ASYMPTOTIC NORMALITY OF THE QMLE ESTIMATOR OF ARCH IN THE NONSTATIONARY CASE

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We establish consistency and asymptotic normality of the quasi-maximum likelihood estimator in the linear ARCH model. Contrary to the existing literature, we allow the parameters to be in the region where no stationary version of the process exists. This implies that the estimator is always asymptotically normal.

KEYWORDS: ARCH, asymptotic normality, asymptotic theory, consistency, GARCH, nonstationarity, quasi-maximum likelihood estimation.

1. INTRODUCTION

Consider the first order ARCH (autoregressive conditional heteroscedastic) model introduced by Engle (1982), as given by

\[ y_t = \sigma_t z_t, \]
\[ \sigma_t^2 = \omega + \alpha \sigma_{t-1}^2, \]

for \( t = 1, \ldots, T, \alpha > 0, \omega > 0, \) and with \( z_t \) an i.i.d. \((0,1)\) process with

\[ V(z_t^2) = E(z_t^4 - 1) = \zeta < \infty. \]

Asymptotic inference for the ARCH(1) and more general ARCH models, including GARCH models, has been studied in, e.g., Weiss (1986), Lee and Hansen (1994), Lumsdaine (1996), and Kristensen and Rahbek (2002). These papers, as well as others in the econometric literature, all assume as a minimal requirement that the ARCH process \( y_t \) is suitably ergodic or stationary such that laws of large numbers apply. We relax the condition on the stability of the \( y_t \) process and allow it to be nonstationary and in particular not to have any moments. Our only condition is that \( V(z_t^2) < \infty \).

In the proofs we have aimed at a detailed presentation of the analysis of the score, information, and third derivatives of the likelihood function. We note that the arguments for the score and information in Lemma 3 and Lemma 4 are easily carried over to the stationary case by using the ergodic theorem similar to Lee and Hansen (1994) and Lumsdaine (1996). Moreover, we note that our result in Lemma 5 regarding the uniform boundedness of the third derivatives of the likelihood function does not use the nonstationary condition, and hence can be applied to the stationary case as well. As can be deduced from Remark 2, the established uniform boundedness corrects and significantly extends existing proofs of asymptotic normality for the ARCH(1) model.

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2. INFERENCE

By Nelson (1990) and Bougerol and Picard (1992), \( y_t \) is stationary (a stationary version exists) and ergodic if and only if \( E \log(\alpha z_t^2) < 0 \). In particular, if \( z_t \) is Gaussian, then the if and only if condition is that \( \alpha < \frac{1}{2} \exp(-\Psi(\frac{1}{2})) \simeq 3.56 \), where \( \Psi(\cdot) \) is the Euler psi function; see Nelson (1990).

As mentioned our analysis is under the assumption that \( y_t \) does not have a stationary version or equivalently,

\[
E \log(\alpha z_t^2) \geq 0.
\]

Consider the likelihood estimator based on maximizing the quasi likelihood

\[
\ell_T(\alpha) = -\frac{1}{2} \sum_{t=1}^{T} \left( \log \sigma_t^2 + \frac{y_t^2}{\sigma_t^2} \right)
\]

from which the QMLE (quasi-maximum likelihood estimator) \( \hat{\alpha} \) is found. Note that this is the true likelihood if \( z_t \) is Gaussian and that we are conditioning on the initial value \( y_0 \). Our main result is the following:

**THEOREM 1:** Assume that the ARCH process \( y_t \) in (1) does not allow a stationary version or equivalently (2) holds. Assume further that the iid \((0, 1)\) process \( z_t \) is such that \( V(z_t^2) = \zeta \) is finite, and the scale parameter \( \omega \) is known. Then as \( T \to \infty \) the sequence of QMLE \( \hat{\alpha} \) is consistent, and asymptotically normal,

\[
\sqrt{T}(\hat{\alpha} - \alpha) \xrightarrow{D} N(0, \sigma^2),
\]

where

\[
\sigma^2 = \zeta \alpha^2 > 0.
\]

**REMARK 1:** Note that if \( z_t \) is Gaussian, then \( \sigma^2 = 2\alpha^2 \) in Theorem 1.

**PROOF:** Together Lemmas 3, 4, and 5 in the next section establish the classical Cramér type conditions; see, e.g., Lehmann (1999). \( Q.E.D. \)

3. DERIVATION

For exposition and without loss of generality we henceforth set \( \omega = 1 \).

