2 & \beta_1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} e_{1i} \\ e_{2i} \end{pmatrix} \\ &= \begin{pmatrix} \beta_2 e_{1i} + \beta_1 e_{2i} \\ (e_{1i} - e_{2i}) \end{pmatrix}. \end{aligned}$$

The projection of q_i on p_i yields

$$\begin{aligned} q_i &= \beta^* p_i + \varepsilon_i \\ E(p_i \varepsilon_i) &= 0 \end{aligned}$$

where

$$\beta^* = \frac{E(p_i q_i)}{E(p_i^2)} = \frac{\beta_2 - \beta_1}{2}$$

Hence if it is estimated by OLS, $\hat{\beta} \to_p \beta^*$, which does not equal either β_1 or β_2. This is called **simultaneous equations bias**.

10.1 Instrumental Variables

Let the equation of interest be

$$y_i = z_i'\beta + e_i \tag{10.1}$$

where z_i is $k \times 1$, and assume that $E(z_i e_i) \neq 0$ so there is **endogeneity**. We call (10.1) the structural equation. In matrix notation, this can be written as

$$Y = Z\beta + e. \tag{10.2}$$

Any solution to the problem of endogeneity requires additional information which we call **instruments**.

Definition 10.1.1 *The $\ell \times 1$ random vector x_i is an instrumental variable for (10.1) if $E(x_i e_i) = 0$.*

In a typical set-up, some regressors in z_i will be uncorrelated with e_i (for example, at least the intercept). Thus we make the partition

$$z_i = \begin{pmatrix} z_{1i} \\ z_{2i} \end{pmatrix} \begin{matrix} k_1 \\ k_2 \end{matrix} \tag{10.3}$$

where $E(z_{1i}e_i) = 0$ yet $E(z_{2i}e_i) \neq 0$. We call z_{1i} exogenous and z_{2i} endogenous. By the above definition, z_{1i} is an instrumental variable for (10.1), so should be included in x_i. So we have the partition

$$x_i = \begin{pmatrix} z_{1i} \\ x_{2i} \end{pmatrix} \begin{matrix} k_1 \\ \ell_2 \end{matrix} \tag{10.4}$$

where $z_{1i} = x_{1i}$ are the **included exogenous variables**, and x_{2i} are the **excluded exogenous variables**. That is x_{2i} are variables which could be included in the equation for y_i (in the sense that they are uncorrelated with e_i) yet can be *excluded*, as they would have true zero coefficients in the equation.

The model is **just-identified** if $\ell = k$ (i.e., if $\ell_2 = k_2$) and **over-identified** if $\ell > k$ (i.e., if $\ell_2 > k_2$).

We have noted that any solution to the problem of endogeneity requires instruments. This does not mean that valid instruments actually exist.

10.2 Reduced Form

The reduced form relationship between the variables or "regressors" z_i and the instruments x_i is found by linear projection. Let

$$\Gamma = E\left(x_i x_i'\right)^{-1} E\left(x_i z_i'\right)$$

be the $\ell \times k$ matrix of coefficients from a projection of z_i on x_i, and define

$$u_i = z_i - x_i'\Gamma$$

as the projection error. Then the reduced form linear relationship between z_i and x_i is

$$z_i = \Gamma' x_i + u_i. \tag{10.5}$$

In matrix notation, we can write (10.5) as

$$Z = X\Gamma + u \tag{10.6}$$

where u is $n \times k$.

By construction,

$$E(x_i u_i') = 0,$$

so (10.5) is a projection and can be estimated by OLS:

$$\begin{aligned} Z &= X\hat{\Gamma} + \hat{u} \\ \hat{\Gamma} &= \left(X'X\right)^{-1}\left(X'Z\right). \end{aligned}$$

Substituting (10.6) into (10.2), we find

$$\begin{aligned} Y &= (X\Gamma + u)\beta + e \\ &= X\lambda + v, \end{aligned} \tag{10.7}$$

where

$$\lambda = \Gamma\beta \tag{10.8}$$

and

$$v = u\beta + e.$$

Observe that

$$E\left(x_i v_i\right) = E\left(x_i u_i'\right)\beta + E\left(x_i e_i\right) = 0.$$

Thus (10.7) is a projection equation and may be estimated by OLS. This is

$$\begin{aligned} Y &= X\hat{\lambda} + \hat{v}, \\ \hat{\lambda} &= \left(X'X\right)^{-1}\left(X'Y\right) \end{aligned}$$

The equation (10.7) is the reduced form for Y. (10.6) and (10.7) together are the **reduced form equations** for the system

$$\begin{aligned} Y &= X\lambda + v \\ Z &= X\Gamma + u. \end{aligned}$$

As we showed above, OLS yields the reduced-form estimates $(\hat{\lambda}, \hat{\Gamma})$

10.3 Identification

The structural parameter β relates to (λ, Γ) through (10.8). The parameter β is **identified**, meaning that it can be recovered from the reduced form, if

$$rank(\Gamma) = k. \tag{10.9}$$

Assume that (10.9) holds. If $\ell = k$, then $\beta = \Gamma^{-1}\lambda$. If $\ell > k$, then for any $W > 0$, $\beta = \left(\Gamma' W \Gamma\right)^{-1}\Gamma' W \lambda$.

If (10.9) is not satisfied, then β cannot be recovered from (λ, Γ). Note that a necessary (although not sufficient) condition for (10.9) is $\ell \geq k$.

Since X and Z have the common variables X_1, we can rewrite some of the expressions. Using (10.3) and (10.4) to make the matrix partitions $X = [X_1, X_2]$ and $Z = [X_1, Z_2]$, we can partition Γ as

$$\begin{aligned}\Gamma &= \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \\ &= \begin{bmatrix} I & \Gamma_{12} \\ 0 & \Gamma_{22} \end{bmatrix}\end{aligned}$$

(10.6) can be rewritten as

$$\begin{aligned}Z_1 &= X_1 \\ Z_2 &= X_1\Gamma_{12} + X_2\Gamma_{22} + u_2. \end{aligned} \tag{10.10}$$

β is identified if $rank(\Gamma) = k$, which is true if and only if $rank(\Gamma_{22}) = k_2$ (by the upper-diagonal structure of Γ). Thus the key to identification of the model rests on the $\ell_2 \times k_2$ matrix Γ_{22} in (10.10).

10.4 Estimation

The model can be written as

$$\begin{aligned} y_i &= z_i'\beta + e_i \\ E\left(x_i e_i\right) &= 0 \end{aligned}$$

or

$$\begin{aligned} Eg\left(w_i, \beta\right) &= 0 \\ g\left(w_i, \beta\right) &= x_i\left(y_i - z_i'\beta\right). \end{aligned}$$

This a moment condition model. Appropriate estimators include GMM and EL. The estimators and distribution theory developed in those Chapter 8 and 9 directly apply. Recall that the GMM estimator, for given weight matrix W_n, is

$$\hat{\beta} = \left(Z'XW_nX'Z\right)^{-1} Z'XW_nX'Y.$$

10.5 Special Cases: IV and 2SLS

If the model is just-identified, so that $k = \ell$, then the formula for GMM simplifies. We find that

$$\begin{aligned}\hat{\beta} &= \left(Z'XW_nX'Z\right)^{-1} Z'XW_nX'Y \\ &= \left(X'Z\right)^{-1} W_n^{-1}\left(Z'X\right)^{-1} Z'XW_nX'Y \\ &= \left(X'Z\right)^{-1} X'Y\end{aligned}$$

This estimator is often called the **instrumental variables estimator** (IV) of β, where X is used as an instrument for Z. Observe that the weight matrix W_n has disappeared. In the just-identified case, the weight matrix places no role. This is also the MME estimator of β, and the EL estimator. Another interpretation stems from the fact that since $\beta = \Gamma^{-1}\lambda$, we can construct the **Indirect Least Squares** (ILS) estimator:

$$\begin{aligned}
\hat{\beta} &= \hat{\Gamma}^{-1}\hat{\lambda} \\
&= \left(\left(X'X\right)^{-1}\left(X'Z\right)\right)^{-1}\left(\left(X'X\right)^{-1}\left(X'Y\right)\right) \\
&= \left(X'Z\right)^{-1}\left(X'X\right)\left(X'X\right)^{-1}\left(X'Y\right) \\
&= \left(X'Z\right)^{-1}\left(X'Y\right).
\end{aligned}$$

which again is the IV estimator.

Recall that the optimal weight matrix is an estimate of the inverse of $\Omega = E\left(x_i x_i' e_i^2\right)$. In the special case that $E\left(e_i^2 \mid x_i\right) = \sigma^2$ (homoskedasticity), then $\Omega = E\left(x_i x_i'\right)\sigma^2 \propto E\left(x_i x_i'\right)$ suggesting the weight matrix $W_n = \left(X'X\right)^{-1}$. Using this choice, the GMM estimator equals

$$\hat{\beta}_{2SLS} = \left(Z'X\left(X'X\right)^{-1}X'Z\right)^{-1}Z'X\left(X'X\right)^{-1}X'Y$$

This is called the **two-stage-least squares** (2SLS) estimator. It was originally proposed by Theil (1953) and Basmann (1957), and is the classic estimator for linear equations with instruments. Under the homoskedasticity assumption, the 2SLS estimator is efficient GMM, but otherwise it is inefficient.

It is useful to observe that writing

$$\begin{aligned}
P_X &= X\left(X'X\right)^{-1}X', \\
\hat{Z} &= P_X Z = X\left(X'X\right)^{-1}X'Z,
\end{aligned}$$

then

$$\begin{aligned}
\hat{\beta} &= \left(Z'P_X Z\right)^{-1}Z'P_X Y \\
&= \left(\hat{Z}'\hat{Z}\right)\hat{Z}'Y.
\end{aligned}$$

The source of the "two-stage" name is since it can be computed as follows

- First regress Z on X, vis., $\hat{\Gamma} = \left(X'X\right)^{-1}\left(X'Z\right)$ and $\hat{Z} = X\hat{\Gamma} = P_X Z$.
- Second, regress Y on $\hat{Z}$, vis., $\hat{\beta} = \left(\hat{Z}'\hat{Z}\right)^{-1}\hat{Z}'Y$.

It is useful to scrutinize the projection $\hat{Z}$. Recall, $Z = [Z_1, Z_2]$ and $X = [Z_1, X_2]$. Then

$$\begin{aligned}
\hat{Z} &= \left[\hat{Z}_1, \hat{Z}_2\right] \\
&= [P_X Z_1, P_X Z_2] \\
&= [Z_1, P_X Z_2] \\
&= \left[Z_1, \hat{Z}_2\right],
\end{aligned}$$

since Z_1 lies in the span of X. Thus in the second stage, we regress Y on Z_1 and $\hat{Z}_2$. So only the endogenous variables Z_2 are replaced by their fitted values:

$$\hat{Z}_2 = X_1\hat{\Gamma}_{12} + X_2\hat{\Gamma}_{22}.$$

10.6 Bekker Asymptotics

Bekker (1994) used an alternative asymptotic framework to analyze the finite-sample bias in the 2SLS estimator. Here we present a simplified version of one of his results. In our notation, the model is

$$\begin{aligned}
Y &= Z\beta + e && (10.11)\\
Z &= X\Gamma + u && (10.12)\\
\xi &= (e, u) \\
E(\xi \mid X) &= 0 \\
E\left(\xi\xi' \mid X\right) &= S
\end{aligned}$$

As before, X is $n \times l$ so there are l instruments.

First, let's analyze the approximate bias of OLS applied to (10.11). Using (10.12),

$$E\left(\frac{1}{n}Z'e\right) = E(z_i e_i) = \Gamma' E(x_i e_i) + E(u_i e_i) = S_{21}$$

and

$$\begin{aligned}
E\left(\frac{1}{n}Z'Z\right) &= E\left(z_i z_i'\right) \\
&= \Gamma' E\left(x_i x_i'\right)\Gamma + E\left(u_i x_i'\right)\Gamma + \Gamma' E\left(x_i u_i'\right) + E\left(u_i u_i'\right) \\
&= \Gamma' Q \Gamma + S_{22}
\end{aligned}$$

where $Q = E\left(x_i x_i'\right)$. Hence by a first-order approximation

$$\begin{aligned}
E\left(\hat{\beta}_{OLS} - \beta\right) &\approx \left(E\left(\frac{1}{n}Z'Z\right)\right)^{-1} E\left(\frac{1}{n}Z'e\right) \\
&= \left(\Gamma' Q \Gamma + S_{22}\right)^{-1} S_{21} && (10.13)
\end{aligned}$$

which is zero only when $S_{21} = 0$ (when Z is exogenous).

We now derive a similar result for the 2SLS estimator.

$$\hat{\beta}_{2SLS} = \left(Z'P_X Z\right)^{-1}\left(Z'P_X Y\right).$$

Let $P_X = X\left(X'X\right)^{-1}X'$. By the spectral decomposition of an idempotent matrix, $P = H\Lambda H'$ where $\Lambda = diag(I_l, 0)$. Let $q = H'\xi S^{-1/2}$ which satisfies $Eqq' = I_n$ and partition $q = (q_1'\ q_2')$ where q_1 is $l \times 1$. Hence

$$\begin{aligned} E\left(\frac{1}{n}\xi' P_X \xi\right) &= \frac{1}{n}S^{1/2\prime}E\left(q'\Lambda q\right)S^{1/2} \\ &= \frac{1}{n}S^{1/2\prime}E\left(\frac{1}{n}q_1'q_1\right)S^{1/2} \\ &= \frac{l}{n}S^{1/2\prime}S^{1/2} \\ &= \alpha S \end{aligned}$$

where

$$\alpha = \frac{l}{n}.$$

Using (10.12) and this result,

$$\frac{1}{n}E\left(Z'P_X e\right) = \frac{1}{n}E\left(\Gamma'X'e\right) + \frac{1}{n}E\left(u'P_X e\right) = \alpha S_{21},$$

and

$$\begin{aligned} \frac{1}{n}E\left(Z'P_X Z\right) &= \Gamma'E\left(x_i x_i'\right)\Gamma + \Gamma'E\left(x_i u_i\right) + E\left(u_i x_i'\right)\Gamma + \frac{1}{n}E\left(u'P_X u\right) \\ &= \Gamma'Q\Gamma + \alpha S_{22}. \end{aligned}$$

Together

$$\begin{aligned} E\left(\hat{\beta}_{2SLS} - \beta\right) &\approx \left(E\left(\frac{1}{n}Z'P_X Z\right)\right)^{-1} E\left(\frac{1}{n}Z'P_X e\right) \\ &= \alpha\left(\Gamma'Q\Gamma + \alpha S_{22}\right)^{-1} S_{21}. \end{aligned} \tag{10.14}$$

In general this is non-zero, except when $S_{21} = 0$ (when Z is exogenous). It is also close to zero when $\alpha = 0$. Bekker (1994) pointed out that it also has the reverse implication – that when $\alpha = l/n$ is large, the bias in the 2SLS estimator will be large. Indeed as $\alpha \to 1$, the expression in (10.14) approaches that in (10.13), indicating that the bias in 2SLS approaches that of OLS as the number of instruments increases.

