Econometrics 710
Midterm Exam
March 10, 2011
Sample Answers

1. The coefficient $\boldsymbol{\gamma}_2$ is the best linear predictor for e_i^2 given $\boldsymbol{x}_i$. The coefficient $\boldsymbol{\gamma}_1$ is the best linear approximation to $\sigma^2(\boldsymbol{x}_i)$. They are the same. To see this explicitly, the FOC for $\boldsymbol{\gamma}_1$ is

$$\mathbf{0} = -2E\left(\boldsymbol{x}_i\sigma^2(\boldsymbol{x}_i)\right) + E\left(\boldsymbol{x}_i\boldsymbol{x}_i'\right)\boldsymbol{\gamma}_1$$

so that

$$\boldsymbol{\gamma}_1 = E\left(\boldsymbol{x}_i\boldsymbol{x}_i'\right)^{-1}E\left(\boldsymbol{x}_i\sigma^2(\boldsymbol{x}_i)\right).$$

The FOC for $\boldsymbol{\gamma}_2$ is

$$\mathbf{0} = -2E\left(\boldsymbol{x}_ie_i^2\right) + E\left(\boldsymbol{x}_i\boldsymbol{x}_i'\right)\boldsymbol{\gamma}_2$$

so that

$$\boldsymbol{\gamma}_2 = E\left(\boldsymbol{x}_i\boldsymbol{x}_i'\right)^{-1}E\left(\boldsymbol{x}_ie_i^2\right)$$

By conditioning and the LIE

$$\begin{aligned} E\left(\boldsymbol{x}_ie_i^2\right) &= E\left(E\left(\boldsymbol{x}_ie_i^2 \mid \boldsymbol{x}_i\right)\right) \\ &= E\left(\boldsymbol{x}_iE\left(e_i^2 \mid \boldsymbol{x}_i\right)\right) \\ &= E\left(\boldsymbol{x}_i\sigma^2(\boldsymbol{x}_i)\right) \end{aligned}$$

and thus

$$\begin{aligned} \boldsymbol{\gamma}_2 &= E\left(\boldsymbol{x}_i\boldsymbol{x}_i'\right)^{-1}E\left(\boldsymbol{x}_ie_i^2\right) \\ &= E\left(\boldsymbol{x}_i\boldsymbol{x}_i'\right)^{-1}E\left(\boldsymbol{x}_i\sigma^2(\boldsymbol{x}_i)\right) \\ &= \boldsymbol{\gamma}_1 \end{aligned}$$

2.

(a) From the analysis of omitted variable bias, we know that $\gamma_1 = \beta_1$ under one of two conditions:
   i. $\beta_2 = 0$ in the long regression
   ii. $E\left(x_ix_i^2\right) = 0$ or equivalently $E\left(x_i^3\right) = 0$
   If $E\left(x_i\right) = 0$, this is equivalent to x_i having zero skewness

(b) From the same argument, $\gamma_1 = \theta_1$ under one of two conditions:
   i. $\theta_2 = 0$ in the long regression
   ii. $E\left(x_ix_i^3\right) = 0$ or equivalently $E\left(x_i^4\right) = 0$. This is impossible. Thus the conditions for $\gamma_1 = \theta_1$ and $\gamma_1 = \beta_1$ are not similar.

3. Substituing $y_i = x_i\beta + e_i$,

$$\begin{aligned} \hat{\beta} &= \frac{\sum_{i=1}^n x_i^3y_i}{\sum_{i=1}^n x_i^4} \\ &= \frac{\sum_{i=1}^n x_i^3\left(x_i\beta + e_i\right)}{\sum_{i=1}^n x_i^4} \\ &= \beta + \frac{\sum_{i=1}^n x_i^3e_i}{\sum_{i=1}^n x_i^4} \end{aligned}$$

Thus

$$\sqrt{n}\left(\hat{\beta} - \beta\right) = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i^3e_i}{\frac{1}{n}\sum_{i=1}^n x_i^4}$$

By the WLLN, if $Ex_i^4 < \infty$, then as $n \to \infty$

$$\frac{1}{n}\sum_{i=1}^{n} x_i^4 \to_p Ex_i^4.$$

By the LIE and $E(e_i \mid x_i) = 0$ then

$$E\left(x_i^3 e_i\right) = E\left(E\left(x_i^3 e_i \mid x_i\right)\right) = E\left(x_i^3 E\left(e_i \mid x_i\right)\right) = 0.$$

Then by the CLT, if $E\left(x_i^6 e_i^2\right) < \infty$, as $n \to \infty$,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i^3 e_i \to_d N(0, E\left(x_i^6 e_i^2\right)).$$

Together

$$\sqrt{n}\left(\hat{\beta} - \beta\right) \to_d \frac{N(0, E\left(x_i^6 e_i^2\right))}{Ex_i^4} = N(0, \frac{E\left(x_i^6 e_i^2\right)}{\left(Ex_i^4\right)^2})$$

4.

(a) The restricted model is $y_i = \alpha + e_i$. The CLS estimator is $\tilde{\alpha} = n^{-1}\sum_{i=1}^{n} y_i$.

(b) Let $(\hat{\alpha}, \widehat{\boldsymbol{\beta}})$ be the unrestricted OLS estimator of $(\hat{\alpha}, \widehat{\boldsymbol{\beta}})$. Let $\widehat{\boldsymbol{V}} = \widehat{\boldsymbol{Q}}^{-1}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{Q}}^{-1}$ be estimator of the asymptotic covariance matrix for $(\hat{\alpha}, \widehat{\boldsymbol{\beta}})$. Letting $\widetilde{\boldsymbol{x}}_i = \begin{pmatrix} 1 \\ \boldsymbol{x}_i \end{pmatrix}$, this is

$$\begin{aligned} \widehat{\boldsymbol{Q}} &= \frac{1}{n}\sum_{i=1}^{n} \widetilde{\boldsymbol{x}}_i \widetilde{\boldsymbol{x}}_i' \\ \widehat{\boldsymbol{\Omega}} &= \frac{1}{n}\sum_{i=1}^{n} \widetilde{\boldsymbol{x}}_i \widetilde{\boldsymbol{x}}_i' \hat{e}_i^2 \end{aligned}$$

where $\hat{e}_i = y_i - \hat{\alpha} - \boldsymbol{x}_i'\widehat{\boldsymbol{\beta}}$. Partition $\widehat{\boldsymbol{V}}$ as

$$\widehat{\boldsymbol{V}} = \begin{bmatrix} \widehat{\boldsymbol{V}}_{11} & \widehat{\boldsymbol{V}}_{12} \\ \widehat{\boldsymbol{V}}_{21} & \widehat{\boldsymbol{V}}_{22} \end{bmatrix}.$$

Using equation (7.22) from the notes,

$$\tilde{\alpha}_{MD} = \hat{\alpha} - \widehat{\boldsymbol{V}}_{12}\widehat{\boldsymbol{V}}_{22}^{-1}\widehat{\boldsymbol{\beta}}.$$

Since

$$\hat{\alpha} = \overline{y} - \overline{\boldsymbol{x}}'\widehat{\boldsymbol{\beta}}$$

we can also write this as

$$\tilde{\alpha}_{MD} = \overline{y} - \left(\overline{\boldsymbol{x}} + \widehat{\boldsymbol{V}}_{22}^{-1}\widehat{\boldsymbol{V}}_{21}\right)'\widehat{\boldsymbol{\beta}}$$