Econometrics 710
Midterm Exam
March 4, 2015
Sample Answers

1. By expanding the square

$$\begin{aligned} T(\theta) &= E\left[(y - x'\theta)^2 \tau(x)\right] \\ &= E\left[y^2\tau(x)\right] - 2E\left[yx'\tau(x)\right]\theta + \theta' E\left[xx'\tau(x)\right]\theta \end{aligned}$$

Taking the first derivative

$$\frac{\partial}{\partial\theta}T(\theta) = 2E\left[xy\tau(x)\right]\theta + 2E\left[xx'\tau(x)\right]\theta$$

Setting it equal to zero and solving for θ

$$\theta = \left(E\left[xx'\tau(x)\right]\right)^{-1} E\left[xy\tau(x)\right]$$

2. Since $e = y - x'\theta$,

$$\begin{aligned} E\left[xe\tau(x)\right] &= E\left[xy\tau(x)\right] - E\left[xx'\tau(x)\right]\theta \\ &= E\left[xy\tau(x)\right] - E\left[xx'\tau(x)\right]\left(E\left[xx'\tau(x)\right]\right)^{-1} E\left[xy\tau(x)\right] \\ &= E\left[xy\tau(x)\right] - E\left[xy\tau(x)\right] \\ &= 0 \end{aligned}$$

For this result, you do not need an additional assumption. For example, $E(e|x) = 0$ is not needed.

3. If the conditional mean is linear $E(y|x) = x'\beta$ then

$$\begin{aligned} \theta &= \left(E\left[xx'\tau(x)\right]\right)^{-1} E\left(E\left[yx\tau(x)|x\right]\right) \\ &= \left(E\left[xx'\tau(x)\right]\right)^{-1} E\left(x\tau(x)E\left[y|x\right]\right) \\ &= \left(E\left[xx'\tau(x)\right]\right)^{-1} E\left(x\tau(x)x'\beta\right) \\ &= \beta \end{aligned}$$

so that $\theta = \beta$. When the conditional mean is linear it equals the best linear predictor, and thus θ equals the best linear predictor as well.

A common answer was: "When $\tau(x) = \tau$ is independent of x". This may appear to be technically correct, for then indeed the problem reduces to that of the best linear predictor. However, this is an uninteresting solution and thus I graded it as a missed answer. I tried to exclude this answer by including the explicit warning "Under what condition other than $\tau(x) = 1$ ..."

4. $\hat{\theta} = \left(\sum_{i=1}^n x_i x_i' \tau(x_i)\right)^{-1} \sum_{i=1}^n x_i y_i \tau(x_i)$. Alternatively, $\hat{\theta} = \left(X'TX\right)^{-1}\left(X'TY\right)$ where $T = diag\{\tau(x_i)\}$

5. If $E(e|x) = 0$ then $E(Y|X) = X\theta$ and

$$\begin{aligned} E(\hat{\theta}|X) &= \left(X'TX\right)^{-1}\left(X'TE(Y|X)\right) \\ &= \left(X'TX\right)^{-1}\left(X'TX\theta\right) \\ &= \theta \end{aligned}$$

By iterated expectations, $E(\hat{\theta}) = E\left(E(\hat{\theta}|X)\right) = \theta$ and $\hat{\theta}$ is unbiased for θ. Thus the estimator is unbiased when the conditional mean is linear.

6. By the WLLN,

$$\frac{1}{n}\sum_{i=1}^n x_i x_i' \tau(x_i) \to_p E\left(x_i x_i' \tau(x_i)\right)$$

and

$$\frac{1}{n}\sum_{i=1}^n x_i y_i \tau(x_i) \to_p E\left(x_i y_i \tau(x_i)\right)$$

By the continuous mapping theorem,

$$\begin{aligned}\hat{\theta} &= \left(\sum_{i=1}^{n} x_i x_i' \tau(x_i)\right)^{-1} \sum_{i=1}^{n} x_i y_i \tau(x_i) \\ &\underset{p}{\longrightarrow} \left(E\left(x_i x_i' \tau(x_i)\right)\right)^{-1} E\left(x_i y_i \tau(x_i)\right) \\ &= \theta\end{aligned}$$

and thus $\hat{\theta}$ is consistent for θ.

Regularity conditions sufficient for this result are:

- $E\left\|x_i\right\|^2 < \infty$
- $Ey_i^2 < \infty$
- $E\left(x_i x_i' \tau(x_i)\right) > 0$

7. Since $y = X\theta + e$, then

$$\begin{aligned}\hat{\theta} &= \left(X'TX\right)^{-1}\left(X'Ty\right) \\ &= \left(X'TX\right)^{-1}\left(X'TX\right)\theta + \left(X'TX\right)^{-1}\left(X'Te\right) \\ &= \theta + \left(X'TX\right)^{-1}\left(X'Te\right).\end{aligned}$$

Then

$$\begin{aligned}\sqrt{n}\left(\hat{\theta} - \theta\right) &= \sqrt{n}\left(X'TX\right)^{-1}\left(X'Te\right) \\ &= \left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i' \tau(x_i)\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i e_i \tau(x_i)\right)\end{aligned}$$

As shown in question 2, $E\left[x_i e_i \tau(x_i)\right] = 0$. By the CLT

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i e_i \tau(x_i) \to_d N(0, S)$$

where $S = E\left(x_i x_i' e_i^2 \tau(x_i)^2\right)$. Then

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \to_d Q^{-1}N(0, S) = N(0, Q^{-1}SQ^{-1})$$

where $Q = E\left(x_i x_i' \tau(x_i)\right)$.

Regularity conditions sufficient for this result are:

- $E\left\|x_i\right\|^4 < \infty$
- $Ey_i^4 < \infty$
- $E\left(x_i x_i' \tau(x_i)\right) > 0$

8. If $E(e^2|x) = \sigma^2$ then $S = E\left(x_i x_i' e_i^2 \tau(x_i)^2\right) = \widetilde{Q}\sigma^2$ where $\widetilde{Q} = E\left(x_i x_i' \tau(x_i)^2\right)$. Then $Q^{-1}SQ^{-1} = Q^{-1}\widetilde{Q}Q^{-1}\sigma^2$ and the asymptotic distribution is $N(0, Q^{-1}\widetilde{Q}Q^{-1}\sigma^2)$.

9. $\hat{V} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i' \tau(x_i)\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i' \tau(x_i)^2 \hat{e}_i^2\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i' \tau(x_i)\right)^{-1}$ where $\hat{e}_i = y_i - x_i'\hat{\theta}$.