Bekker (1994) showed further that under the alternative asymptotic approximation that α is fixed as $n \to \infty$ (so that the number of instruments goes to infinity proportionately with sample size) then the expression in (10.14) is the probability limit of $\hat{\beta}_{2SLS} - \beta$

10.7 Identification Failure

Recall the reduced form equation

$$Z_2 = X_1\Gamma_{12} + X_2\Gamma_{22} + u_2.$$

The parameter β fails to be identified if Γ_{22} has deficient rank. The consequences of identification failure for inference are quite severe.

Take the simplest case where $k = l = 1$ (so there is no X_1). Then the model may be written as

$$\begin{aligned} y_i &= z_i\beta + e_i \\ z_i &= x_i\gamma + u_i \end{aligned}$$

and $\Gamma_{22} = \gamma = E(x_i z_i)/Ex_i^2$. We see that β is identified if and only if $\Gamma_{22} = \gamma \neq 0$, which occurs when $E(z_i x_i) \neq 0$. Thus identification hinges on the existence of correlation between the excluded exogenous variable and the included endogenous variable.

Suppose this condition fails, so $E(z_i x_i) = 0$. Then by the CLT

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i e_i \to_d N_1 \sim N\left(0, E\left(x_i^2 e_i^2\right)\right) \tag{10.15}$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i z_i = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i u_i \to_d N_2 \sim N\left(0, E\left(x_i^2 u_i^2\right)\right) \tag{10.16}$$

therefore

$$\hat{\beta} - \beta = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i e_i}{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i z_i} \to_d \frac{N_1}{N_2} \sim Cauchy,$$

since the ratio of two normals is Cauchy. This is particularly nasty, as the Cauchy distribution does not have a finite mean. This result carries over to more general settings, and was examined by Phillips (1989) and Choi and Phillips (1992).

Suppose that identification does not complete fail, but is *weak*. This occurs when Γ_{22} is full rank, but *small*. This can be handled in an asymptotic analysis by modeling it as local-to-zero, viz

$$\Gamma_{22} = n^{-1/2}C,$$

where C is a full rank matrix. The $n^{-1/2}$ is picked because it provides just the right balancing to allow a rich distribution theory.

To see the consequences, once again take the simple case $k = l = 1$. Here, the instrument x_i is weak for z_i if

$$\gamma = n^{-1/2}c.$$

Then (10.15) is unaffected, but (10.16) instead takes the form

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i z_i &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i^2\gamma + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i u_i \\ &= \frac{1}{n}\sum_{i=1}^{n} x_i^2 c + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i u_i \\ &\to_d Qc + N_2\end{aligned}$$

therefore

$$\hat{\beta} - \beta \to_d \frac{N_1}{Qc + N_2}.$$

As in the case of complete identification failure, we find that $\hat{\beta}$ is inconsistent for β and the asymptotic distribution of $\hat{\beta}$ is non-normal. In addition, standard test statistics have non-standard distributions, meaning that inferences about parameters of interest can be misleading.

The distribution theory for this model was developed by Staiger and Stock (1997) and extended to nonlinear GMM estimation by Stock and Wright (2000). Further results on testing were obtained by Wang and Zivot (1998).

The bottom line is that it is highly desirable to avoid identification failure. Once again, the equation to focus on is the reduced form

$$Z_2 = X_1\Gamma_{12} + X_2\Gamma_{22} + u_2$$

and identification requires $rank(\Gamma_{22}) = k_2$. If $k_2 = 1$, this requires $\Gamma_{22} \neq 0$, which is straightforward to assess using a hypothesis test on the reduced form. Therefore in the case of $k_2 = 1$ (one RHS endogenous variable), one constructive recommendation is to explicitly estimate the reduced form equation for Z_2, construct the test of $\Gamma_{22} = 0$, and at a minimum check that the test rejects $H_0 : \Gamma_{22} = 0$.

When $k_2 > 1$, $\Gamma_{22} \neq 0$ is not sufficient for identification. It is not even sufficient that each column of Γ_{22} is non-zero (each column corresponds to a distinct endogenous variable in X_2). So while a minimal check is to test that each columns of Γ_{22} is non-zero, this cannot be interpreted as definitive proof that Γ_{22} has full rank. Unfortunately, tests of deficient rank are difficult to implement. In any event, it appears reasonable to explicitly estimate and report the reduced form equations for X_2, and attempt to assess the likelihood that Γ_{22} has deficient rank.

Chapter 11

The Bootstrap

11.1 The Empirical Distribution Function

The observations are w_i, $i, ..., n$, with CDF $F(w) = P(w_i \leq w)$ and true value F_0. Note that $F_0(w) = P(w_i \leq w) = E1(w_i \leq w)$, where $1(\cdot)$ is the indicator function, so $F_0(w)$ can be expressed as a population moment. The MME is the corresponding sample moment:

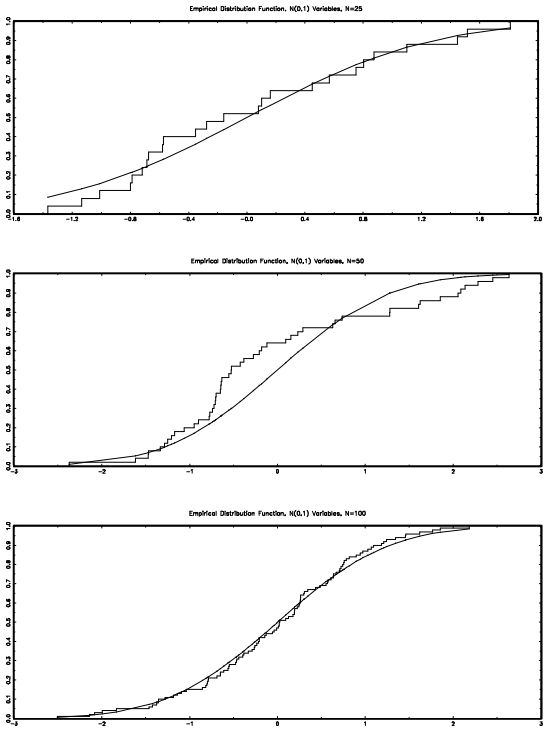
$$F_n(w) = \frac{1}{n}\sum_{i=1}^{n} 1(w_i \leq w).$$

$F_n(w)$ is called the empirical distribution function (EDF). F_n is a nonparametric estimate of F_0. Note that while F_0 may be either discrete or continuous, F_n is by construction a (discontinuous) step function.

The EDF is a consistent estimator of the CDF. To see this, note that for any w, $1(w_i \leq w)$ is an iid random variable with expectation $F_0(w)$. Thus by the WLLN, $F_n(w) \to_p F_0(w)$. It also converges at rate $\sqrt{n}$. By the CLT,

$$\sqrt{n}(F_n(w) - F_0(w)) \to^d N(0, F_0(w)(1 - F_0(w))).$$

To see the effect of sample size on the EDF, in Figure 2, I have plotted the EDF and true CDF for three random samples of size $n = 25$, 50, and 100. The random draws are from the $N(0, 1)$ distribution. For $n = 25$, the EDF is only a crude approximation to the CDF, but the approximation appears to improve for the large n. Yet even for $n = 100$, there is a significant divergence between the EDF and CDF around $w = -.6$ in this sample. In general, as the sample size gets larger, the EDF step function gets uniformly close to the true CDF.



11.2 Definition of the Bootstrap

Let $T_n = T_n(w_1, ..., w_n, \theta)$ be a statistic of interest with CDF $G_n(x, F) = P(T_n \leq x \mid F)$. For exact inference on θ, we desire $G_n(x, F_0)$. This is impossible since F_0 is unknown.

In a seminal contribution, Efron (1979) observed that since F_0 can be estimated by F_n, $G_n(x, F_0)$ can be estimated by plugging F_n into $G_n(x, F)$ to get the estimator

$$G_n^*(x) = G_n(x, F_n). \tag{11.1}$$

Bootstrap inference is based on $G_n^*(x)$.

Indeed, bootstrap inference can be based on any estimate F_n of F. (Choices other than the EDF introduced in the previous section are discussed later.) When F_n is the EDF $G_n^*(x)$ is called the nonparametric bootstrap.

The bootstrap distribution substitutes F_n for F_0 in the formula $G_n(x, F)$. As such, it not only

pretends that the distribution of w_i is F_n rather than F_0, but it also pretends that the true value of the parameter[1] is the sample estimate $\hat{\theta}$, rather than θ_0.

The EDF is a valid discrete probability distribution which puts probability mass $1/n$ at each point $w_i = (y_i, x_i)$, $i = 1, ..., n$. Notationally, it will be helpful to think of a random variable w_i^* with the distribution F_n. That is,

$$P(w_i^* \leq x) = F_n(x).$$

We can easily calculate the moments of w_i^* :

$$\begin{aligned} Eh(w_i^*) &= \int h(w) dF_n(w) \\ &= \sum_{i=1}^{n} h(w_i) P\left(w^* = w_i\right) \\ &= \frac{1}{n} \sum_{i=1}^{n} h(w_i), \end{aligned}$$

the empirical sample average.

Let $T_n^* = T_n(w_1^*, ..., w_n^*, \hat{\theta})$ be a random variable with distribution G_n^*. That is,

$$P(T_n^* \leq x) = G_n^*(x).$$

We call w_i^* and T_n^* the bootstrap distribution of the data and statistic. T_n^* is the correct analog of T_n when the true CDF is F_n, as the data w_i^* are sampled from the CDF F_n and the parameter $\hat{\theta}_n$ is the true value under F_n.

11.3 Computation

Since the EDF F_n is a multinomial (with n support points), in principle the distribution of T_n^* could be calculated by direct methods. Since there are 2^n possible samples $\{w_1^*, ..., w_n^*\}$, however, such a calculation is computationally infeasible unless n is very small. The popular alternative is to use Monte Carlo simulation to approximate the distribution. The algorithm is identical to our discussion of Monte Carlo simulation, with the following points of clarification:

- The sample size n used for the simulation is the same as the sample size
- The random vectors w_i^* are drawn randomly from the distribution function F_n
- $T_n^* = T_n(w_1^*, ..., w_n^*, \hat{\theta})$ is evaluated at the sample estimate $\hat{\theta}$.

[1]More precisely, the bootstrap pretends that the true value of θ is the value consistent with F_n, the estimate of F used to construct the bootstrap. In most cases, and the ones we consider, this value of θ is the sample estimate $\hat{\theta}$.

The generation of random vectors from F_n depends on the form of F_n. When F_n equals the EDF, this is particularly simple. Recall that F_n is a discrete probability distribution putting probability mass $1/n$ at each sample point w_i. Thus a random draw from F_n is just a random draw from the sample $\{w_1, ..., w_n\}$. For a bootstrap sample we need n independent random draws from F_n. This requires that we make n independent random draws from the sample $\{w_1, ..., w_n\}$ **with replacement**. In consequence, a bootstrap sample $\{w_1^*, ..., w_n^*\}$ will necessarily have some ties and multiple values, which is generally not a problem.

A theory for the determination of the number of bootstrap replications B has been developed by Andrews and Buchinsky (2000).

11.4 Bootstrap Estimation of Bias

The bias of $\hat{\theta}$ is

$$\tau_n = E(\hat{\theta} - \theta_0).$$

Let $T_n(\theta) = \hat{\theta} - \theta$. Then

$$\tau_n = E(T_n(\theta_0)).$$

The bootstrap counterparts are $\hat{\theta}^* = \hat{\theta}(w_1^*, ..., w_n^*)$ and $T_n^* = \hat{\theta}^* - \theta_n = \hat{\theta}^* - \hat{\theta}$. The bootstrap estimate of τ_n is

$$\tau_n^* = E(T_n^*).$$

If this is calculated by the simulation described in the previous subsection, the estimate of τ_n^* is

$$\begin{aligned} \hat{\tau}_n^* &= \frac{1}{B}\sum_{b=1}^{B} T_{nb}^* \\ &= \frac{1}{B}\sum_{b=1}^{B} \hat{\theta}_b^* - \hat{\theta} \\ &= \overline{\hat{\theta}^*} - \hat{\theta}. \end{aligned}$$

If $\hat{\theta}$ is biased, it might be desirable to construct a biased-corrected estimator (one with reduced bias). Ideally, this would be

$$\tilde{\theta} = \hat{\theta} - \tau_n,$$

but τ_n is unknown. The (estimated) bootstrap biased-corrected estimator is

$$\begin{aligned} \tilde{\theta}^* &= \hat{\theta} - \hat{\tau}_n^* \\ &= \hat{\theta} - (\overline{\hat{\theta}^*} - \hat{\theta}) \\ &= 2\hat{\theta} - \overline{\hat{\theta}^*}. \end{aligned}$$

Note, in particular, that the biased-corrected estimator is *not* $\overline{\hat{\theta}^*}$. Intuitively, the bootstrap makes the following experiment. Suppose that $\hat{\theta}$ is the truth. Then what is the average value of $\hat{\theta}$ calculated from such samples? The answer is $\overline{\hat{\theta}^*}$. If this is lower than $\hat{\theta}$, this suggests that the estimator is *downward-biased*, so a biased-corrected estimator of θ should be *larger* than $\hat{\theta}$, and the best guess is the difference between $\hat{\theta}$ and $\overline{\hat{\theta}^*}$. Similarly if $\overline{\hat{\theta}^*}$ is higher than $\hat{\theta}$, then the estimator is upward-biased and the biased-corrected estimator should be lower than $\hat{\theta}$.

11.5 Bootstrap Estimation of Variance

Let $T_n = \hat{\theta}$. The variance of $\hat{\theta}$ is

$$V_n = E(T_n - ET_n)^2.$$

Let $T_n^* = \hat{\theta}^*$. It has variance

$$V_n^* = E(T_n^* - ET_n^*)^2.$$

The simulation estimate is

$$\hat{V}_n^* = \frac{1}{B}\sum_{b=1}^{B}\left(\hat{\theta}_b^* - \overline{\hat{\theta}^*}\right)^2.$$

A bootstrap standard error for $\hat{\theta}$ is the square root of the bootstrap estimate of variance, $s(\hat{\beta}) = \sqrt{\hat{V}_n^*}$.

While this standard error may be calculated and reported, it is not clear if it is useful. The primary use of asymptotic standard errors is to construct asymptotic confidence intervals, which are based on the asymptotic normal approximation to the t-ratio. However, the use of the bootstrap presumes that such asymptotic approximations might be poor, in which case the normal approximation is suspect. It appears superior to calculate bootstrap confidence intervals, and we turn to this next.

