Econometrics 710
Midterm Exam
March 16, 2017
Sample Answers

1.

(a) $\widehat{C} = R'\widehat{\beta} \pm z\sqrt{R'\widehat{V}_{\widehat{\beta}}R} = \left[R'\widehat{\beta} - z\sqrt{R'\widehat{V}_{\widehat{\beta}}R}, \qquad R'\widehat{\beta} + z\sqrt{R'\widehat{V}_{\widehat{\beta}}R}\right].$

(b) $\theta_0 \in \widehat{C}$ if $\left|R'\widehat{\beta} - \theta_0\right| \leq z\sqrt{R'\widehat{V}_{\widehat{\beta}}R}$ or equivalently if $|T| \leq z$ where $T = \left(R'\widehat{\beta} - \theta_0\right)/\sqrt{R'\widehat{V}_{\widehat{\beta}}R}$.

By the CLT, $T \rightarrow N(0,1)$ when θ_0 is the true mean.The rule "Reject H_0 if $\theta_0 \notin \widehat{C}$" is the same as "Reject H_0 if $|T| > z$". Under H_0 this has probability

$$P(|T| > z) \rightarrow P(|N(0,1)| > z) = 0.05$$

Thus the test has asymptotic size 5%

2. There are two possibilities. First, using the FWL theorem, $\widehat{\beta}_1 = \left(X_1'M_2X_1\right)^{-1} X_1'M_2y$. This equals $\widetilde{\beta}_1 = \left(X_1'X_1\right)^{-1} X_1'y$ if $M_2X_1 = X_1$ or $X_2'X_1 = 0$. Equivalently if $\sum_{i=1}^n x_{1i}x_{2i} = 0$. Similarly, $\widehat{\beta}_2 = \widetilde{\beta}_2$ under the same condition.

Second, if both $X_1'y = 0$ and $X_2'y = 0$ then all the coefficients equal zero and thus also equal one another. So, either $X_2'X_1 = 0$ or both $X_1'y = 0$ and $X_2'y = 0$.

Notice, this is a problem about the estimates, not about the population coefficients.

3.

(a) Since $\left(\widehat{\beta} - \beta\right)^2 = \widehat{\beta}^2 + \beta^2 - 2\beta\widehat{\beta}$, we have the relalationship

$$\widehat{\theta} = \widehat{\beta}^2 = \left(\widehat{\beta} - \beta\right)^2 - \beta^2 + 2\beta\widehat{\beta}$$

Taking expectations

$$\begin{aligned}
E\left(\widehat{\theta}\right) &= E\left(\left(\left(\widehat{\beta} - \beta\right)^2\right)\right) - \beta^2 + 2\beta E\left(\widehat{\beta}\right) \\
&= E\left(\left(\widehat{\beta} - \beta\right)^2\right) - \beta^2 + 2\beta\beta \\
&= \text{var}(\widehat{\beta}) + \beta^2 \\
&= V_{\widehat{\beta}} + \theta
\end{aligned}$$

since $E\left(\widehat{\beta}\right) = \beta$ under the assumption $E(e_i \mid x_i) = 0$

(b) $\widehat{\theta}^* = \widehat{\theta} - \widehat{V}_{\widehat{\beta}}$

(c) The Horn-Horn-Duncan covariance matrix estimate $\overline{V}_{\widehat{\beta}}$ is most appropriate, as it is unbiased for $V_{\widehat{\beta}}$ under conditional homoskedasticity, but remains a valid estimator under heteroskedasticity. Then $\widehat{\theta}^* = \widehat{\theta} - \overline{V}_{\widehat{\beta}}$ and

$$\begin{aligned}
E\left(\widehat{\theta}^*\right) &= E\left(\widehat{\theta}\right) - E\left(\overline{V}_{\widehat{\beta}}\right) \\
&= V_{\widehat{\beta}} + \theta - E\left(\overline{V}_{\widehat{\beta}}\right) \\
&= V_{\widehat{\beta}} + \theta - V_{\widehat{\beta}} \\
&= \theta
\end{aligned}$$

Thus $\widehat{\theta}^*$ when Horn-Horn-Duncan covariance matrix estimate $\overline{V}_{\widehat{\beta}}$ is used, and $E(e_i^2|x_i) = \sigma^2$.

4.

(a) If $E\left(x_i^8\right) < \infty$ and $E\left(e_i^8\right) < \infty$ then $V_\Omega = \operatorname{var}\left(x_i^2 e_i^2\right) < \infty$. By the CLT

$$\sqrt{n}\left(\widetilde{\Omega} - \Omega\right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(x_i^2 e_i^2 - E\left(x_i^2 e_i^2\right)\right) \rightarrow_d N(0, V_\Omega)$$

where $V_\Omega = \operatorname{var}\left(x_i^2 e_i^2\right) = E\left(x_i^4 e_i^2\right) - \left(E\left(x_i^2 e_i^2\right)\right)^2$

(b) Use the expansion

$$\begin{aligned} \widehat{e}_i^2 &= \left(y_i - x_i \widehat{\beta}\right)^2 \\ &= \left(e_i - x_i\left(\widehat{\beta} - \beta\right)\right)^2 \\ &= e_i^2 + x_i^2\left(\widehat{\beta} - \beta\right)^2 - 2 e_i x_i \left(\widehat{\beta} - \beta\right) \end{aligned}$$

Then

$$\begin{aligned} \widehat{\Omega} &= \frac{1}{n} \sum_{i=1}^{n} x_i^2 \widehat{e}_i^2 \\ &= \widetilde{\Omega} + \frac{1}{n} \sum_{i=1}^{n} x_i^4 \left(\widehat{\beta} - \beta\right)^2 - 2 \frac{1}{n} \sum_{i=1}^{n} x_i^3 e_i \left(\widehat{\beta} - \beta\right) \end{aligned}$$

and

$$\sqrt{n}\left(\widehat{\Omega} - \Omega\right) = \sqrt{n}\left(\widetilde{\Omega} - \Omega\right) + \frac{1}{n} \sum_{i=1}^{n} x_i^4 \sqrt{n}\left(\widehat{\beta} - \beta\right)^2 - 2 \frac{1}{n} \sum_{i=1}^{n} x_i^3 e_i \sqrt{n}\left(\widehat{\beta} - \beta\right) \tag{1}$$

Since $\frac{1}{n} \sum_{i=1}^{n} x_i^4 \rightarrow_p E x_i^4$ and $\sqrt{n}\left(\widehat{\beta} - \beta\right)^2 \rightarrow_p 0$ the second term on the right-hand-side converges in probability to zero. Since $\frac{1}{n} \sum_{i=1}^{n} x_i^3 e_i \rightarrow_p E\left(x_i^3 e_i\right) = 0$ when $E(e_i | x_i) = 0$ and $\sqrt{n}\left(\widehat{\beta} - \beta\right) = O_p(1)$, the third term on the right-hand-side converges in probability to zero. We find that

$$\sqrt{n}\left(\widehat{\Omega} - \Omega\right) = \sqrt{n}\left(\widetilde{\Omega} - \Omega\right) + o_p(1) \rightarrow_d N(0, V_\Omega)$$

(c) The regression assumption plays an important role, as otherwise the third term on the right-hand-side does not converge in probability to zero, but rather to $E\left(x_i^3 e_i\right)$ multiplied by a normal variable.