With the likelihood function given by (3), the score, information, and the third derivative of the log-likelihood with respect to \( \alpha \) are found to be given by

\[
\frac{\partial}{\partial \alpha} \ell_T(\alpha) = -\frac{1}{2} \sum_{t=1}^{T} \left( 1 - \frac{y_t^2}{\sigma_t^2} \right) \frac{y_t^2}{\sigma_t^2},
\]

\[
\frac{\partial^2}{\partial \alpha^2} \ell_T(\alpha) = \frac{1}{2} \sum_{t=1}^{T} \left( 1 - 2 \frac{y_t^2}{\sigma_t^2} \right) \frac{y_t^4}{\sigma_t^4},
\]

\[
\frac{\partial^3}{\partial \alpha^3} \ell_T(\alpha) = -\sum_{t=1}^{T} \left( 1 - 3 \frac{y_t^2}{\sigma_t^2} \right) \frac{y_t^6}{\sigma_t^6}.
\]
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In the following we study the asymptotic behavior of these in order to establish consistency and asymptotic normality of the QMLE. First, consider the asymptotic behavior of $y_t$:

**LEMMA 1:** Assume that (2) holds; then

$$y_t^2 \xrightarrow{a.s.} \infty$$

as $t \to \infty$.

**PROOF:** This follows by Theorem 2 of Nelson (1990). Q.E.D.

Next, consider the asymptotic behavior of the following type of averages:

**LEMMA 2:** Assume that (2) holds; then with $m \leq k$ positive integers,

$$\frac{y_{m-1}^2}{(1 + \alpha y_{m-1}^2)^k} \xrightarrow{a.s.} \begin{cases} 
1 & \text{if } m = k, \\
0 & \text{if } m < k,
\end{cases}$$

and likewise, as $T \to \infty$,

$$\frac{1}{T} \sum_{t=1}^{T} \frac{y_{m-1}^2}{(1 + \alpha y_{m-1}^2)^k} \xrightarrow{a.s.} \begin{cases} 
1 & \text{if } m = k, \\
0 & \text{if } m < k.
\end{cases}$$

**PROOF:** The results follow by Lemma 1. Q.E.D.

Next turn to the score and the information:

**LEMMA 3:** Under the assumptions of Theorem 1, then with $\frac{\partial \ell_T(\alpha)}{\partial \alpha}$ given by (4),

$$\left(\frac{1}{\sqrt{T}}\right) \frac{\partial}{\partial \alpha} \ell_T(\alpha) \xrightarrow{D} N\left(0, \frac{\xi}{4\alpha^2}\right)$$

as $T \to \infty$.

**PROOF:** By definition $(1/\sqrt{T})\frac{\partial \ell_T(\alpha)}{\partial \alpha} = (1/\sqrt{T}) \sum_{t=1}^{T} s_t$, where

$$s_t = -\frac{1}{2} \left(1 - \frac{y_t^2}{\sigma_i^2}\right) \frac{y_{t-1}^2}{\sigma_i^2}.$$ 

The process $s_t$ is a Martingale difference sequence with respect to $\mathcal{F}_t = \sigma\{y_t, y_{t-1}, \ldots, y_0\}$ as $E|s_t| \leq E|1 - z_i^2|/2\alpha < \infty$ and

$$E(s_t | \mathcal{F}_{t-1}) = -\frac{1}{2} E(1 - z_i^2) \frac{y_{t-1}^2}{\sigma_i^2} = 0.$$

Next, using (8),

$$\frac{1}{T} \sum_{t=1}^{T} E(s_t^2 | \mathcal{F}_{t-1}) = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{y_{t-1}^2}{1 + \alpha y_{t-1}^2}\right) \xrightarrow{a.s.} \frac{\xi}{4\alpha^2} > 0,$$
where \( \xi = E(1 - z_i^2)^2 = V(z_i^2) \). Furthermore, as \( z_i^2 \) is bounded by \( \mu_i^2 = (1 - z_i^2)^2/4\alpha^2 \) we derive the Lindeberg type condition,

\[
\frac{1}{T} \sum_{t=1}^{T} E(s_i^2 \mathbb{1}[|s_i| > \sqrt{T}\delta]) \leq E(\mu_i^2 \mathbb{1}[|\mu_i| > \sqrt{T}\delta]) \to 0,
\]

for some \( \delta > 0 \) and as \( T \) tends to \( \infty \) using \( V(z_i^2) = \xi < \infty \). By the central limit theorem in Brown (1971) the desired result follows.

**Q.E.D.**

**LEMMA 4:** Under the assumptions of Theorem 1, then with the observed information \( \partial^2 \ell_T(\alpha)/\partial \alpha^2 \) given by (5),

\[
\frac{1}{T} \left( -\frac{\partial^2}{\partial \alpha^2} \ell_T(\alpha) \right) \xrightarrow{a.s.} \frac{1}{2\alpha^2} > 0
\]

as \( T \to \infty \).

