Econometrics 710
Midterm Exam
March 11, 2008
Sample Answers

1.

(a) This is easiest solved using matrix notation. Write the model as $y = X_1\beta_1 + X_2\beta_2 e$ and the short regression as $y = X_1\hat{\beta}_1 + \hat{e}$. Let $M_1 = I - X_1\left(X_1'X_1\right)^{-1}X_1'$. By the properties of least-squares and the fact that $M_1 X_1 = 0$,

$$\begin{aligned}\hat{e} &= M_1 y \\ &= M_1\left(X_1\beta_1 + X_2\beta_2 + e\right) \\ &= M_1\left(X_2\beta_2 + e\right)\end{aligned}$$

Thus since M_1 is idempotent

$$\begin{aligned}\left(n-k_1\right)s^2 &= \left(X_2\beta_2 + e\right)' M_1 M_1 \left(X_2\beta_2 + e\right) \\ &= \left(X_2\beta_2 + e\right)' M_1 \left(X_2\beta_2 + e\right) \\ &= e'M_1 e + \beta_2' X_2 M_1 X_2 \beta_2 + 2\beta_2' X_2' M_1 e.\end{aligned}$$

Since X_2 and M_1 are functions of X, and $E\left(e \mid X\right) = 0$

$$\begin{aligned}\left(n-k_1\right)E\left(s^2 \mid X\right) &= E\left(e'M_1 e \mid X\right) + \beta_2' X_2 M_1 X_2 \beta_2 \\ &= \left(n-k_1\right)\sigma^2 + \beta_2' X_2' M_1 X_2 \beta_2.\end{aligned}$$

the second equality since $E\left(ee' \mid X\right) = I\sigma^2$ and $\operatorname{tr}\left[M_1\right] = rank(M_1) = n - k_1$ imply that

$$E\left(e'M_1 e \mid X\right) = \operatorname{tr}\left[M_1 E\left(ee' \mid X\right)\right] = \operatorname{tr}\left[M_1\right]\sigma^2 = \left(n-k_1\right)\sigma^2$$

Therefore we find

$$E\left(s^2 \mid X\right) = \sigma^2 + \frac{1}{n-k_1}\beta_2' X_2' M_1 X_2 \beta_2.$$

Common errors:

i. Assuming (implicitly or explicitly) that $\beta_2 = 0$
ii. Pretending that $\hat{e}'\hat{e} = e'e$
iii. Assuming that s^2 must be unbiased because it is in a correctly-specified model.

(b) Note that

$$s^2 = \left(\frac{n}{n-k_1}\right)\left(\frac{1}{n}e'M_1 e + \frac{1}{n}\beta_2' X_2' M_1 X_2 \beta_2 + 2\frac{1}{n}\beta_2' X_2' M_1 e\right)$$

and $\frac{n}{n-k_1} \longrightarrow 1$. We learned in class that

$$\frac{1}{n}e'M_1 e \longrightarrow_p \sigma^2.$$

Indeed,

$$\frac{1}{n}e'M_1 e = \frac{1}{n}e'e - \frac{1}{n}e'X_1\left(\frac{1}{n}X_1'X_1\right)^{-1}\frac{1}{n}X_1'e \longrightarrow_p \sigma^2$$

since $\frac{1}{n}X_1'X_1 \longrightarrow_p Q_{11}$ and $\frac{1}{n}X_1'e \longrightarrow_p 0$.

Next,

$$\begin{aligned}\frac{1}{n}\beta_2' X_2' M_1 e &= \beta_2' \left(\frac{1}{n} X_2' e - \frac{1}{n} X_2' X_1 \left(\frac{1}{n} X_1' X_1 \right)^{-1} \frac{1}{n} X_1' e \right) \\ &\xrightarrow{p} \beta_2' \left(0 - Q_{21} \left(Q_{11} \right)^{-1} 0 \right) \\ &= 0\end{aligned}$$

Finally,

$$\begin{aligned}\frac{1}{n}\beta_2' X_2' M_1 X_2 \beta_2 &= \beta_2' \left[\frac{1}{n} X_2' X_2 - \frac{1}{n} X_2' X_1 \left(\frac{1}{n} X_1' X_1 \right)^{-1} \frac{1}{n} X_1' X_2 \right] \beta_2 \\ &\xrightarrow{p} \beta_2' \left[Q_{22} - Q_{21} \left(Q_{11} \right)^{-1} Q_{12} \right] \beta_2\end{aligned}$$

In sum

$$s^2 \xrightarrow{p} \sigma^2 + \beta_2' \left[Q_{22} - Q_{21} \left(Q_{11} \right)^{-1} Q_{12} \right] \beta_2.$$

Common errors:

i. Confusing probability limits and expectations
ii. Assuming that s^2 must be consistent because it is under correct specification.

