Econometrics 710
Midterm Exam Sample Answers
Spring 2009

1. $\hat{\sigma}^2 = \tilde{\sigma}^2$.

   Since the two regressor matrices span the same linear space, the two residual vectors are identical, so the variance estimates are identical. To see this algebraically, observe that

$$Z = XC$$

   where

$$C = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}.$$

   which is a full rank matrix. (It is upper diagonal with positive diagonal elements. The regression residuals are

$$\begin{aligned} \tilde{e} &= Y - Z\left(Z'Z\right)^{-1} Z'Y \\ &= Y - XC\left(C'X'XC\right)^{-1} C'X'Y \\ &= Y - XCC^{-1}\left(X'X\right)^{-1} C'^{-1}C'X'Y \\ &= Y - X\left(X'X\right)^{-1} X'Y \\ &= \hat{e} \end{aligned}$$

   The two residual vectors are identical. So

$$\tilde{\sigma}^2 = \frac{1}{n}\tilde{e}'\tilde{e} = \frac{1}{n}\hat{e}'\hat{e} = \hat{\sigma}^2$$

2.

   (a) $q = 5$. The dimension of β_2 is 5, and the hypothesis is that all components of β_2 are zero, which is five restrictions.

   (b) No.

   The Wald test rejects for large values of W_n, when $W_n \geq c$ for some c. A test which rejects for small W_n can have correct Type I error, but will have low power. An asymptotic α% test rejects H_0 if $W_n \geq c_\alpha$ where c_α is the $1-\alpha$ quantile (the upper α quantile), that is $P\left(W_n \geq c_\alpha\right) = 1 - P\left(W_n < c_\alpha\right) = \alpha$. An asymptotic 5% test rejects if $W_n \geq 11.07$, and an asymptotic 1% test rejects for $W_n \geq 15.09$. Since W_n is far smaller, H_0 is not rejected.

   The question about the 1% quantile is misleading. It is not the 1% critical value.

   It is also not sensible to talk about test with 99% Type I error. So there is no sense in which 0.55 is a reasonable critical value for a test.

3. The model is a linear regression. We know this because the question specifies the conditional mean of y_i given x_i. This is a regression, so it does not need to be assumed. As the conditional variance is unspecified the regression is heteroskedastic.

   (a) Note that you can write the model as a linear equation

$$\begin{aligned} g(x) &= E\left(y_i \mid x_i = x\right) \\ &= \beta_0 + \beta_1 x + \beta_2 x^2 \\ &= z'\beta \end{aligned}$$

where

$$\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}, \qquad z = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$$

It is useful to designate the regressor vector as z, rather than x. It is sloppy to use x for both the individual scalar and the vector $(1, x, x^2)$.

As the equation is linear in the parameters, it can be estimated by least-squares

$$\begin{aligned} y_i &= \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{\beta}_2 x_i^2 + \hat{e}_i \\ &= \hat{\beta}' z_i + \hat{e}_i \end{aligned}$$

where

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}, \qquad z_i = \begin{pmatrix} 1 \\ x_i \\ x_i^2 \end{pmatrix}$$

(Note, this is not a problem in NLLS.)

Since the model is a regression, FGLS is also possible, although since the model did not suggest a functional form for the variance, this would require an approximate variance equation, and it may just be easiest and best to use least-squares.

The estimate of the conditional mean function at x is

$$\hat{g}(x) = \hat{\beta}_0 + \hat{\beta}_1 x + \hat{\beta}_2 x^2 = z'\hat{\beta}$$

We know that

$$\sqrt{n}\left(\hat{\beta} - \beta\right) \to_d N(0, V)$$

where $V = Q^{-1}\Omega Q^{-1}$. Thus

$$\sqrt{n}\left(\hat{g}(x) - g(x)\right) = \sqrt{n}\left(z'\hat{\beta} - z'\beta\right) = z'\sqrt{n}\left(\hat{\beta} - \beta\right) \to_d N\left(0, z'Vz\right)$$

An estimate of V is

$$\begin{aligned} \hat{V} &= \hat{Q}^{-1}\hat{\Omega}\hat{Q}^{-1} \\ \hat{Q} &= \frac{1}{n}\sum_{i=1}^{n} z_i z_i' \\ \hat{\Omega} &= \frac{1}{n}\sum_{i=1}^{n} z_i z_i' \hat{e}_i^2 \end{aligned}$$

A standard error for $\hat{g}(x) = z'\hat{\beta}$ is $\left(n^{-1}z'\hat{V}z\right)^{1/2}$. An asymptotic 95% confidence interval is

$$\begin{aligned} \hat{g}(x) \pm 2\left(n^{-1}z'\hat{V}z\right)^{1/2} &= z'\hat{\beta} \pm 2\left(n^{-1}z'\hat{V}z\right)^{1/2} \\ &= \hat{\beta}_0 + \hat{\beta}_1 x + \hat{\beta}_2 x^2 \pm 2\left(n^{-1}z'\hat{V}z\right)^{1/2} \end{aligned}$$

Since the function $z'\hat{\beta}$ is linear in the parameters, this Wald confidence interval approach is appropriate.

Also, note that the problem asked for an estimate of $g(x)$ and a confidence interval for $g(x)$. It did not ask for a forecast interval for y given x. That is a different problem which has a different answer.

4. This model is a projection, not a regression. The question was vague about how the sample split was made. My intention, and the common interpretation, was that the split is made randomly. Thus observations are split into two random groups, and are thus from the same distribution.