**PROOF:** Rewrite minus the observed information as

\[
\frac{1}{T} \left( -\frac{\partial^2}{\partial \alpha^2} \ell_T(\alpha) \right) = \frac{1}{2T} \sum_{t=1}^{T} \kappa_t \gamma_t
\]

with

\[
\kappa_t = 2z_i^2 - 1 \quad \text{and} \quad \gamma_t = \frac{y_t^4}{\sigma_t^4} = \frac{y_{t-1}^4}{(1 + ay_{t-1}^2)^2}.
\]

The strong law of large numbers implies

\[
\frac{1}{T} \sum_{t=1}^{T} \kappa_t \xrightarrow{a.s.} 1,
\]

\[
\frac{1}{T} \sum_{t=1}^{T} |\kappa_t| \xrightarrow{a.s.} \kappa < \infty,
\]

while (7) implies \( \gamma_t \xrightarrow{a.s.} (1/\alpha^2) \) and hence the desired result follows. **Q.E.D.**

Finally, we turn to the uniform boundedness of the third derivative of the likelihood function. Note that the proof does not require nonstationarity of the process.

**LEMMA 5:** Denote by \( I(\alpha, \delta) \) the interval \([\alpha - \delta, \alpha + \delta] \), \( 0 < \delta < \alpha \). Then with \( \partial^3 \ell_T(\alpha)/\partial \alpha^3 \) given by (6), it holds that

\[
\sup_{\tilde{\alpha} \in I(\alpha, \delta)} \left| \frac{1}{T} \frac{\partial^3}{\partial \alpha^3} \ell_T(\tilde{\alpha}) \right| \leq g(\alpha, \delta, T) \xrightarrow{a.s.} \beta < \infty
\]

as \( T \to \infty \).
PROOF: With \( \alpha_l = \alpha - \delta \),
\[
\left| \frac{1}{T} \frac{\partial^3}{\partial \alpha^3} \ell_T(\tilde{\alpha}) \right| = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{3 \gamma_t^2}{\sigma_t^2} - 1 \right) \frac{y_t^6}{\sigma_t^6} \leq \frac{1}{T} \sum_{t=1}^{T} \left( \frac{3 \gamma_t^2}{\sigma_t^2} - 1 \right) \frac{1}{\alpha_t^3}
\]
\[
= \frac{1}{T} \sum_{t=1}^{T} \left( \frac{3 \{1 + \alpha y_{t-1}^2\}}{\{1 + \tilde{\alpha} y_{t-1}^2\}} z_t^2 - 1 \right) \frac{1}{\alpha_t^3}
\]
\[
\leq \frac{1}{T} \sum_{t=1}^{T} \left( 3 \{1 + \frac{\alpha}{\alpha_t} z_t^2 \} \right) \frac{1}{\alpha_t^3} := g(\alpha, \delta, T)
\]
and the results follows by the law of large numbers. \( Q.E.D. \)

REMARK 2: When deriving consistency and asymptotic normality, the classical sufficient condition regarding bounds of the third derivatives of the likelihood function is that
\[
E \sup_{\tilde{\alpha} \in I(\alpha, \delta)} \left| \frac{1}{T} \frac{\partial^3}{\partial \alpha^3} \ell_T(\tilde{\alpha}) \right| < \infty.
\]
In Basawa, Feigin, and Heyde (1976, condition (B.7)) this is incorrectly stated as
\[
\sup_{\tilde{\alpha} \in I(\alpha, \delta)} E \left| \frac{1}{T} \frac{\partial^3}{\partial \alpha^3} \ell_T(\tilde{\alpha}) \right| < \infty.
\]
The mistake is reproduced in Weiss (1986) and next in Lumsdaine (1996), and their proofs may therefore not be complete.

4. CONCLUSION

In this paper we have shown that for the ARCH(1) model the QMLE is always asymptotically Gaussian so long as the fourth order moment of the innovations \( z_t \) is finite. This is somewhat surprising as most researchers have assumed one needs strict stationarity.

For financial applications most often it is the GARCH(1, 1) as opposed to the ARCH(1) model that is applied and hence of much interest. In a forthcoming paper, which follows on from the developments given in this paper, Jensen and Rahbek (2003) show that indeed the results hold for the GARCH(1, 1) model as well. That is, whether or not the process is stationary, asymptotic normality holds and hence there are no “knife edge results like [in] the unit root case” as conjectured by Lumsdaine (1996, p. 580). The derivations for the GARCH(1, 1) case are more involved and lengthy due to the added complexity of the lagged variance in the \( \sigma_t^2 \) specification.

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