2.

(a) By definition,

$$V = \left(E x_i^2 \right)^{-1} \left(E \left(x_i^2 e_i^2 \right) \right) \left(E x_i^2 \right)^{-1} = E \left(x_i^2 e_i^2 \right)$$

and

$$V^0 = \left(E x_i^2 \right)^{-1} E e_i^2 = \sigma^2$$

where $\sigma^2 = E e_i^2$.

(b) By the definition of covariance and the above equations,

$$\begin{aligned} C &= cov \left(x_i^2, e_i^2 \right) \\ &= E \left(x_i^2 e_i^2 \right) - E \left(x_i^2 \right) E \left(e_i^2 \right) \\ &= V - V^0 \end{aligned}$$

Thus $C = V - V^0$ (or $V = C + V^0$).

Common errors:

i. Assuming that e_i is homoskedastic (e.g., stating that the assumptions imply homoskedasticity)
ii. Assuming that $C = 0$

3. A point forecast of y_{n+1} takes the form $x'\hat{\beta}$ for some estimate $\hat{\beta}$ of β. A complete answer requires describing the choice of estimator $\hat{\beta}$, and it is best if this choice is justified.

(a) One option is least-squares $\hat{\beta} = \left(X'X \right)^{-1} X'y$. While this estimator is not semiparametrically efficient in the model, it can be justified as simple and robust to misspecification.

(b) Another option is FGLS. $\hat{\beta} = \left(X'\hat{D}^{-1}X \right)^{-1} X'\hat{D}^{-1}y$ where $\hat{D} = diag \left(\hat{\sigma}_{\hat{\imath}}^2 \right)$, $\hat{\sigma}_{\hat{\imath}}^2 = z_i'\hat{\gamma}$ and $\hat{\gamma} = \left(Z'Z \right)^{-1} Z'\hat{\eta}$ where $\hat{\eta}$ is the vector with $i'th$ entry $\hat{e}_i^2$ where $\hat{e}_i = y_i - x_i'\hat{\beta}$ and $\hat{\beta}$ is the OLS estimator. Given that the model is specified as a regression with a parametric variance equation, the FGLS estimator is semiparametrically efficient.

A standard forecast interval takes the form

$$x'\hat{\beta} \pm 2\sqrt{z'\hat{\gamma} + \frac{1}{n}x'\hat{V}x'}$$

where $\hat{\gamma}$ is an estimate of γ and $\hat{V}$ is an estimate of the asymptotic variance of the estimator $\hat{\beta}$. The natural estimator for $\hat{\gamma}$ is described above. The estimate $\hat{V}$ depends on the choice for $\hat{\beta}$. If $\hat{\beta}$ is OLS, then either $\hat{V} = \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}\sum_i x_i x_i' \hat{e}_i^2\right)\left(\frac{1}{n}X'X\right)^{-1}$ or $\hat{V} = \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}\sum_i x_i x_i' \left(z_i'\hat{\gamma}\right)\right)\left(\frac{1}{n}X'X\right)^{-1}$. If $\hat{\beta}$ is FGLS, a good choice is $\hat{V} = \left(\frac{1}{n}X'\hat{D}^{-1}X\right)^{-1}$.

On a cautionary note, it may be observed that this forecast interval is correct only when the error e_i is normally distributed.

Common errors:

i. Assuming that the error is homoskedastic (even though the question explicitly assumes a heteroskedastic variance equation).
ii. Assuming that γ is known
iii. Assuming that β is known
iv. Stating that the forecast is $\hat{\beta}'x$ without describing $\hat{\beta}$.
v. Picking the least-squares estimator but not describing why this choice is made.
vi. Specifying the forecast interval as $x'\hat{\beta} \pm 2\sqrt{\hat{\sigma}^2 + \frac{1}{n}x'\hat{V}x'}$ or $x'\hat{\beta} \pm 2\sqrt{\frac{1}{n}x'\hat{V}x'}$

4. It was not stated explicitly, but implicit in the notation we can see that γ is real valued. A convenient way to write the estimator $\hat{\gamma}$ is

$$\hat{\gamma} = \left(\hat{\beta}'X'X\hat{\beta}\right)^{-1}\hat{\beta}'X'Z$$

Since $Z = X\beta\gamma + u$, we see

$$\begin{aligned}
\hat{\gamma} &= \left(\hat{\beta}'X'X\hat{\beta}\right)^{-1}\hat{\beta}'X'\left(X\beta\gamma + u\right) \\
&= \left(\hat{\beta}'\frac{1}{n}X'X\hat{\beta}\right)^{-1}\hat{\beta}'\frac{1}{n}X'X\beta\gamma + \left(\hat{\beta}'\frac{1}{n}X'X\hat{\beta}\right)^{-1}\hat{\beta}'\frac{1}{n}X'u
\end{aligned}$$