11.6 Efron's Percentile Interval

For a distribution function $G_n(x, F)$, let $q_n(\alpha, F)$ denote its quantile function. This is the function which solves

$$G_n(q_n(\alpha, F), F) = \alpha.$$

[When $G_n(x, F)$ is discrete, $q_n(\alpha, F)$ may be non-unique, but we will ignore such complications.] Let $q_n(\alpha) = q_n(\alpha, F_0)$ denote the quantile function of the true sampling distribution, and $q_n^*(\alpha) = q_n(\alpha, F_n)$ denote the quantile function of the bootstrap distribution. Note that this function will change depending on the underlying statistic T_n whose distribution is G_n.

Let $T_n = \hat{\theta}$, an estimate of a parameter of interest. In $(1-\alpha)\%$ of samples, $\hat{\theta}$ lies in the region $[q_n(\alpha/2), q_n(1-\alpha/2)]$. This motivates a confidence interval proposed by Efron:

$$C_1 = [q_n^*(\alpha/2), \quad q_n^*(1-\alpha/2)].$$

This is often called the *percentile confidence interval*.

Computationally, the quantile $q_n^*(x)$ is estimated by $\hat{q}_n^*(x)$, the x'th sample quantile of the simulated statistics $\{T_{n1}^*, ..., T_{nB}^*\}$, as discussed in the section on Monte Carlo simulation. The $(1-\alpha)\%$ Efron percentile interval is then $[\hat{q}_n^*(\alpha/2), \quad \hat{q}_n^*(1-\alpha/2)]$.

The interval C_1 is a popular bootstrap confidence interval often used in empirical practice. This is because it is easy to compute, simple to motivate, was popularized by Efron early in the history of the bootstrap, and also has the feature that it is translation invariant. That is, if we define $\phi = f(\theta)$ as the parameter of interest for a monotonic function f, then percentile method applied to this problem will produce the confidence interval $[f(q_n^*(\alpha/2)), \quad f(q_n^*(1-\alpha/2))]$, which is a naturally good property.

However, as we show now, C_1 is in a deep sense very poorly motivated.

It will be useful if we introduce an alternative definition C_1. Let $T_n(\theta) = \hat{\theta} - \theta$ and let $q_n(\alpha)$ be the quantile function of its distribution. (These are the original quantiles, with θ subtracted.) Then C_1 can alternatively be written as

$$C_1 = [\hat{\theta} + q_n^*(\alpha/2), \quad \hat{\theta} + q_n^*(1-\alpha/2)].$$

This is a bootstrap estimate of the "ideal" confidence interval

$$C_1^0 = [\hat{\theta} + q_n(\alpha/2), \quad \hat{\theta} + q_n(1-\alpha/2)].$$

The latter has coverage probability

$$\begin{aligned} P\left(\theta_0 \in C_1^0\right) &= P\left(\hat{\theta} + q_n(\alpha/2) \leq \theta_0 \leq \hat{\theta} + q_n(1-\alpha/2)\right) \\ &= P\left(-q_n(1-\alpha/2) \leq \hat{\theta} - \theta_0 \leq -q_n(\alpha/2)\right) \\ &= G_n(-q_n(\alpha/2), F_0) - G_n(-q_n(1-\alpha/2), F_0) \end{aligned}$$

which generally is not $1-\alpha$! There is one important exception. If $\hat{\theta}-\theta_0$ has a symmetric distribution, then $G_n(-x, F_0) = 1 - G_n(x, F_0)$, so

$$\begin{aligned} P\left(\theta_0 \in C_1^0\right) &= G_n(-q_n(\alpha/2), F_0) - G_n(-q_n(1-\alpha/2), F_0) \\ &= (1 - G_n(q_n(\alpha/2), F_0)) - (1 - G_n(q_n(1-\alpha/2), F_0)) \\ &= \left(1 - \frac{\alpha}{2}\right) - \left(1 - \left(1 - \frac{\alpha}{2}\right)\right) \\ &= 1 - \alpha \end{aligned}$$

and this idealized confidence interval is accurate. Therefore, C_1^0 and C_1 are designed for the case that $\hat{\theta}$ has a symmetric distribution about θ_0.

When $\hat{\theta}$ does not have a symmetric distribution, C_1 may perform quite poorly.

However, by the translation invariance argument presented above, it also follows that if there exists some monotonic transformation $f(\cdot)$ such that $f(\hat{\theta})$ is symmetrically distributed about $f(\theta_0)$, then the idealized percentile bootstrap method will be accurate.

Based on these arguments, many argue that the percentile interval should not be used unless the sampling distribution is close to unbiased and symmetric.

11.7 Alternative Percentile Interval

The problems with the percentile method outlined above can be circumvented by an alternative method.

Let $T_n(\theta) = \hat{\theta} - \theta$. Then

$$\begin{aligned} 1-\alpha &= P\left(q_n(\alpha/2) \leq T_n(\theta_0) \leq q_n(1-\alpha/2)\right) \\ &= P\left(\hat{\theta} - q_n(1-\alpha/2) \leq \theta_0 \leq \hat{\theta} - q_n(\alpha/2)\right), \end{aligned}$$

so an exact $(1-\alpha)\%$ confidence interval for θ_0 would be

$$C_2^0 = [\hat{\theta} - q_n(1-\alpha/2), \quad \hat{\theta} - q_n(\alpha/2)].$$

This motivates a bootstrap analog

$$C_2 = [\hat{\theta} - q_n^*(1-\alpha/2), \quad \hat{\theta} - q_n^*(\alpha/2)].$$

Notice that generally this is very different from the Efron interval C_1! They coincide in the special case that $G_n^*(x)$ is symmetric about $\hat{\theta}$, but otherwise they differ.

Computationally, this interval can be estimated from a bootstrap simulation by sorting the bootstrap statistics $T_n^* = \left(\hat{\theta}^* - \hat{\theta}\right)$, which are centered at the sample estimate $\hat{\theta}$. These are sorted to yield the quantile estimates $\hat{q}_n^*(.025)$ and $\hat{q}_n^*(.975)$. The 95% confidence interval is then $[\hat{\theta} - \hat{q}_n^*(.975), \quad \hat{\theta} - \hat{q}_n^*(.025)]$.

This confidence interval is discussed in most theoretical treatments of the bootstrap, but is not widely used in practice.

11.8 One-Sided Hypothesis Tests

Suppose we want to test $H_0 : \theta = \theta_0$ against $H_1 : \theta < \theta_0$ at size α. We would set $T_n(\theta) = \left(\hat{\theta} - \theta\right)/s(\hat{\theta})$ and reject H_0 in favor of H_1 if $T_n(\theta_0) < c$, where c would be selected so that

$$P\left(T_n(\theta_0) < c\right) = \alpha.$$

Thus $c = q_n(\alpha)$. Since this is unknown, a bootstrap test replaces $q_n(\alpha)$ with the bootstrap estimate $q_n^*(\alpha)$, and the test rejects if $T_n(\theta_0) < q_n^*(\alpha)$.

Similarly, if the alternative is $H_1 : \theta > \theta_0$, the bootstrap test rejects if $T_n(\theta_0) > q_n^*(1-\alpha)$.

Computationally, these critical values can be estimated from a bootstrap simulation by sorting the bootstrap t-statistics $T_n^* = \left(\hat{\theta}^* - \hat{\theta}\right)/s(\hat{\theta}^*)$. Note, and this is important, that the bootstrap test statistic is centered at the estimate $\hat{\theta}$, and the standard error $s(\hat{\theta}^*)$ is calculated on the bootstrap sample. These t-statistics are sorted to find the estimated quantiles $\hat{q}_n^*(\alpha)$ and/or $\hat{q}_n^*(1-\alpha)$.

11.9 Percentile-t Equal-Tailed Interval

Let $T_n(\theta) = \left(\hat{\theta} - \theta\right) / s(\hat{\theta})$. Then

$$\begin{aligned} 1-\alpha &= P\left(q_n(\alpha/2) \leq T_n(\theta_0) \leq q_n(1-\alpha/2)\right) \\ &= P\left(q_n(\alpha/2) \leq \left(\hat{\theta} - \theta_0\right) / s(\hat{\theta}) \leq q_n(1-\alpha/2)\right) \\ &= P\left(\hat{\theta} - s(\hat{\theta}) q_n(1-\alpha/2) \leq \theta_0 \leq \hat{\theta} - s(\hat{\theta}) q_n(\alpha/2)\right), \end{aligned}$$

so an exact $(1-\alpha)\%$ confidence interval for θ_0 would be

$$C_3^0 = [\hat{\theta} - s(\hat{\theta}) q_n(1-\alpha/2), \quad \hat{\theta} - s(\hat{\theta}) q_n(\alpha/2)].$$

This motivates a bootstrap analog

$$C_3 = [\hat{\theta} - s(\hat{\theta}) q_n^*(1-\alpha/2), \quad \hat{\theta} - s(\hat{\theta}) q_n^*(\alpha/2)].$$

This is often called a *percentile-t confidence interval*. It is *equal-tailed* or *central* since the probability that θ_0 is below the left endpoint approximately equals the probability that θ_0 is above the right endpoint, each $\alpha/2$.

Computationally, this is based on the critical values from the one-sided hypothesis tests, discussed above.

11.10 Two-Sided Hypothesis Tests

Suppose we want to test $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ at size α. We would set $T_n(\theta) = \left(\hat{\theta} - \theta\right) / s(\hat{\theta})$ and reject H_0 in favor of H_1 if $|T_n(\theta_0)| > c$, where c would be selected so that

$$P\left(|T_n(\theta_0)| > c\right) = \alpha.$$

Note that

$$\begin{aligned} P\left(|T_n(\theta_0)| < c\right) &= P\left(-c < T_n(\theta_0) < c\right) \\ &= G_n(c) - G_n(-c) \\ &\equiv \overline{G}_n(c), \end{aligned}$$

which is a symmetric distribution function. The ideal critical value $c = q_n(\alpha)$ solves the equation

$$\overline{G}_n(q_n(\alpha)) = 1 - \alpha.$$

Equivalently, $q_n(\alpha)$ is the $1-\alpha$ quantile of the distribution of $|T_n(\theta_0)|$.

The bootstrap estimate is $q_n^*(\alpha)$, the $1-\alpha$ quantile of the distribution of $|T_n^*|$, or the number which solves the equation

$$\overline{G}_n^*(q_n^*(\alpha)) = G_n^*(q_n^*(\alpha)) - G_n^*(-q_n^*(\alpha)) = 1-\alpha.$$

Computationally, $q_n^*(\alpha)$ is estimated from a bootstrap simulation by sorting the bootstrap t-statistics $|T_n^*| = \left|\hat{\theta}^* - \hat{\theta}\right| / s(\hat{\theta}^*)$, and taking the upper $\alpha\%$ quantile.

The bootstrap test rejects if $|T_n(\theta_0)| > q_n^*(\alpha)$.

11.11 Symmetric Percentile-t Intervals

Let

$$C_4 = [\hat{\theta} - s(\hat{\theta})q_n^*(\alpha), \quad \hat{\theta} + s(\hat{\theta})q_n^*(\alpha)],$$

where $q_n^*(\alpha)$ is the bootstrap critical value for a two-sided hypothesis test, as defined above. C_4 is called the symmetric percentile-t interval. It is designed to work well since

$$\begin{aligned} P(\theta_0 \in C_4) &= P\left(\hat{\theta} - s(\hat{\theta})q_n^*(\alpha) \le \theta_0 \le \hat{\theta} + s(\hat{\theta})q_n^*(\alpha)\right) \\ &= P\left(|T_n(\theta_0)| < q_n^*(\alpha)\right) \\ &\simeq P\left(|T_n(\theta_0)| < q_n(\alpha)\right) \\ &= 1-\alpha. \end{aligned}$$

11.12 Vector Tests

If θ is a vector, then to test $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ at size α, we would use a Wald statistic

$$W_n(\theta) = \left(\hat{\theta} - \theta\right)' \hat{V}_\theta^{-1} \left(\hat{\theta} - \theta\right)$$

or some other asymptotically chi-square statistic. Thus here $T_n(\theta) = W_n(\theta)$. The ideal test rejects if $W_n \ge q_n(\alpha)$, where $q_n(\alpha)$ is the $(1-\alpha)\%$ quantile of the distribution of W_n. The bootstrap test rejects if $W_n \ge q_n^*(\alpha)$, where $q_n^*(\alpha)$ is the $(1-\alpha)\%$ quantile of the distribution of

$$W_n^* = \left(\hat{\theta}^* - \hat{\theta}\right)' \hat{V}_\theta^{*-1} \left(\hat{\theta}^* - \hat{\theta}\right).$$

Computationally, the critical value $q_n^*(\alpha)$ is found as the quantile from simulated values of W_n^*. Note in the simulation that the Wald statistic is a quadratic form in $\left(\hat{\theta}^* - \hat{\theta}\right)$, not $\left(\hat{\theta}^* - \theta_0\right)$. [This is a typical mistake made by practitioners.]

11.13 Asymptotic Expansions

Let T_n be a statistic such that

$$T_n \to_d N(0, v^2). \tag{11.2}$$

If $T_n = \sqrt{n}\left(\hat{\theta} - \theta_0\right)$ then $v = V$ while of T_n is a t-ratio than $v = 1$. Equivalently, writing $T_n \sim G_n(x, F)$ then

$$\lim_{n\to\infty} G_n(x, F) = \Phi\left(\frac{x}{v}\right),$$

or

$$G_n(x, F) = \Phi\left(\frac{x}{v}\right) + o(1). \tag{11.3}$$

While (11.3) says that G_n converges to $\Phi\left(\frac{x}{v}\right)$ as $n \to \infty$, it says nothing, however, about the rate of convergence, or the size of the divergence for any particular sample size n. A better asymptotic approximation may be obtained through an *asymptotic expansion.*

The following notation will be helpful. Let a_n be a sequence.

Definition 11.13.1 $a_n = o(1)$ *if* $a_n \to 0$ *as* $n \to \infty$

Definition 11.13.2 $a_n = O(1)$ *if* $|a_n|$ *is uniformly bounded.*

Definition 11.13.3 $a_n = o(n^{-r})$ *if* $n^r |a_n| \to 0$ *as* $n \to \infty$.

Basically, $a_n = O(n^{-r})$ if it declines to zero like n^{-r}.

We say that a function $g(x)$ is *even* if $g(-x) = g(x)$, and a function $h(x)$ is *odd* if $h(-x) = -h(x)$. The derivative of an even function is odd, and vice-versa.

