Econometrics 710
Midterm Exam
March 8, 2016

The exam questions all concern the following setting. The random variables are (y, x) with $y \in \mathbb{R}$ and $x \in \mathbb{R}$ and there is a random sample $\{y_i, x_i : i = 1, ..., n\}$ from (y, x). Define the conditional mean $m(x) = E(y_i | x_i = x)$. A researcher is interested in estimating the average derivative

$$\theta = E\left[\frac{\partial}{\partial x} m(x_i)\right].$$

Assume that the true conditional mean takes the form

$$m(x) = c_0 + c_1 x + c_2 x^2 \tag{1}$$

but this is **not necessarily known** by the researcher. Also, write the moments of x_i as $\mu_x = Ex_i$, $\sigma_x^2 = \text{var}(x)$, and $s_x = E(x_i - \mu_x)^3$.

1. Given (1), find an expression for θ in terms of c_0, c_1, c_2 and the moments of x_i.

2. Suppose that the researcher estimates θ by linear OLS. Specifically, they estimate by least-squares

$$y_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_i + \widehat{e}_i$$

   and then set $\widehat{\theta} = \widehat{\beta}_1$. Let (β_0, β_1) denote the population version of this regression (the best linear prediction coefficients). Find an expression for the bias in β_1 for θ, e.g. the difference $\beta_1 - \theta$, in terms of c_0, c_1, c_2 and the moments of x_i.

   Hint: The answer requires a few lines of algebra. It will be convenient to know that $Ex_i^2 = \sigma_x^2 + \mu_x^2$ and $Ex_i^3 = s_x + 3\mu_x\sigma_x^2 + \mu_x^3$. If you get bogged down in the algebra, you may wish to skip to question 4, which doesn't depend on the answer to question 2.

3. Describe the conditions under which $\beta_1 = \theta$.

4. Now suppose that the researcher knows that the quadratic specification (1) is the correct conditional mean, and estimates a quadratic regression by least-squares

$$y_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_i + \widehat{\beta}_2 x_i^2 + \widehat{e}_i$$

   For simplicity, assume the researcher knows (from prior information) the mean $\mu_x = Ex_i$. Describe (be precise) the appropriate estimator $\widehat{\theta}$ for θ given this information.

5. Is $\widehat{\theta}$ unbiased for θ?

   [Hint: There is a simple argument.]

6. Show that $\widehat{\theta}$ is consistent for θ as $n \to \infty$.

   [Hint: Again, a simple answer is sufficient.]

7. Find the asymptotic distribution of $\sqrt{n}\left(\widehat{\theta} - \theta\right)$ as $n \to \infty$. It is sufficient to write your answer in terms of the asymptotic covariance matrix of the OLS estimator.

8. Now suppose that $\mu_x = Ex_i$ is unknown. Describe the appropriate estimator $\widehat{\theta}$ for θ.

9. Is $\widehat{\theta}$ consistent for θ?

10. Optional (Only attempt if you have extra time). How would we find the asymptotic distribution of $\sqrt{n}\left(\widehat{\theta} - \theta\right)$ as $n \to \infty$?

    (a) What is the additional challenge when μ_x is estimated?
    [Why is the answer from part 7 insufficient?]

    (b) Describe a strategy for obtaining the correct asymptotic distribution.

    (c) Find it.