Then since $\hat{\beta} \longrightarrow_p \beta$ and $\frac{1}{n}X'u$ $\frac{1}{n}X'u \longrightarrow_p 0$.

$$\hat{\gamma} \overset{p}{\longrightarrow} \left(\beta'Q\beta\right)^{-1}\beta'Q\beta\gamma + \left(\beta'Q\beta\right)^{-1}\beta'0 = \gamma$$

(Technically, this result requires that $\beta'Q\beta > 0$, otherwise γ is not identified.)

Another way to solve this is to write $\hat{\beta} = \left(X'X\right)^{-1}X'y$ and then

$$\begin{aligned}
\hat{\gamma} &= \left(y'X\left(X'X\right)^{-1}\left(X'X\right)\left(X'X\right)^{-1}X'y\right)^{-1}\left(y'X\left(X'X\right)^{-1}X'Z\right) \\
&= \left(y'X\left(X'X\right)^{-1}X'y\right)^{-1}\left(y'X\left(X'X\right)^{-1}X'Z\right) \\
&= \left(\left(\frac{1}{n}y'X\right)\left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}X'y\right)\right)^{-1}\left(\left(\frac{1}{n}y'X\right)\left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}X'Z\right)\right) \\
&\overset{p}{\longrightarrow} \left(E\left(y_i x_i'\right)\left(E\left(x_i x_i'\right)\right)^{-1}E\left(x_i y_i\right)\right)^{-1}\left(E\left(y_i x_i'\right)\left(E\left(x_i x_i'\right)\right)^{-1}E\left(x_i z_i\right)\right) \qquad (1)
\end{aligned}$$

We want to show that this equals γ. Since $y = x_i'\beta + e_i$ and $Ex_ie_i = 0$,

$$E(x_iy_i) = E\left(x_i\left(x_i'\beta + e_i\right)\right) = E\left(x_ix_i'\right)\beta$$

and since $z_i = x_i'\beta\gamma + u_i$ and $Ex_iu_i = 0$,

$$E(x_iz_i) = E\left(x_i\left(x_i'\beta\gamma + u_i\right)\right) = E\left(x_ix_i'\right)\beta\gamma$$

Therefore theright-hand-side of (1) equals

$$\left(\beta'E\left(x_ix_i'\right)\left(E\left(x_ix_i'\right)\right)^{-1}E\left(x_ix_i'\right)\beta\right)^{-1}\left(\beta'E\left(x_ix_i'\right)\left(E\left(x_ix_i'\right)\right)^{-1}E\left(x_ix_i'\right)\beta\gamma\right) = \gamma$$

The extra credit problem asked for the asymptotic distribution of $\hat{\gamma}$. In general this is tricky as you have to handle the joint distribution of $\hat{\beta}$ and $\hat{\gamma}$. But when $\gamma = 0$ the problem simplifies. Note that from the above equation when $\gamma = 0$

$$\begin{aligned}
\sqrt{n}\hat{\gamma} &= \left(\hat{\beta}'\frac{1}{n}X'X\hat{\beta}\right)^{-1}\hat{\beta}'\frac{1}{\sqrt{n}}X'u \\
&\xrightarrow{d} \left(\beta'Q\beta\right)^{-1}\beta'N(0,\Omega_u) \\
&= N\left(0, \frac{\beta'E\left(x_ix_i'u_i^2\right)\beta}{\left(\beta'E\left(x_ix_i'\right)\beta\right)^2}\right)
\end{aligned}$$

Common errors:

(a) Attempting to demonstrate consistency by taking expectations

(b) Treating $\hat{\beta}$ as a constant rather than a random variable

(c) Treating $\hat{\beta}$ as an invertible matrix

(d) Treating $\hat{\beta}$ as if it is a function of X. e.g. $E(\hat{\beta}'x_iu_i \mid X) = \hat{\beta}'x_iE(u_i \mid X)$ (this is incorrect since $\hat{\beta}$ is a function of X and y and the problem does not make an assumption about the relationship between e_i and u_i)

(e) Saying that the WLLN asserts that $n^{-1}\sum_{i=1}^n(\hat{\beta}'x_i)^2 \xrightarrow{p} E\left(\hat{\beta}'x_i\right)^2$ ($\hat{\beta}'x_i$ is not iid, as $\hat{\beta}$ depends on the full sample).