Theorem 11.13.1 *Under regularity conditions and (11.2),*

$$G_n(x, F) = \Phi\left(\frac{x}{v}\right) + \frac{1}{n^{1/2}} g_1(x, F) + \frac{1}{n} g_2(x, F) + O(n^{-3/2})$$

uniformly over x*, where* g_1 *is an even function of* x*, and* g_2 *is an odd function of* x*. Moreover,* g_1 *and* g_2 *are differentiable functions of* x *and continuous in* F *relative to the supremum norm on the space of distribution functions.*

We can interpret Theorem 11.13.1 as follows. First, $G_n(x, F)$ converges to the normal limit at rate $n^{1/2}$. To a second order of approximation,

$$G_n(x, F) \approx \Phi\left(\frac{x}{v}\right) + n^{-1/2} g_1(x, F).$$

Since the derivative of g_1 is odd, the density function is skewed. To a third order of approximation,

$$G_n(x, F) \approx \Phi\left(\frac{x}{v}\right) + n^{-1/2} g_1(x, F) + n^{-1} g_2(x, F)$$

which adds a symmetric non-normal component to the approximate density (for example, adding leptokurtosis).

11.14 One-Sided Tests

Using the expansion of Theorem 11.13.1, we can assess the accuracy of one-sided hypothesis tests and confidence regions based on an asymptotically normal t-ratio T_n. An asymptotic test is based on $\Phi(x)$.

To the second order, the exact distribution is

$$P\left(T_n < x\right) = G_n(x, F_0) = \Phi(x) + \frac{1}{n^{1/2}} g_1(x, F_0) + O(n^{-1})$$

since $v = 1$. The difference is

$$\begin{aligned} \Phi(x) - G_n(x, F_0) &= \frac{1}{n^{1/2}} g_1(x, F_0) + O(n^{-1}) \\ &= O(n^{-1/2}), \end{aligned}$$

so the order of the error is $O(n^{-1/2})$.

A bootstrap test is based on $G_n^*(x)$, which from Theorem 11.13.1 has the expansion

$$G_n^*(x) = G_n(x, F_n) = \Phi(x) + \frac{1}{n^{1/2}} g_1(x, F_n) + O(n^{-1}).$$

Because $\Phi(x)$ appears in both expansions, the difference between the bootstrap distribution and the true distribution is

$$G_n^*(x) - G_n(x, F_0) = \frac{1}{n^{1/2}} \left(g_1(x, F_n) - g_1(x, F_0)\right) + O(n^{-1}).$$

Since F_n converges to F at rate $\sqrt{n}$, and g_1 is continuous with respect to F, the difference $(g_1(x, F_n) - g_1(x, F_0))$ converges to 0 at rate $\sqrt{n}$. Heuristically,

$$\begin{aligned} g_1(x, F_n) - g_1(x, F_0) &\approx \frac{\partial}{\partial F} g_1(x, F_0) \left(F_n - F_0\right) \\ &= O(n^{-1/2}), \end{aligned}$$

The "derivative" $\frac{\partial}{\partial F} g_1(x, F)$ is only heuristic, as F is a function. We conclude that

$$G_n^*(x) - G_n(x, F_0) = O(n^{-1}),$$

or

$$P\left(T_n^* \leq x\right) = P\left(T_n \leq x\right) + O(n^{-1}),$$

which is an improved rate of convergence over the asymptotic test (which converged at rate $O(n^{-1/2})$). This rate can be used to show that one-tailed bootstrap inference based on the t-ratio achieves a so-called *asymptotic refinement* – the Type I error of the test converges at a faster rate than an analogous asymptotic test.

11.15 Symmetric Two-Sided Tests

If a random variable X has distribution function $H(x) = P(X \leq x)$, then the random variable $|X|$ has distribution function

$$\overline{H}(x) = H(x) - H(-x)$$

since

$$\begin{aligned} P(|X| \leq x) &= P(-x \leq X \leq x) \\ &= P(X \leq x) - P(X \leq -x) \\ &= H(x) - H(-x). \end{aligned}$$

For example, if $Z \sim N(0,1)$, then $|Z|$ has distribution function

$$\overline{\Phi}(x) = \Phi(x) - \Phi(-x) = 2\Phi(x) - 1.$$

Similarly, if T_n has exact distribution $G_n(x, F)$, then $|T_n|$ has the distribution function

$$\overline{G}_n(x, F) = G_n(x, F) - G_n(-x, F).$$

A two-sided hypothesis test rejects H_0 for large values of $|T_n|$. Since $T_n \to_d Z$, then $|T_n| \to_d |Z| \sim \overline{\Phi}$. Thus asymptotic critical values are taken from the $\overline{\Phi}$ distribution, and exact critical values are taken from the $\overline{G}_n(x, F_0)$ distribution. From Theorem 11.13.1, we can calculate that

$$\begin{aligned} \overline{G}_n(x, F) &= G_n(x, F) - G_n(-x, F) \\ &= \left(\Phi(x) + \frac{1}{n^{1/2}} g_1(x, F) + \frac{1}{n} g_2(x, F)\right) \\ &\quad - \left(\Phi(-x) + \frac{1}{n^{1/2}} g_1(-x, F) + \frac{1}{n} g_2(-x, F)\right) + O(n^{-3/2}) \\ &= \overline{\Phi}(x) + \frac{2}{n} g_2(x, F) + O(n^{-3/2}), \end{aligned} \tag{11.4}$$

where the simplifications are because g_1 is even and g_2 is odd. Hence the difference between the asymptotic distribution and the exact distribution is

$$\overline{\Phi}(x) - \overline{G}_n(x, F_0) = \frac{2}{n} g_2(x, F_0) + O(n^{-3/2}) = O(n^{-1}).$$

The order of the error is $O(n^{-1})$.

Interestingly, the asymptotic two-sided test has a better coverage rate than the asymptotic one-sided test. This is because the first term in the asymptotic expansion, g_1, is an even function, meaning that the errors in the two directions exactly cancel out.

Applying (11.4) to the bootstrap distribution, we find

$$\overline{G}_n^*(x) = \overline{G}_n(x, F_n) = \overline{\Phi}(x) + \frac{2}{n} g_2(x, F_n) + O(n^{-3/2}).$$

Thus the difference between the bootstrap and exact distributions is

$$\begin{aligned}\overline{G}_n^*(x) - \overline{G}_n(x, F_0) &= \frac{2}{n}\left(g_2(x, F_n) - g_2(x, F_0)\right) + O(n^{-3/2}) \\ &= O(n^{-3/2}),\end{aligned}$$

the last equality because F_n converges to F_0 at rate $\sqrt{n}$, and g_2 is continuous in F. Another way of writing this is

$$P\left(|T_n^*| < x\right) = P\left(|T_n| < x\right) + O(n^{-3/2})$$

so the error from using the bootstrap distribution (relative to the true unknown distribution) is $O(n^{-3/2})$. This is in contrast to the use of the asymptotic distribution, whose error is $O(n^{-1})$. Thus a two-sided bootstrap test also achieves an asymptotic refinement, similar to a one-sided test.

A reader might get confused between the two simultaneous effects. Two-sided tests have better rates of convergence than the one-sided tests, and bootstrap tests have better rates of convergence than asymptotic tests.

The analysis shows that there may be a trade-off between one-sided and two-sided tests. Two-sided tests will have more accurate size (Reported Type I error), but one-sided tests might have more power against alternatives of interest. Confidence intervals based on the bootstrap can be asymmetric if based on one-sided tests (equal-tailed intervals) and can therefore be more informative and have smaller length than symmetric intervals. Therefore, the choice between symmetric and equal-tailed confidence intervals is unclear, and needs to be determined on a case-by-case basis.

11.16 Percentile Confidence Intervals

To evaluate the coverage rate of the percentile interval, set $T_n = \sqrt{n}\left(\hat{\theta} - \theta_0\right)$. We know that $T_n \to_d N(0, V)$, which is not pivotal, as it depends on the unknown V. Theorem 11.13.1 shows that a first-order approximation

$$G_n(x, F) = \Phi\left(\frac{x}{v}\right) + O(n^{-1/2}),$$

where $v = \sqrt{V}$, and for the bootstrap

$$G_n^*(x) = G_n(x, F_n) = \Phi\left(\frac{x}{\hat{v}}\right) + O(n^{-1/2}),$$

where $\hat{V} = V(F_n)$ is the bootstrap estimate of V. The difference is

$$\begin{aligned}G_n^*(x) - G_n(x, F_0) &= \Phi\left(\frac{x}{\hat{v}}\right) - \Phi\left(\frac{x}{v}\right) + O(n^{-1/2}) \\ &= -\phi\left(\frac{x}{\hat{v}}\right)\frac{x}{\hat{v}}\left(\hat{v} - v\right) + O(n^{-1/2}) \\ &= O(n^{-1/2})\end{aligned}$$

Hence the order of the error is $O(n^{-1/2})$.

The good news is that the percentile-type methods (if appropriately used) can yield $\sqrt{n}$-convergent asymptotic inference. Yet these methods do not require the calculation of standard errors! This means that in contexts where standard errors are not available or are difficult to calculate, the percentile bootstrap methods provide an attractive inference method.

The bad news is that the rate of convergence is disappointing. It is no better than the rate obtained from an asymptotic one-sided confidence region. Therefore if standard errors are available, it is unclear if there are any benefits from using the percentile bootstrap over simple asymptotic methods.

Based on these arguments, the theoretical literature (e.g. Hall, 1992, Horowitz, 2002) tends to advocate the use of the percentile-t bootstrap methods rather than percentile methods.

11.17 Bootstrap Methods for Regression Models

The bootstrap methods we have discussed have set $G_n^*(x) = G_n(x, F_n)$, where F_n is the EDF. Any other consistent estimate of F_0 may be used to define a feasible bootstrap estimator. The advantage of the EDF is that it is fully nonparametric, it imposes no conditions, and works in nearly any context. But since it is fully nonparametric, it may be inefficient in contexts where more is known about F. We discuss some bootstrap methods appropriate for the case of a regression model where

$$\begin{aligned} y_i &= x_i'\beta + e_i \\ E(e_i \mid x_i) &= 0. \end{aligned}$$

The non-parametric bootstrap distribution resamples the observations (y_i^*, x_i^*) from the EDF, which implies

$$\begin{aligned} y_i^* &= x_i^{*\prime}\hat{\beta} + e_i^* \\ E(x_i^* e_i^*) &= 0 \end{aligned}$$

but generally

$$E(e_i^* \mid x_i^*) \neq 0.$$

The the bootstrap distribution does not impose the regression assumption, and is thus an inefficient estimator of the true distribution (when in fact the regression assumption is true.)

One approach to this problem is to impose the very strong assumption that the error e_i is independent of the regressor x_i. The advantage is that in this case it is straightforward to construct bootstrap distributions. The disadvantage is that the bootstrap distribution may be a poor approximation when the error is not independent of the regressors.

To impose independence, it is sufficient to sample the x_i^* and e_i^* independently, and then create $y_i^* = x_i^{*\prime}\hat{\beta} + e_i^*$. There are different ways to impose independence. A non-parametric method is to sample the bootstrap errors e_i^* randomly from EDF ore OLS residuals $\{\hat{e}_1, ..., \hat{e}_n\}$. A parametric

method is to generate the bootstrap errors e_i^* from a parametric distribution, such as the normal $e_i^* \sim N(0, \hat{\sigma}^2)$.

For the regressors x_i^*, a nonparametric method is to sample the x_i^* randomly from the EDF or sample values $\{x_1, ..., x_n\}$. A parametric method is to sample x_i^* from an estimated parametric distribution. A third approach sets $x_i^* = x_i$. This is equivalent to treating the regressors as *fixed in repeated samples*. If this is done, then all inferential statements are made conditionally on the observed values of the regressors, which is a valid statistical approach. It does not really matter, however, whether or not the x_i are really "fixed" or random.

The methods discussed above are unattractive for most applications in econometrics because they impose the stringent assumption that x_i and e_i are independent. Typically what is desirable is to impose only the regression condition $E(e_i \mid x_i) = 0$. Unfortunately this is a harder problem.

One proposal which imposes the regression condition without independence is the *Wild Bootstrap*. The idea is to construct a conditional distribution for e_i^* so that

$$\begin{aligned} E(e_i^* \mid x_i) &= 0 \\ E\left(e_i^{*2} \mid x_i\right) &= \hat{e}_i^2 \\ E\left(e_i^{*3} \mid x_i\right) &= \hat{e}_i^3. \end{aligned}$$

A conditional distribution with these features will preserve the main important features of the data. This can be achieved using a two-point distribution of the form

$$\begin{aligned} P\left(e_i^* = \left(\frac{1+\sqrt{5}}{2}\right)\hat{e}_i\right) &= \frac{\sqrt{5}-1}{2\sqrt{5}} \\ P\left(e_i^* = \left(\frac{1-\sqrt{5}}{2}\right)\hat{e}_i\right) &= \frac{\sqrt{5}+1}{2\sqrt{5}} \end{aligned}$$

For each x_i, you sample e_i^* using this two-point distribution.

11.18 Bootstrap GMM Inference

Let $w_i = (y_i, x_i, z_i)$ and let $\hat{\beta}$ be the 2SLS or GMM estimator of β. Using the EDF of w_i, we can apply the bootstrap methods discussed before to compute estimates of the bias and variance of $\hat{\beta}$, and construct confidence intervals for β, identically as in the regression model. However, caution should be applied when interpreting such results.

A straightforward application of the nonparametric bootstrap works in the sense of consistently achieving the first-order asymptotic distribution. This has been shown by Hahn (1996). However, it fails to achieve an asymptotic refinement when the model is over-identified, jeopardizing the theoretical justification for percentile-t methods. Furthermore, the bootstrap applied J test will yield the wrong answer.

The problem is that in the sample, $\hat{\beta}$ is the "true" value and yet $\overline{g}_n(\hat{\beta}) \neq 0$. Thus according to random variables w_i^* drawn from the EDF F_n,

$$E\left(g_i\left(\hat{\beta}\right)\right) = \overline{g}_n(\hat{\beta}) \neq 0.$$

This means that w_i^* do not satisfy the same moment conditions as the population distribution.

A correction suggested by Hall and Horowitz (1996) can solve the problem. Given the bootstrap sample (Y^*, X^*, Z^*), define the bootstrap GMM criterion

$$J^*(\beta) = n \cdot \left(\overline{g}_n^*(\beta) - \overline{g}_n(\hat{\beta})\right)' W_n^* \left(\overline{g}_n^*(\beta) - \overline{g}_n(\hat{\beta})\right)$$

where $\overline{g}_n(\hat{\beta})$ is from the in-sample data, not from the bootstrap data.