   (a) We know that a full sample estimate has the asymptotic distribution

$$\sqrt{2n}\left(\hat{\beta}-\beta\right) \rightarrow_d N(0,V)$$

   If the sample is split randomly, each sub-sample estimate has the same distribution, but with n replacing $2n$:

$$\sqrt{n}\left(\hat{\beta}_1-\beta\right) \rightarrow_d N(0,V)$$
$$\sqrt{n}\left(\hat{\beta}_2-\beta\right) \rightarrow_d N(0,V)$$

   where $V = Q^{-1}\Omega Q^{-1}$.

   Notice: The coefficients β are identical in the two samples by assumption, and the asymptotic variances are the same when the sample splitting is random. If the sample split is not random (the question is not specific) then the variances are different, and see the answer to (b) below.

   Should the asymptotic variances V have subscripts, e.g. be V_1 and V_2? Perhaps. But in this case so should β_1 and β_2. So if you had set $\beta_1 = \beta_2 = \beta$ it follows that you should also have set $V_1 = V_2 = V$.

   Since the observations are iid, the estimates $\hat{\beta}_1$ and $\hat{\beta}_2$ are independent. Thus they are asymptotically jointly normally distributed with zero covariance:

$$\begin{pmatrix} \sqrt{n}\left(\hat{\beta}_1-\beta\right) \\ \sqrt{n}\left(\hat{\beta}_2-\beta\right) \end{pmatrix} \rightarrow_d N\left(0, \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}\right)$$

   The asymptotic distribution of $\sqrt{n}\left(\hat{\beta}_1-\hat{\beta}_2\right)$ is then

$$\sqrt{n}\left(\hat{\beta}_1-\hat{\beta}_2\right) \rightarrow_d N(0,V+V) = N(0,2V).$$

   (b) This question presumes that there is some meaning to the coefficients in the two samples, so this changes the nature of the problem. The sample split must not be random. We can think of the augmented model as

$$\begin{aligned} y_{1i} &= x_{1i}'\beta_1 + e_{1i} \\ E(x_{1i}e_{1i}) &= 0 \end{aligned}$$

   and

$$\begin{aligned} y_{2i} &= x_{2i}'\beta_2 + e_{2i} \\ E(x_{2i}e_{2i}) &= 0 \end{aligned}$$

   Assuming that the split is made conditional on the regressors, the estimates are still independent. We have

$$\begin{pmatrix} \sqrt{n}\left(\hat{\beta}_1-\beta_1\right) \\ \sqrt{n}\left(\hat{\beta}_2-\beta_2\right) \end{pmatrix} \rightarrow_d N\left(0, \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}\right)$$

where

$$\begin{aligned} V_1 &= Q_1^{-1}\Omega_1 Q_1^{-1} \\ Q_1 &= E\left(x_{1i}x_{1i}'\right) \\ \Omega_1 &= E\left(x_{1i}x_{1i}'e_{1i}^2\right) \end{aligned}$$

$$\begin{aligned} V_2 &= Q_2^{-1}\Omega_2 Q_2^{-1} \\ Q_2 &= E\left(x_{2i}x_{2i}'\right) \\ \Omega_2 &= E\left(x_{2i}x_{2i}'e_{2i}^2\right) \end{aligned}$$

There are two approaches to testing in this context. Either we can assume $V_1 = V_2$, or allow them to differ.

The first approach ($V_1 = V_2$) is appropriate if your hypothesis is that the two subsamples are iid – come from the same distribution.

The second approach ($V_1 \neq V_2$) is appropriate if you want to test that the coefficients are the same, but remain agnostic about the marginal distributions of the regressors. I will describe testing for this second approach. In this case we find

$$\sqrt{n}\left(\left(\hat{\beta}_1 - \beta_1\right) - \left(\hat{\beta}_2 - \beta_2\right)\right) \rightarrow_d N\left(0, V_1 + V_2\right)$$

Under the hypothesis $H_0 : \beta_1 = \beta_2$,

$$\sqrt{n}\left(\hat{\beta}_1 - \hat{\beta}_2\right) \rightarrow_d N\left(0, V_1 + V_2\right)$$

An appropriate Wald statistic is

$$W_n = n\left(\hat{\beta}_1 - \hat{\beta}_2\right)'\left(\hat{V}_1 + \hat{V}_2\right)^{-1}\left(\hat{\beta}_1 - \hat{\beta}_2\right)$$

where

$$\begin{aligned} \hat{V}_1 &= \hat{Q}_1^{-1}\hat{\Omega}_1\hat{Q}_1^{-1} \\ \hat{Q}_1 &= \frac{1}{n}\sum_{i=1}^{n} x_{1i}x_{1i}' \\ \hat{\Omega}_1 &= \frac{1}{n}\sum_{i=1}^{n} x_{1i}x_{1i}'\hat{e}_{1i}^2 \\ \hat{e}_{1i} &= y_{1i} - x_{1i}'\hat{\beta}_1 \end{aligned}$$

and

$$\begin{aligned} \hat{V}_2 &= \hat{Q}_2^{-1}\hat{\Omega}_2\hat{Q}_2^{-1} \\ \hat{Q}_2 &= \frac{1}{n}\sum_{i=1}^{n} x_{2i}x_{2i}' \\ \hat{\Omega}_2 &= \frac{1}{n}\sum_{i=1}^{n} x_{2i}x_{2i}'\hat{e}_{2i}^2 \\ \hat{e}_{2i} &= y_{2i} - x_{2i}'\hat{\beta}_2 \end{aligned}$$

An asymptotic $\acute{\alpha}\%$ test rejects H_0 in favor of $H_1 : \beta_1 \neq \beta_2$ when $W_n \geq c$ where c is the α critical value from the χ_k^2 distribution, its $1 - \alpha$ quantile, where $k = \dim(x_i)$.