Let $\hat{\beta}^*$ minimize $J^*(\beta)$, and define all statistics and tests accordingly. In the linear model, this implies that the bootstrap estimator is

$$\hat{\beta}^* = \left(Z^{*\prime} X^* W_n^* X^{*\prime} Z^*\right)^{-1} \left(Z^{*\prime} X^* W_n^* \left(X^{*\prime} Y^* - X^{\prime} \hat{e}\right)\right).$$

where $\hat{e} = Y - Z\hat{\beta}$ are the in-sample residuals. The bootstrap J statistic is $J^*(\hat{\beta}^*)$.

Brown and Newey (2001) have an alternative solution. They note that we can sample from the observations $\{w_1, ..., w_n\}$ with the empirical likelihood probabilities $\{\hat{p}_i\}$ described in Chapter 9. Since $\sum_{i=1}^n \hat{p}_i g_i\left(\hat{\beta}\right) = 0$, this sampling scheme preserves the moment conditions of the model, so no recentering or adjustments are needed. Brown and Newey argue that this bootstrap procedure will be more efficient than the Hall-Horowitz GMM bootstrap.

To date, there are very few empirical applications of bootstrap GMM, as this is a very new area of research.

Chapter 12

Univariate Time Series

A time series y_t is a process observed in sequence over time, $t = 1, ..., T$. To indicate the dependence on time, we adopt new notation, and use the subscript t to denote the individual observation, and T to denote the number of observations.

Because of the sequential nature of time series, we expect that Y_t and Y_{t-1} are *not* independent, so classical assumptions are not valid.

We can separate time series into two categories: univariate ($y_t \in R$ is scalar); and multivariate ($y_t \in R^m$ is vector-valued). The primary model for univariate time series is autoregressions (ARs). The primary model for multivariate time series is vector autoregressions (VARs).

12.1 Stationarity and Ergodicity

Definition 12.1.1 *$\{Y_t\}$ is covariance (weakly) stationary if*

$$E(Y_t) = \mu$$

is independent of t, and

$$Cov\,(Y_t, Y_{t-k}) = \gamma(k)$$

is independent of t for all k.

$\gamma(k)$ is called the autocovariance function.

Definition 12.1.2 *$\{Y_t\}$ is strictly stationary if the joint distribution of $(Y_t, ..., Y_{t-k})$ is independent of t for all k.*

Definition 12.1.3 *$\rho(k) = \gamma(k)/\gamma(0) = Corr(Y_t, Y_{t-k})$ is the autocorrelation function.*

Definition 12.1.4 *(loose). A stationary time series is ergodic if $\gamma(k) \to 0$ as $k \to \infty$.*

The following two theorems are essential to the analysis of stationary time series. There proofs are rather difficult, however.

Theorem 12.1.1 *If Y_t is strictly stationary and ergodic and $X_t = f(Y_t, Y_{t-1}, ...)$ is a random variable, then X_t is strictly stationary and ergodic.*

Theorem 12.1.2 *(Ergodic Theorem). If X_t is strictly stationary and ergodic and $E|X_t| < \infty$, then as $T \to \infty$,*

$$\frac{1}{T}\sum_{t=1}^{T} X_t \to_p E(X_t).$$

This allows us to consistently estimate parameters using time-series moments:
The sample mean:

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} Y_t$$

The sample autocovariance

$$\hat{\gamma}(k) = \frac{1}{T}\sum_{t=1}^{T} (Y_t - \hat{\mu})(Y_{t-k} - \hat{\mu}).$$

The sample autocorrelation

$$\hat{\rho}(k) = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)}.$$

Theorem 12.1.3 *If Y_t is strictly stationary and ergodic and $EY_t^2 < \infty$, then as $T \to \infty$,*

1. $\hat{\mu} \to_p E(Y_t)$;
2. $\hat{\gamma}(k) \to_p \gamma(k)$;
3. $\hat{\rho}(k) \to_p \rho(k)$.

Proof. Part (1) is a direct consequence of the Ergodic theorem. For Part (2), note that

$$\begin{aligned}\hat{\gamma}(k) &= \frac{1}{T}\sum_{t=1}^{T} (Y_t - \hat{\mu})(Y_{t-k} - \hat{\mu}) \\ &= \frac{1}{T}\sum_{t=1}^{T} Y_t Y_{t-k} - \frac{1}{T}\sum_{t=1}^{T} Y_t\hat{\mu} - \frac{1}{T}\sum_{t=1}^{T} Y_{t-k}\hat{\mu} + \hat{\mu}^2.\end{aligned}$$

By Theorem 12.1.1 above, the sequence $Y_t Y_{t-k}$ is strictly stationary and ergodic, and it has a finite mean by the assumption that $EY_t^2 < \infty$. Thus an application of the Ergodic Theorem yields

$$\frac{1}{T}\sum_{t=1}^{T} Y_t Y_{t-k} \to_p E(Y_t Y_{t-k}).$$

Thus

$$\hat{\gamma}(k) \to_p E(Y_t Y_{t-k}) - \mu^2 - \mu^2 + \mu^2 = E(Y_t Y_{t-k}) - \mu^2 = \gamma(k).$$

Part (3) follows by the continuous mapping theorem: $\hat{\rho}(k) = \hat{\gamma}(k)/\hat{\gamma}(0) \to_p \gamma(k)/\gamma(0) = \rho(k)$. ■

12.2 Autoregressions

In time-series, the series $\{..., Y_1, Y_2, ..., Y_T, ...\}$ are jointly random. We consider the conditional expectation

$$E\left(Y_t \mid I_{t-1}\right)$$

where $I_{t-1} = \{Y_{t-1}, Y_{t-2}, ...\}$ is the past history of the series.

An autoregressive (AR) model specifies that only a finite number of past lags matter:

$$E\left(Y_t \mid I_{t-1}\right) = E\left(Y_t \mid Y_{t-1}, ..., Y_{t-k}\right).$$

A linear AR model (the most common type used in practice) specifies linearity:

$$E\left(Y_t \mid I_{t-1}\right) = \alpha + \rho_1 Y_{t-1} + \rho_2 Y_{t-1} + \cdots + \rho_k Y_{t-k}.$$

Letting

$$e_t = Y_t - E\left(Y_t \mid I_{t-1}\right),$$

then we have the autoregressive model

$$\begin{aligned} Y_t &= \alpha + \rho_1 Y_{t-1} + \rho_2 Y_{t-1} + \cdots + \rho_k Y_{t-k} + e_t \\ E\left(e_t \mid I_{t-1}\right) &= 0. \end{aligned}$$

The last property defines a special time-series process.

Definition 12.2.1 *e_t is a martingale difference sequence (MDS) if $E\left(e_t \mid I_{t-1}\right) = 0$.*

Regression errors are naturally a MDS. Some time-series processes may be a MDS as a consequence of optimizing behavior. For example, some versions of the life-cycle hypothesis imply that either changes in consumption, or consumption growth rates, should be a MDS. Most asset pricing models imply that asset returns should be the sum of a constant plus a MDS.

The MDS property for the regression error plays the same role in a time-series regression as does the conditional mean-zero property for the regression error in a cross-section regression. In fact, it is even more important in the time-series context, as it is difficult to derive distribution theories without this property.

A useful property of a MDS is that e_t is uncorrelated with any function of the lagged information I_{t-1}. Thus for $k > 0$, $E(Y_{t-k} e_t) = 0$.

12.3 Stationarity of AR(1) Process

A mean-zero AR(1) is

$$Y_t = \rho Y_{t-1} + e_t.$$

Assume that e_t is iid, $E(e_t) = 0$ and $Ee_t^2 = \sigma^2 < \infty$.

By back-substitution, we find

$$\begin{aligned} Y_t &= e_t + \rho e_{t-1} + \rho^2 e_{t-2} + ... \\ &= \sum_{k=0}^{\infty} \rho^k e_{t-k}. \end{aligned}$$

Loosely speaking, this series converges if the sequence $\rho^k e_{t-k}$ gets small as $k \to \infty$. This occurs when $|\rho| < 1$.

Theorem 12.3.1 *If $|\rho| < 1$ then Y_t is strictly stationary and ergodic.*

We can compute the moments of Y_t using the infinite sum:

$$EY_t = \sum_{k=0}^{\infty} \rho^k E\left(e_{t-k}\right) = 0$$

$$Var(Y_t) = \sum_{k=0}^{\infty} \rho^{2k} Var\left(e_{t-k}\right) = \frac{\sigma^2}{1-\rho^2}.$$

If the equation for Y_t has an intercept, the above results are unchanged, except that the mean of Y_t can be computed from the relationship

$$EY_t = \alpha + \rho EY_{t-1},$$

and solving for $EY_t = EY_{t-1}$ we find $EY_t = \alpha/(1-\rho)$.

12.4 Lag Operator

An algebraic construct which is useful for the analysis of autoregressive models is the lag operator.

Definition 12.4.1 *The lag operator L satisfies $LY_t = Y_{t-1}$.*

Defining $L^2 = LL$, we see that $L^2 Y_t = LY_{t-1} = Y_{t-2}$. In general, $L^k Y_t = Y_{t-k}$.

The AR(1) model can be written in the format

$$Y_t - \rho Y_{t-1} + e_t$$

or

$$(1 - \rho L) Y_{t-1} = e_t.$$

The operator $\rho(L) = (1 - \rho L)$ is a polynomial in the operator L. We say that the *root* of the polynomial is $1/\rho$, since $\rho(z) = 0$ when $z = 1/\rho$. We call $\rho(L)$ the autoregressive polynomial of Y_t.

From Theorem 12.3.1, an AR(1) is stationary iff $|\rho| < 1$. Note that an equivalent way to say this is that an AR(1) is stationary iff the root of the autoregressive polynomial is larger than one (in absolute value).

12.5 Stationarity of AR(k)

The AR(k) model is

$$Y_t = \rho_1 Y_{t-1} + \rho_2 Y_{t-1} + \cdots + \rho_k Y_{t-k} + e_t.$$

Using the lag operator,

$$Y_t - \rho_1 L Y_t - \rho_2 L^2 Y_t - \cdots - \rho_k L^k Y_t = e_t,$$

or

$$\rho(L) Y_t = e_t$$

where

$$\rho(L) = 1 - \rho_1 L - \rho_2 L^2 - \cdots - \rho_k L^k.$$

We call $\rho(L)$ the autoregressive polynomial of Y_t.

The *Fundamental Theorem of Algebra* says that any polynomial can be factored as

$$\rho(z) = \left(1 - \lambda_1^{-1} z\right)\left(1 - \lambda_2^{-1} z\right) \cdots \left(1 - \lambda_k^{-1} z\right)$$

where the $\lambda_1, ..., \lambda_k$ are the complex *roots* of $\rho(z)$, which satisfy $\rho(\lambda_j) = 0$.

We know that an AR(1) is stationary iff the absolute value of the root of its autoregressive polynomial is larger than one. For an AR(k), the requirement is that all roots are larger than one. Let $|\lambda|$ denote the modulus of a complex number λ.

Theorem 12.5.1 *The AR(k) is strictly stationary and ergodic if and only if* $|\lambda_j| > 1$ *for all* j.

One way of stating this is that "All roots lie outside the unit circle."

If one of the roots equals 1, we say that $\rho(L)$, and hence Y_t, "has a unit root". This is a special case of non-stationarity, and is of great interest in applied time series.

12.6 Estimation

Let

$$\begin{aligned} x_t &= \begin{pmatrix} 1 & Y_{t-1} & Y_{t-2} & \cdots & Y_{t-k} \end{pmatrix}' \\ \beta &= \begin{pmatrix} \alpha & \rho_1 & \rho_2 & \cdots & \rho_k \end{pmatrix}'. \end{aligned}$$

Then the model can be written as

$$y_t = x_t'\beta + e_t.$$

The OLS estimator is

$$\hat{\beta} = \left(X'X\right)^{-1} X'Y.$$

To study $\hat{\beta}$, it is helpful to define the process $u_t = x_t e_t$. Note that u_t is a MDS, since

$$E\left(u_t \mid I_{t-1}\right) = E\left(x_t e_t \mid I_{t-1}\right) = x_t E\left(e_t \mid I_{t-1}\right) = 0.$$

By Theorem 12.1.1, it is also strictly stationary and ergodic. Thus

$$\frac{1}{T}\sum_{t=1}^{T} x_t e_t = \frac{1}{T}\sum_{t=1}^{T} u_t \to_p E\left(u_t\right) = 0. \tag{12.1}$$

Theorem 12.6.1 *If the AR(k) process Y_t is strictly stationary and ergodic and $EY_t^2 < \infty$, then $\hat{\beta} \to_p \beta$ as $T \to \infty$.*

Proof. The vector x_t is strictly stationary and ergodic, and by Theorem 12.1.1, so is $x_t x_t'$. Thus by the Ergodic Theorem,

$$\frac{1}{T}\sum_{t=1}^{T} x_t x_t' \to_p E\left(x_t x_t'\right) = Q.$$

Combined with (12.1) and the continuous mapping theorem, we see that

$$\hat{\beta} = \beta + \left(\frac{1}{T}\sum_{t=1}^{T} x_t x_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} x_t e_t\right) \to_p Q^{-1}0 = 0.$$

■

12.7 Asymptotic Distribution

Theorem 12.7.1 *MDS CLT. If u_t is a strictly stationary and ergodic MDS and $E(u_t u_t') = \Omega < \infty$, then as $T \to \infty$,*

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} u_t \to_d N(0, \Omega).$$

Since $x_t e_t$ is a MDS, we can apply Theorem 12.7.1 to see that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_t e_t \to_d N(0, \Omega),$$

where

$$\Omega = E(x_t x_t' e_t^2).$$

Theorem 12.7.2 *If the AR(k) process Y_t is strictly stationary and ergodic and $EY_t^4 < \infty$, then as $T \to \infty$,*

$$\sqrt{T}\left(\hat{\beta} - \beta\right) \to_d N\left(0, Q^{-1}\Omega Q^{-1}\right).$$

This is identical in form to the asymptotic distribution of OLS in cross-section regression. The implication is that asymptotic inference is the same. In particular, the asymptotic covariance matrix is estimated just as in the cross-section case.

12.8 Bootstrap for Autoregressions

In the non-parametric bootstrap, we constructed the bootstrap sample by randomly resampling from the data values $\{y_t, x_t\}$. This creates an iid bootstrap sample. Clearly, this cannot work in a time-series application, as this imposes inappropriate independence.

Briefly, there are two popular methods to implement bootstrap resampling for time-series data.

Method 1: Model-Based (Parametric) Bootstrap.

1. Estimate $\hat{\beta}$ and residuals $\hat{e}_t$.
2. Fix an initial condition $(Y_{-k+1}, Y_{-k+2}, ..., Y_0)$.
3. Simulate iid draws e_i^* from the empirical distribution of the residuals $\{\hat{e}_1, ..., \hat{e}_T\}$.
4. Create the bootstrap series Y_t^* by the recursive formula

$$Y_t^* = \hat{\alpha} + \hat{\rho}_1 Y_{t-1}^* + \hat{\rho}_2 Y_{t-2}^* + \cdots + \hat{\rho}_k Y_{t-k}^* + e_t^*.$$

This construction imposes homoskedasticity on the errors e_i^*, which may be different than the properties of the actual e_i. It also presumes that the AR(k) structure is the truth.

Method 1: Block Resampling

1. Divide the sample into T/m blocks of length m.
2. Resample complete blocks. For each simulated sample, draw T/m blocks.
3. Paste the blocks together to create the bootstrap time-series Y_t^*.
4. This allows for arbitrary stationary serial correlation, heteroskedasticity, and for model-misspecification.
5. The results may be sensitive to the block length, and the way that the data is partitioned into blocks.
6. May not work well in small samples.

12.9 Trend Stationarity

$$\begin{aligned} Y_t &= \mu_0 + \mu_1 t + S_t \quad (12.2) \\ S_t &= \rho_1 S_{t-1} + \rho_2 S_{t-2} + \cdots + \rho_k S_{t-l} + e_t, \quad (12.3) \end{aligned}$$

or

$$Y_t = \alpha_0 + \alpha_1 t + \rho_1 Y_{t-1} + \rho_2 Y_{t-1} + \cdots + \rho_k Y_{t-k} + e_t. \quad (12.4)$$

There are two essentially equivalent ways to estimate the autoregressive parameters $(\rho_1, ..., \rho_k)$.

- You can estimate (12.4) by OLS.
- You can estimate (12.2)-(12.3) sequentially by OLS. That is, first estimate (12.2), get the residual $\hat{S}_t$, and then perform regression (12.3) replacing S_t with $\hat{S}_t$. This procedure is sometimes called *Detrending*.

The reason why these two procedures are (essentially) the same is the Frisch-Waugh-Lovell theorem.

Seasonal Effects

There are three popular methods to deal with seasonal data.

- Include dummy variables for each season. This presumes that "seasonality" does not change over the sample.

- Use "seasonally adjusted" data. The seasonal factor is typically estimated by a two-sided weighted average of the data for that season in neighboring years. Thus the seasonally adjusted data is a "filtered" series. This is a flexible approach which can extract a wide range of seasonal factors. The seasonal adjustment, however, also alters the time-series correlations of the data.

- First apply a seasonal differencing operator. If s is the number of seasons (typically $s = 4$ or $s = 12$),
$$\Delta_s Y_t = Y_t - Y_{t-s},$$
or the season-to-season change. The series $\Delta_s Y_t$ is clearly free of seasonality. But the long-run trend is also eliminated, and perhaps this was of relevance.

12.10 Testing for Omitted Serial Correlation

For simplicity, let the null hypothesis be an AR(1):

$$Y_t = \alpha + \rho Y_{t-1} + u_t. \tag{12.5}$$

We are interested in the question if the error u_t is serially correlated. We model this as an AR(1):

$$u_t = \theta u_{t-1} + e_t \tag{12.6}$$

with e_t a MDS. The hypothesis of no omitted serial correlation is

$$\begin{aligned} H_0 &: \theta = 0 \\ H_1 &: \theta \neq 0. \end{aligned}$$

We want to test H_0 against H_1.

To combine (12.5) and (12.6), we take (12.5) and lag the equation once:

$$Y_{t-1} = \alpha + \rho Y_{t-2} + u_{t-1}.$$

We then multiply this by θ and subtract from (12.5), to find

$$Y_t - \theta Y_{t-1} = \alpha - \theta\alpha + \rho Y_{t-1} - \theta\rho Y_{t-1} + u_t - \theta u_{t-1},$$

or

$$Y_t = \alpha(1-\theta) + (\rho + \theta)\, Y_{t-1} - \theta\rho Y_{t-2} + e_t = AR(2).$$

Thus under H_0, Y_t is an AR(1), and under H_1 it is an AR(2). H_0 may be expressed as the restriction that the coefficient on Y_{t-2} is zero.

An appropriate test of H_0 against H_1 is therefore a Wald test that the coefficient on Y_{t-2} is zero. (A simple exclusion test).

In general, if the null hypothesis is that Y_t is an AR(k), and the alternative is that the error is an AR(m), this is the same as saying that under the alternative Y_t is an AR(k+m), and this is equivalent to the restriction that the coefficients on $Y_{t-k-1}, ..., Y_{t-k-m}$ are jointly zero. An appropriate test is the Wald test of this restriction.

12.11 Model Selection

What is the appropriate choice of k in practice? This is a problem of model selection.

One approach to model selection is to choose k based on a Wald tests.

Another is to minimize the AIC or BIC information criterion, e.g.

$$AIC(k) = \log \hat{\sigma}^2(k) + \frac{2k}{T},$$

where $\hat{\sigma}^2(k)$ is the estimated residual variance from an AR(k)

One ambiguity in defining the AIC criterion is that the sample available for estimation changes as k changes. (If you increase k, you need more initial conditions.) This can induce strange behavior in the AIC. The best remedy is to fix a upper value $\overline{k}$, and then reserve the first $\overline{k}$ as initial conditions, and then estimate the models AR(1), AR(2), ..., AR($\overline{k}$) on this (unified) sample.

12.12 Autoregressive Unit Roots

The AR(k) model is

$$\begin{aligned} \rho(L)Y_t &= \mu + e_t \\ \rho(L) &= 1 - \rho_1 L - \cdots - \rho_k L^k. \end{aligned}$$

As we discussed before, Y_t has a unit root when $\rho(1) = 0$, or

$$\rho_1 + \rho_2 + \cdots + \rho_k = 1.$$

In this case, Y_t is non-stationary. The ergodic theorem and MDS CLT do not apply, and test statistics are asymptotically non-normal.

A helpful way to write the equation is the so-called Dickey-Fuller reparameterization:

$$\Delta Y_t = \mu + \alpha_0 Y_{t-1} + \alpha_1 \Delta Y_{t-1} + \cdots + \alpha_{k-1} \Delta Y_{t-(k-1)} + e_t. \tag{12.7}$$

These models are equivalent linear transformations of one another. The DF parameterization is convenient because the parameter α_0 summarizes the information about the unit root, since $\rho(1) = -\alpha_0$. To see this, observe that the lag polynomial for the Y_t computed from (12.7) is

$$(1 - L) - \alpha_0 L - \alpha_1 (L - L^2) - \cdots - \alpha_{k-1}(L^{k-1} - L^k)$$

But this must equal $\rho(L)$, as the models are equivalent. Thus

$$\rho(1) = (1 - 1) - \alpha_0 - (1 - 1) - \cdots - (1 - 1) = -\alpha_0.$$

Hence, the hypothesis of a unit root in Y_t can be stated as

$$H_0 : \alpha_0 = 0.$$

Note that the model is stationary if $\alpha_0 < 0$. So the natural alternative is

$$H_1 : \alpha_0 < 0.$$

Under H_0, the model for Y_t is

$$\Delta Y_t = \mu + \alpha_1 \Delta Y_{t-1} + \cdots + \alpha_{k-1} \Delta Y_{t-(k-1)} + e_t,$$

which is an AR(k-1) in the first-difference ΔY_t. Thus if Y_t has a (single) unit root, then ΔY_t is a stationary AR process. Because of this property, we say that if Y_t is non-stationary but $\Delta^d Y_t$ is stationary, then Y_t is "integrated of order d", or $I(d)$. Thus a time series with unit root is $I(1)$.

Since α_0 is the parameter of a linear regression, the natural test statistic is the t-statistic for H_0 from OLS estimation of (12.7). Indeed, this is the most popular unit root test, and is called the Augmented Dickey-Fuller (ADF) test for a unit root.

It would seem natural to assess the significance of the ADF statistic using the normal table. However, under H_0, Y_t is non-stationary, so conventional normal asymptotics are invalid. An alternative asymptotic framework has been developed to deal with non-stationary data. We do not have the time to develop this theory in detail, but simply assert the main results.

Theorem 12.12.1 *(Dickey-Fuller Theorem). Assume $\alpha_0 = 0$. As $T \to \infty$,*

$$T\hat{\alpha}_0 \to_d (1 - \alpha_1 - \alpha_2 - \cdots - \alpha_{k-1})\, DF_\alpha$$

$$ADF = \frac{\hat{\alpha}_0}{s(\hat{\alpha}_0)} \to DF_t.$$

The limit distributions DF_α and DF_t are non-normal. They are skewed to the left, and have negative means.

The first result states that $\hat{\alpha}_0$ converges to its true value (of zero) at rate T, rather than the conventional rate of $T^{1/2}$. This is called a "super-consistent" rate of convergence.

The second result states that the t-statistic for $\hat{\alpha}_0$ converges to a limit distribution which is non-normal, but does not depend on the parameters α. This distribution has been extensively tabulated, and may be used for testing the hypothesis H_0. Note: The standard error $s(\hat{\alpha}_0)$ is the conventional ("homoskedastic") standard error. But the theorem does not require an assumption of homoskedasticity. Thus the Dickey-Fuller test is robust to heteroskedasticity.

Since the alternative hypothesis is one-sided, the ADF test rejects H_0 in favor of H_1 when $ADF < c$, where c is the critical value from the ADF table. If the test rejects H_0, this means that the evidence points to Y_t being stationary. If the test does not reject H_0, a common conclusion is that the data suggests that Y_t is non-stationary. This is not really a correct conclusion, however. All we can say is that there is insufficient evidence to conclude whether the data is stationary or not.

We have described the test for the setting of with an intercept. Another popular setting includes as well a linear time trend. This model is

$$\Delta Y_t = \mu_1 + \mu_2 t + \alpha_0 Y_{t-1} + \alpha_1 \Delta Y_{t-1} + \cdots + \alpha_{k-1} \Delta Y_{t-(k-1)} + e_t. \tag{12.8}$$

This is natural when the alternative hypothesis is that the series is stationary about a linear time trend. If the series has a linear trend (e.g. GDP, Stock Prices), then the series itself is non-stationary, but it may be stationary around the linear time trend. In this context, it is a silly waste of time to fit an AR model to the level of the series without a time trend, as the AR model cannot conceivably describe this data. The natural solution is to include a time trend in the fitted OLS equation. When conducting the ADF test, this means that it is computed as the t-ratio for α_0 from OLS estimation of (12.8).

If a time trend is included, the test procedure is the same, but different critical values are required. The ADF test has a different distribution when the time trend has been included, and a different table should be consulted.

Most texts include as well the critical values for the extreme polar case where the intercept has been omitted from the model. These are included for completeness (from a pedagogical perspective) but have no relevance for empirical practice where intercepts are always included.

Chapter 13

Multivariate Time Series

A multivariate time series Y_t is a vector process $m \times 1$. Let $I_{t-1} = (Y_{t-1}, Y_{t-2}, ...)$ be all lagged information at time t. The typical goal is to find the conditional expectation $E(Y_t \mid I_{t-1})$. Note that since Y_t is a vector, this conditional expectation is also a vector.

13.1 Vector Autoregressions (VARs)

A VAR model specifies that the conditional mean is a function of only a finite number of lags:

$$E(Y_t \mid I_{t-1}) = E(Y_t \mid Y_{t-1}, ..., Y_{t-k}).$$

A linear VAR specifies that this conditional mean is linear in the arguments:

$$E(Y_t \mid Y_{t-1}, ..., Y_{t-k}) = A_0 + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots A_k Y_{t-k}.$$

Observe that A_0 is $m \times 1$,and each of A_1 through A_k are $m \times m$ matrices.

Defining the $m \times 1$ regression error

$$e_t = Y_t - E(Y_t \mid I_{t-1}),$$

we have the VAR model

$$\begin{aligned} Y_t &= A_0 + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots A_k Y_{t-k} + e_t \\ E(e_t \mid I_{t-1}) &= 0. \end{aligned}$$

Alternatively, defining the $mk + 1$ vector

$$x_t = \begin{pmatrix} 1 \\ Y_{t-1} \\ Y_{t-2} \\ \vdots \\ Y_{t-k} \end{pmatrix}$$

and the $m \times (mk+1)$ matrix

$$A = \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_k \end{pmatrix},$$

then

$$Y_t = Ax_t + e_t.$$

The VAR model is a system of m equations. One way to write this is to let a_j' be the jth row of A. Then the VAR system can be written as the equations

$$Y_{jt} = a_j' x_t + e_{jt}.$$

Unrestricted VARs were introduced to econometrics by Sims (1980).

13.2 Estimation

Consider the moment conditions

$$E\left(x_t e_{jt}\right) = 0,$$

$j = 1, ..., m$. These are implied by the VAR model, either as a regression, or as a linear projection.

The GMM estimator corresponding to these moment conditions is equation-by-equation OLS

$$\hat{a}_j = (X'X)^{-1} X'Y_j.$$

An alternative way to compute this is as follows. Note that

$$\hat{a}_j' = Y_j' X (X'X)^{-1}.$$

And if we stack these to create the estimate $\hat{A}$, we find

$$\begin{aligned} \hat{A} &= \begin{pmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_{m+1}' \end{pmatrix} X (X'X)^{-1} \\ &= Y'X(X'X)^{-1}, \end{aligned}$$

where

$$Y = \begin{pmatrix} Y_1 & Y_2 & \cdots & Y_m \end{pmatrix}$$

the $T \times m$ matrix of the stacked y_t'.

This (system) estimator is known as the SUR (Seemingly Unrelated Regressions) estimator, and was originally derived by Zellner (1962)

13.3 Restricted VARs

The unrestricted VAR is a system of m equations, each with the same set of regressors. A restricted VAR imposes restrictions on the system. For example, some regressors may be excluded from some of the equations. Restrictions may be imposed on individual equations, or across equations. The GMM framework gives a convenient method to impose such restrictions on estimation.

13.4 Single Equation from a VAR

Often, we are only interested in a single equation out of a VAR system. This takes the form

$$Y_{jt} = a_j' x_t + e_t,$$

and x_t consists of lagged values of Y_{jt} and the other $Y_{lt}'s$. In this case, it is convenient to re-define the variables. Let $y_t = Y_{jt}$, and Z_t be the other variables. Let $e_t = e_{jt}$ and $\beta = a_j$. Then the single equation takes the form

$$y_t = x_t'\beta + e_t, \tag{13.1}$$

and

$$x_t = \left[\begin{pmatrix} 1 & Y_{t-1} & \cdots & Y_{t-k} & Z_{t-1}' & \cdots & Z_{t-k}' \end{pmatrix}'\right].$$

This is just a conventional regression, with time series data.

13.5 Testing for Omitted Serial Correlation

Consider the problem of testing for omitted serial correlation in equation (13.1). Suppose that e_t is an AR(1). Then

$$\begin{aligned} y_t &= x_t'\beta + e_t \\ e_t &= \theta e_{t-1} + u_t \\ E\left(u_t \mid I_{t-1}\right) &= 0. \end{aligned} \tag{13.2}$$

Then the null and alternative are

$$H_0 : \theta = 0 \qquad H_1 : \theta \neq 0.$$

Take the equation $y_t = x_t'\beta + e_t$, and subtract off the equation once lagged multiplied by θ, to get

$$\begin{aligned} y_t - \theta y_{t-1} &= \left(x_t'\beta + e_t\right) - \theta\left(x_{t-1}'\beta + e_{t-1}\right) \\ &= x_t'\beta - \theta x_{t-1}\beta + e_t - \theta e_{t-1}, \end{aligned}$$

or

$$y_t = \theta y_{t-1} + x_t'\beta + x_{t-1}'\gamma + u_t, \tag{13.3}$$

which is a valid regression model.

So testing H_0 versus H_1 is equivalent to testing for the significance of adding (y_{t-1}, x_{t-1}) to the regression. This can be done by a Wald test. We see that an appropriate, general, and simple way to test for omitted serial correlation is to test the significance of extra lagged values of the dependent variable and regressors.

You may have heard of the Durbin-Watson test for omitted serial correlation, which once was very popular, and is still routinely reported by conventional regression packages. The DW test is appropriate only when regression $y_t = x_t'\beta + e_t$ is not dynamic (has no lagged values on the RHS), and e_t is iid $N(0,1)$. Otherwise it is invalid.

Another interesting fact is that (13.2) is a special case of (13.3), under the restriction $\gamma = -\beta\theta$. This restriction, which is called a common factor restriction, may be tested if desired. If valid, the model (13.2) may be estimated by iterated GLS. (A simple version of this estimator is called Cochrane-Orcutt.) Since the common factor restriction appears arbitrary, and is typically rejected empirically, direct estimation of (13.2) is uncommon in recent applications.

13.6 Selection of Lag Length in an VAR

If you want a data-dependent rule to pick the lag length k in a VAR, you may either use a testing-based approach (using, for example, the Wald statistic), or an information criterion approach. The formula for the AIC and BIC are

$$\begin{aligned} AIC(k) &= \log\det\left(\hat{\Omega}(k)\right) + 2\frac{p}{T} \\ BIC(k) &= \log\det\left(\hat{\Omega}(k)\right) + \frac{p\log(T)}{T} \\ \hat{\Omega}(k) &= \frac{1}{T}\sum_{t=1}^{T}\hat{e}_t(k)\hat{e}_t(k)' \\ p &= m(km+1) \end{aligned}$$

where p is the number of parameters in the model, and $\hat{e}_t(k)$ is the OLS residual vector from the model with k lags. The log determinant is the criterion from the multivariate normal likelihood.

13.7 Granger Causality

Partition the data vector into (Y_t, Z_t). Define the two information sets

$$\begin{aligned} I_{1t} &= (Y_t, Y_{t-1}, Y_{t-2}, ...) \\ I_{2t} &= (Y_t, Z_t, Y_{t-1}, Z_{t-1}, Y_{t-2}, Z_{t-2}, , ...) \end{aligned}$$

The information set I_{1t} is generated only by the history of Y_t, and the information set I_{2t} is generated by both Y_t and Z_t. The latter has more information.

We say that Z_t does not *Granger-cause* Y_t if

$$E\left(Y_t \mid I_{1,t-1}\right) = E\left(Y_t \mid I_{2,t-1}\right).$$

That is, conditional on information in lagged Y_t, lagged Z_t does not help to forecast Y_t. If this condition does not hold, then we say that Z_t Granger-causes Y_t.

The reason why we call this "Granger Causality" rather than "causality" is because this is not a physical or structure definition of causality. If Z_t is some sort of forecast of the future, such as a futures price, then Z_t may help to forecast Y_t even though it does not "cause" Y_t. This definition of causality was developed by Granger (1969) and Sims (1972).

In a linear VAR, the equation for Y_t is

$$Y_t = \alpha + \rho_1 Y_{t-1} + \cdots + \rho_k Y_{t-k} + Z_{t-1}'\gamma_1 + \cdots + Z_{t-k}'\gamma_k + e_t.$$

In this equation, Z_t does not Granger-cause Y_t if and only if

$$H_0 : \gamma_1 = \gamma_2 = \cdots = \gamma_k = 0.$$

This may be tested using an exclusion (Wald) test.

This idea can be applied to blocks of variables. That is, Y_t and/or Z_t can be vectors. The hypothesis can be tested by using the appropriate multivariate Wald test.

If it is found that Z_t does not Granger-cause Y_t, then we deduce that our time-series model of $E\left(Y_t \mid I_{t-1}\right)$ does not require the use of Z_t. Note, however, that Z_t may still be useful to explain other features of Y_t, such as the conditional variance.

13.8 Cointegration

The idea of cointegration is due to Granger (1981), and was articulated in detail by Engle and Granger (1987).

Definition 13.8.1 *The $m \times 1$ series Y_t is cointegrated if Y_t is $I(1)$ yet there exists β, $m \times r$, of rank r, such that $z_t = \beta' Y_t$ is $I(0)$. The r vectors in β are called the cointegrating vectors.*

If the series Y_t is not cointegrated, then $r = 0$. If $r = m$, then Y_t is $I(0)$. For $0 < r < m$, Y_t is $I(1)$ and cointegrated.

In some cases, it may be believed that β is known a priori. Often, $\beta = (1 \quad -1)'$. For example, if Y_t is a pair of interest rates, then $\beta = (1 \quad -1)'$ specifies that the spread (the difference in returns) is stationary. If $Y = (\log(Consumption) \quad \log(Income))'$, then $\beta = (1 \quad -1)'$ specifies that $\log(Consumption/Income)$ is stationary.

In other cases, β may not be known.

If Y_t is cointegrated with a single cointegrating vector ($r = 1$), then it turns out that β can be consistently estimated by an OLS regression of one component of Y_t on the others. Thus $Y_t =$

(Y_{1t}, Y_{2t}) and $\beta = (\beta_1\ \beta_2)$ and normalize $\beta_1 = 1$. Then $\hat{\beta}_2 = (Y_2'Y_2)^{-1}Y_2Y_1 \to_p \beta_2$. Furthermore this estimation is super-consistent: $T(\hat{\beta}_2 - \beta_2) \to_d Limit$, as first shown by Stock (1987). This is not, in general, a good method to estimate β, but it is useful in the construction of alternative estimators and tests.

We are often interested in testing the hypothesis of no cointegration:

$$\begin{aligned} H_0 &: r = 0 \\ H_1 &: r > 0. \end{aligned}$$

Suppose that β is known, so $z_t = \beta' Y_t$ is known. Then under H_0 z_t is $I(1)$, yet under H_1 z_t is $I(0)$. Thus H_0 can be tested using a univariate ADF test on z_t.

When β is unknown, Engle and Granger (1987) suggested using an ADF test on the estimated residual $\hat{z}_t = \hat{\beta}' Y_t$, from OLS of Y_{1t} on Y_{2t}. Their justification was Stock's result that $\hat{\beta}$ is super-consistent under H_1. Under H_0, however, $\hat{\beta}$ is not consistent, so the ADF critical values are not appropriate. The asymptotic distribution was worked out by Phillips and Ouliaris (1990).

When the data have time trends, it may be necessary to include a time trend in the estimated cointegrating regression. Whether or not the time trend is included, the asymptotic distribution of the test is affected by the presence of the time trend. The asymptotic distribution was worked out in B. Hansen (1992).

13.9 Cointegrated VARs

We can write a VAR as

$$\begin{aligned} A(L)Y_t &= e_t \\ A(L) &= I - A_1 L - A_2 L^2 - \cdots - A_k L^k \end{aligned}$$

or alternatively as

$$\Delta Y_t = \Pi Y_{t-1} + D(L)\Delta Y_{t-1} + e_t$$

where

$$\begin{aligned} \Pi &= -A(1) \\ &= -I + A_1 + A_2 + \cdots + A_k. \end{aligned}$$

Theorem 13.9.1 *(Granger Representation Theorem). Y_t is cointegrated with $m \times r$ β if and only if $rank(\Pi) = r$ and $\Pi = \alpha\beta'$ where α is $m \times r$, $rank(\alpha) = r$.*

Thus cointegration imposes a restriction upon the parameters of a VAR. The restricted model can be written as

$$\begin{aligned} \Delta Y_t &= \alpha\beta' Y_{t-1} + D(L)\Delta Y_{t-1} + e_t \\ \Delta Y_t &= \alpha z_{t-1} + D(L)\Delta Y_{t-1} + e_t. \end{aligned}$$

If β is known, this can be estimated by OLS of ΔY_t on z_{t-1} and the lags of ΔY_t.

If β is unknown, then estimation is done by "reduced rank regression", which is least-squares subject to the stated restriction. Equivalently, this is the MLE of the restricted parameters under the assumption that e_t is iid $N(0, \Omega)$.

One difficulty is that β is not identified without normalization. When $r = 1$, we typically just normalize one element to equal unity. When $r > 1$, this does not work, and different authors have adopted different identification schemes.

In the context of a cointegrated VAR estimated by reduced rank regression, it is simple to test for cointegration by testing the rank of Π. These tests are constructed as likelihood ratio (LR) tests. As they were discovered by Johansen (1988, 1991, 1995), they are typically called the "Johansen Max and Trace" tests. Their asymptotic distributions are non-standard, and are similar to the Dickey-Fuller distributions.

Chapter 14

Limited Dependent Variables

A "limited dependent variable" Y is one which takes a "limited" set of values. The most common cases are

- Binary: $Y = \{0, 1\}$
- Multinomial: $Y = \{0, 1, 2, ..., k\}$
- Integer: $Y = \{0, 1, 2, ...\}$
- Censored: $Y = \{x : x \geq 0\}$

The traditional approach to the estimation of limited dependent variable (LDV) models is parametric maximum likelihood. A parametric model is constructed, allowing the construction of the likelihood function. A more modern approach is semi-parametric, eliminating the dependence on a parametric distributional assumption. We will discuss only the first (parametric) approach, due to time constraints. They still constitute the majority of LDV applications. If, however, you were to write a thesis involving LDV estimation, you would be advised to considered employing a semi-parametric estimation approach.

For the parametric approach, estimation is by MLE. A major practical issue is construction of the likelihood function.

14.1 Binary Choice

The dependent variable $Y_i = \{0, 1\}$. This represents a Yes/No outcome. Given some regressors x_i, the goal is to describe $P(Y_i = 1 \mid x_i)$, as this is the full conditional distribution.

The linear probability model specifies that

$$P(Y_i = 1 \mid x_i) = x_i'\beta.$$

As $P(Y_i = 1 \mid x_i) = E(Y_i \mid x_i)$, this yields the regression: $Y_i = x_i'\beta + e_i$ which can be estimated by OLS. However, the linear probability model does not impose the restriction that $0 \leq P(Y_i \mid x_i) \leq 1$. Even so estimation of a linear probability model is a useful starting point for subsequent analysis.

The standard alternative is to use a function of the form

$$P(Y_i = 1 \mid x_i) = F\left(x_i'\beta\right)$$

where $F(\cdot)$ is a known CDF, typically assumed to be symmetric about zero, so that $F(z) = 1 - F(-z)$. The two standard choices for F are

- Logistic: $F(u) = (1 + e^{-u})^{-1}$.
- Normal: $F(u) = \Phi(u)$.

If F is logistic, we call this the *logit* model, and if F is normal, we call this the *probit* model.

This model is identical to the latent variable model

$$\begin{array}{rcl} Y_i^* & = & x_i'\beta + e_i \\ e_i & \sim & F(\cdot) \\ Y_i & = & \begin{cases} 1 & \text{if } Y_i^* > 0 \\ 0 & \text{otherwise} \end{cases} . \end{array}$$

For then

$$\begin{array}{rcl} P(Y_i = 1 \mid x_i) & = & P(Y_i^* > 0 \mid x_i) \\ & = & P\left(x_i'\beta + e_i > 0 \mid x_i\right) \\ & = & P\left(e_i > -x_i'\beta \mid x_i\right) \\ & = & 1 - F\left(-x_i'\beta\right) \\ & = & F\left(x_i'\beta\right). \end{array}$$

Estimation is by maximum likelihood. To construct the likelihood, we need the conditional distribution of an individual observation. Recall that if Y is Bernoulli, such that $P(Y = 1) = p$ and $P(Y = 0) = 1 - p$, then we can write the density of Y as

$$f(y) = p^y(1-p)^{1-y}, \qquad y = 0, 1.$$

In the Binary choice model, Y_i is conditionally Bernoulli with $P(Y_i = 1 \mid x_i) = p_i = F(x_i'\beta)$. Thus the conditional density is

$$\begin{array}{rcl} f(y_i \mid x_i) & = & p_i^{y_i}(1-p_i)^{1-y_i} \\ & = & F\left(x_i'\beta\right)^{y_i}\left(1 - F\left(x_i'\beta\right)\right)^{1-y_i}. \end{array}$$

Hence the log-likelihood function is

$$\begin{aligned} l_n(\beta) &= \sum_{i=1}^{n} \log f(y_i \mid x_i) \\ &= \sum_{i=1}^{n} \log \left(F\left(x_i'\beta\right)^{y_i} \left(1 - F\left(x_i'\beta\right)\right)^{1-y_i} \right) \\ &= \sum_{i=1}^{n} \left[y_i \log F\left(x_i'\beta\right) + (1 - y_i) \log(1 - F\left(x_i'\beta\right)) \right] \\ &= \sum_{y_i=1} \log F\left(x_i'\beta\right) + \sum_{y_i=0} \log(1 - F\left(x_i'\beta\right)). \end{aligned}$$

The MLE $\hat{\beta}$ is the value of β which maximizes $l_n(\beta)$. Standard errors and test statistics are computed by asymptotic approximations. Details of such calculations are left to more advanced courses.

14.2 Count Data

If $Y = \{0, 1, 2, ...\}$, a typical approach is to employ *Poisson regression.* This model specifies that

$$\begin{aligned} P\left(Y_i = k \mid x_i\right) &= \frac{\exp\left(-\lambda_i\right) \lambda_i^k}{k!}, \qquad k = 0, 1, 2, ... \\ \lambda_i &= \exp(x_i'\beta). \end{aligned}$$

The conditional density is the Poisson with parameter λ_i. The functional form for λ_i has been picked to ensure that $\lambda_i > 0$.

The log-likelihood function is

$$l_n(\beta) = \sum_{i=1}^{n} \log f(y_i \mid x_i) = \sum_{i=1}^{n} \left(-\exp(x_i'\beta) + y_i x_i'\beta - \log(y_i!) \right).$$

The MLE is the value $\hat{\beta}$ which maximizes $l_n(\beta)$.

Since

$$E\left(Y_i \mid x_i\right) = \lambda_i = \exp(x_i'\beta)$$

is the conditional mean, this motivates the label Poisson "regression."

Also observe that the model implies that

$$Var\left(Y_i \mid x_i\right) = \lambda_i = \exp(x_i'\beta),$$

so the model imposes the restriction that the conditional mean and variance of Y_i are the same. This may be considered restrictive. A generalization is the negative binomial.

14.3 Censored Data

The idea of "censoring" is that some data above or below a threshold are mis-reported at the threshold. Thus the model is that there is some latent process y_i^* with unbounded support, but we observe only

$$y_i = \begin{cases} y_i^* & \text{if } y_i \geq 0 \\ 0 & \text{if } y_i < 0 \end{cases} . \tag{14.1}$$

(This is written for the case of the threshold being zero, any known value can substitute.) The observed data y_i therefore come from a mixed continuous/discrete distribution.

Censored models are typically applied when the data set has a meaningful proportion (say 5% or higher) of data at the boundary of the sample support. The censoring process may be explicit in data collection, or it may be a by-product of economic constraints.

An example of a data collection censoring is top-coding of income. In surveys, incomes above a threshold are typically reported at the threshold.

The first censored regression model was developed by Tobin (1958) to explain consumption of durable goods. Tobin observed that for many households, the consumption level (purchases) in a particular period was zero. He proposed the latent variable model

$$\begin{aligned} y_i^* &= x_i'\beta + e_i \\ e_i &\sim iid\ N(0, \sigma^2) \end{aligned}$$

with the observed variable y_i generated by the censoring equation (14.1). This model (now called the Tobit) specifies that the latent (or ideal) value of consumption may be negative (the household would prefer to sell than buy). All that is reported is that the household purchased zero units of the good.

The naive approach to estimate β is to regress y_i on x_i. This does not work because regression estimates $E(Y_i \mid x_i)$, not $E(Y_i^* \mid x_i) = x_i'\beta$, and the latter is of interest. Thus OLS will be biased for the parameter of interest β.

[Note: it is still possible to estimate $E(Y_i \mid x_i)$ by LS techniques. The Tobit framework postulates that this is not inherently interesting, that the parameter of β is defined by an alternative statistical structure.]

Consistent estimation will be achieved by the MLE. To construct the likelihood, observe that the probability of being censored is

$$\begin{aligned} P(y_i = 0 \mid x_i) &= P(y_i^* < 0 \mid x_i) \\ &= P\left(x_i'\beta + e_i < 0 \mid x_i\right) \\ &= P\left(\frac{e_i}{\sigma} < -\frac{x_i'\beta}{\sigma} \mid x_i\right) \\ &= \Phi\left(-\frac{x_i'\beta}{\sigma}\right). \end{aligned}$$

The conditional distribution function above zero is Gaussian:

$$P\left(y_i = y \mid x_i\right) = \int_0^y \sigma^{-1}\phi\left(\frac{z - x_i'\beta}{\sigma}\right) dz, \qquad y > 0.$$

Therefore, the density function can be written as

$$f\left(y \mid x_i\right) = \Phi\left(-\frac{x_i'\beta}{\sigma}\right)^{1(y=0)} \left[\sigma^{-1}\phi\left(\frac{z - x_i'\beta}{\sigma}\right)\right]^{1(y>0)},$$

where $1\left(\cdot\right)$ is the indicator function.

Hence the log-likelihood is a mixture of the probit and the normal:

$$\begin{aligned} l_n(\beta) &= \sum_{i=1}^n \log f(y_i \mid x_i) \\ &= \sum_{y_i=0} \log \Phi\left(-\frac{x_i'\beta}{\sigma}\right) + \sum_{y_i>0} \log\left[\sigma^{-1}\phi\left(\frac{y_i - x_i'\beta}{\sigma}\right)\right]. \end{aligned}$$

The MLE is the value $\hat{\beta}$ which maximizes $l_n(\beta)$.

14.4 Sample Selection

The problem of sample selection arises when the sample is a non-random selection of potential observations. This occurs when the observed data is systematically different from the population of interest. For example, if you ask for volunteers for an experiment, and they wish to extrapolate the effects of the experiment on a general population, you should worry that the people who volunteer may be systematically different from the general population. This has great relevance for the evaluation of anti-poverty and job-training programs, where the goal is to assess the effect of "training" on the general population, not just on the volunteers.

A simple sample selection model can be written as the latent model

$$\begin{aligned} y_i &= x_i'\beta + e_{1i} \\ T_i &= 1\left(z_i'\gamma + e_{0i} > 0\right) \end{aligned}$$

where $1\left(\cdot\right)$ is the indicator function. The dependent variable y_i is observed if (and only if) $T_i = 1$. Else it is unobserved.

For example, y_i could be a wage, which can be observed only if a person is employed. The equation for T_i is an equation specifying the probability that the person is employed.

The model is often completed by specifying that the errors are jointly normal

$$\begin{pmatrix} e_{0i} \\ e_{1i} \end{pmatrix} \sim N\left(0, \begin{pmatrix} 1 & \rho \\ \rho & \sigma^2 \end{pmatrix}\right).$$

It is presumed that we observe $\{x_i, z_i, T_i\}$ for all observations.

Under the normality assumption,

$$e_{1i} = \rho e_{0i} + v_i,$$

where v_i is independent of $e_{0i} \sim N(0,1)$. A useful fact about the standard normal distribution is that

$$E\left(e_{0i} \mid e_{0i} > -x\right) = \lambda(x) = \frac{\phi(x)}{\Phi(x)},$$

and the function $\lambda(x)$ is called the inverse Mills ratio.

The naive estimator of β is OLS regression of y_i on x_i for those observations for which y_i is available. The problem is that this is equivalent to conditioning on the event $\{T_i = 1\}$. However,

$$\begin{aligned} E\left(e_{1i} \mid T_i = 1, Z_i\right) &= E\left(e_{1i} \mid \{e_{0i} > -z_i'\gamma\}, Z_i\right) \\ &= \rho E\left(e_{0i} \mid \{e_{0i} > -z_i'\gamma\}, Z_i\right) + E\left(v_i \mid \{e_{0i} > -z_i'\gamma\}, Z_i\right) \\ &= \rho\lambda\left(z_i'\gamma\right), \end{aligned}$$

which is non-zero. Thus

$$e_{1i} = \rho\lambda\left(z_i'\gamma\right) + u_i,$$

where

$$E\left(u_i \mid T_i = 1, Z_i\right) = 0.$$

Hence

$$y_i = x_i'\beta + \rho\lambda\left(z_i'\gamma\right) + u_i \tag{14.2}$$

is a valid regression equation for the observations for which $T_i = 1$.

Heckman (1979) observed that we could consistently estimate β and ρ from this equation, if γ were known. It is unknown, but also can be consistently estimated by a Probit model for selection. The "Heckit" estimator is thus calculated as follows

- Estimate $\hat{\gamma}$ from a Probit, using regressors z_i. The binary dependent variable is T_i.
- Estimate $\left(\hat{\beta}, \hat{\rho}\right)$ from OLS of y_i on x_i and $\lambda(z_i'\hat{\gamma})$.
- The OLS standard errors will be incorrect, as this is a two-step estimator. They can be corrected using a more complicated formula. Or, alternatively, by viewing the Probit/OLS estimation equations as a large joint GMM problem.

The Heckit estimator is frequently used to deal with problems of sample selection. However, the estimator is built on the assumption of normality, and the estimator can be quite sensitive to this assumption. Some modern econometric research is exploring how to relax the normality assumption.

The estimator can also work quite poorly if $\lambda(z_i'\hat{\gamma})$ does not have much in-sample variation. This can happen if the Probit equation does not "explain" much about the selection choice. Another potential problem is that if $z_i = x_i$, then $\lambda(z_i'\hat{\gamma})$ can be highly collinear with x_i, so the second step OLS estimator will not be able to precisely estimate β. Based this observation, it is typically recommended to find a valid exclusion restriction: a variable should be in z_i which is not in x_i. If this is valid, it will ensure that $\lambda(z_i'\hat{\gamma})$ is not collinear with x_i, and hence improve the second stage estimator's precision.

Chapter 15

Panel Data

A panel is a set of observations on individuals, collected over time. An observation is the pair $\{y_{it}, x_{it}\}$, where the i subscript denotes the individual, and the t subscript denotes time. A panel may be *balanced*:

$$\{y_{it}, x_{it}\} : t = 1, ..., T; \quad i = 1, ..., n,$$

or *unbalanced*:

$$\{y_{it}, x_{it}\} : \text{For } i = 1, ..., n, \quad t = \underline{t}_i, ..., \overline{t}_i.$$

15.1 Individual-Effects Model

The standard panel data specification is that there is an individual-specific effect which enters linearly in the regression

$$y_{it} = x_{it}'\beta + u_i + e_{it}.$$

The typical maintained assumptions are that the individuals i are mutually independent, that u_i and e_{it} are independent, that e_{it} is iid across individuals and time, and that e_{it} is uncorrelated with x_{it}.

OLS of y_{it} on x_{it} is called pooled estimation. It is consistent if

$$E(x_{it}u_i) = 0 \tag{15.1}$$

If this condition fails, then OLS is inconsistent. (15.1) fails if the individual-specific unobserved effect u_i is correlated with the observed explanatory variables x_{it}. This is often believed to be plausible if u_i is an omitted variable.

If (15.1) is true, however, OLS can be improved upon via a GLS technique. In either event, OLS appears a poor estimation choice.

Condition (15.1) is called the *random effects hypothesis.* It is a strong assumption, and most applied researchers try to avoid its use.

15.2 Fixed Effects

This is the most common technique for estimation of non-dynamic linear panel regressions.

The motivation is to allow u_i to be arbitrary, and have arbitrary correlated with x_i. The goal is to eliminate u_i from the estimator, and thus achieve invariance.

There are several derivations of the estimator.

First, let

$$d_{ij} = \begin{cases} 1 & \text{if } i = j \\ \\ 0 & \text{else} \end{cases},$$

and

$$d_i = \begin{pmatrix} d_{i1} \\ \vdots \\ d_{in} \end{pmatrix},$$

an $n \times 1$ dummy vector with a "1" in the $i'th$ place. Let

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

Then note that

$$u_i = d_i' u,$$

and

$$y_{it} = x_{it}'\beta + d_i' u + e_{it}. \tag{15.2}$$

Observe that

$$E\left(e_{it} \mid x_{it}, d_i\right) = 0,$$

so (15.2) is a valid regression, with d_i as a regressor along with x_i.

OLS on (15.2) yields estimator $(\hat{\beta}, \hat{u})$. Conventional inference applies.

Observe that

- This is generally consistent.
- If x_{it} contains an intercept, it will be collinear with d_i, so the intercept it typically omitted from x_{it}.
- Any regressor in x_{it} which is constant over time for all individuals (e.g., their gender) will be collinear with d_i, so will have to be omitted.
- There are $n + k$ regression parameters, which is quite large as typically n is very large.

Computationally, you do not want to actually implement conventional OLS estimation, as the parameter space is too large. OLS estimation of β proceeds by the FWL theorem. Stacking the observations together:

$$Y = X\beta + Du + e,$$

then by the FWL theorem,

$$\begin{aligned}\hat{\beta} &= \left(X'\left(1 - P_D\right)X\right)^{-1}\left(X'\left(1 - P_D\right)Y\right) \\ &= \left(X^{*\prime}X^{*}\right)^{-1}\left(X^{*\prime}Y^{*}\right),\end{aligned}$$

where

$$\begin{aligned}Y^{*} &= Y - D(D'D)^{-1}D'Y \\ X^{*} &= X - D(D'D)^{-1}D'X.\end{aligned}$$

Since the regression of y_{it} on d_i is a regression onto individual-specific dummies, the predicted value from these regressions is the individual specific mean $\overline{y}_i$, and the residual is the demean value

$$y_{it}^{*} = y_{it} - \overline{y}_i.$$

The fixed effects estimator $\hat{\beta}$ is OLS of y_{it}^{*} on x_{it}^{*}, the dependent variable and regressors in deviation-from mean form.

Another derivation of the estimator is to take the equation

$$y_{it} = x_{it}'\beta + u_i + e_{it},$$

and then take individual-specific means by taking the average for the $i'th$ individual:

$$\frac{1}{T_i}\sum_{t=\underline{t}_i}^{\overline{t}_i} y_{it} = \frac{1}{T_i}\sum_{t=\underline{t}_i}^{\overline{t}_i} x_{it}'\beta + u_i + \frac{1}{T_i}\sum_{t=\underline{t}_i}^{\overline{t}_i} e_{it}$$

or

$$\overline{y}_i = \overline{x}_i'\beta + u_i + \overline{e}_i.$$

Subtracting, we find

$$y_{it}^{*} = x_{it}^{*\prime}\beta + e_{it}^{*},$$

which is free of the individual-effect u_i.

15.3 Dynamic Panel Regression

A dynamic panel regression has a lagged dependent variable

$$y_{it} = \alpha y_{it-1} + x_{it}'\beta + u_i + e_{it}. \tag{15.3}$$

This is a model suitable for studying dynamic behavior of individual agents.

Unfortunately, the fixed effects estimator is inconsistent, at least if T is held finite as $n \to \infty$. This is because the sample mean of y_{it-1} is correlated with that of e_{it}.

The standard approach to estimate a dynamic panel is to combine first-differencing with IV or GMM. Taking first-differences of (15.3) eliminates the individual-specific effect:

$$\Delta y_{it} = \alpha \Delta y_{it-1} + \Delta x_{it}' \beta + \Delta e_{it}. \tag{15.4}$$

However, if e_{it} is iid, then it will be correlated with Δy_{it-1} :

$$E\left(\Delta y_{it-1} \Delta e_{it}\right) = E\left(\left(y_{it-1} - y_{it-2}\right)\left(e_{it} - e_{it-1}\right)\right) = -E\left(y_{it-1} e_{it-1}\right) = -\sigma_e^2.$$

So OLS on (15.4) will be inconsistent.

But if there are valid instruments, then IV or GMM can be used to estimate the equation. Typically, we use lags of the dependent variable, two periods back, as y_{t-2} is uncorrelated with Δe_{it}. Thus values of y_{it-k}, $k \geq 2$, are valid instruments.

Hence a valid estimator of α and β is to estimate (15.4) by IV using y_{t-2} as an instrument for Δy_{t-1} (which is just identified). Alternatively, GMM using y_{t-2} and y_{t-3} as instruments (which is overidentified, but loses a time-series observation).

A more sophisticated GMM estimator recognizes that for time-periods later in the sample, there are more instruments available, so the instrument list should be different for each equation. This is conveniently organized by the GMM principle, as this enables the moments from the different time-periods to be stacked together to create a list of all the moment conditions. A simple application of GMM yields the parameter estimates and standard errors.

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