Econometrics 710
Midterm Exam, Spring 2005
Sample Answers

1. The answer is $\hat{V}^* = C^{-1}\hat{V}C^{-1\prime}$. Note that since C is $k \times k$ and full rank,

$$\begin{aligned}
\hat{\beta}^* &= \left(X^{*\prime}X^*\right)^{-1}\left(X^{*\prime}Y\right) \\
&= \left(C'X'XC\right)^{-1}\left(C'X'Y\right) \\
&= C^{-1}\left(X'X\right)^{-1}\left(C'\right)^{-1}C'X'Y \\
&= C^{-1}\left(X'X\right)^{-1}X'Y \\
&= C^{-1}\hat{\beta}
\end{aligned}$$

Note also that

$$\begin{aligned}
\hat{e}^* &= Y - X^*\hat{\beta}^* \\
&= Y - XCC^{-1}\hat{\beta} \\
&= Y - X\hat{\beta} \\
&= \hat{e}
\end{aligned}$$

which implies $\hat{e}_i^* = \hat{e}_i$. Then

$$\begin{aligned}
\hat{\Omega}^* &= \frac{1}{n}\sum_{i=1}^{n} x_i^* x_i^{*\prime} \hat{e}_i^{*2} \\
&= C'\frac{1}{n}\sum_{i=1}^{n} x_i x_i' \hat{e}_i^2 C \\
&= C'\hat{\Omega}C,
\end{aligned}$$

$$\begin{aligned}
\hat{Q}^* &= \frac{1}{n}X^{*\prime}X^* \\
&= C'\frac{1}{n}X'XC \\
&= C'\hat{Q}C,
\end{aligned}$$

and

$$\hat{Q}^{*-1} = \left(C'\hat{Q}C\right)^{-1} = C^{-1}\hat{Q}^{-1}C'^{-1}.$$

Thus

$$
\begin{aligned}
\hat{V}^* &= \hat{Q}^{*-1}\hat{\Omega}^*\hat{Q}^{*-1} \\
&= C^{-1}\hat{Q}^{-1}C'^{-1}C'\hat{\Omega}CC^{-1}\hat{Q}^{-1}C'^{-1} \\
&= C^{-1}\hat{Q}^{-1\prime}\hat{\Omega}\hat{Q}^{-1}C'^{-1} \\
&= C^{-1}\hat{V}C^{-1\prime}
\end{aligned}
$$

2. Since $E\left(y_i \mid x_i\right) = x_i\beta_1 + x_i^2\beta_2$, then $E\left(y_i \mid x_i = 40\right) = 40\beta_1 + 40^2\beta_2$. The hypothesis is thus

$$H_0 : 40\beta_1 + 40^2\beta_2 = 20$$

which is a linear restriction. If desired, this can be rewritten as

$$H_0 : 2\beta_1 + 80\beta_2 = 1$$

Let $(\hat{\beta}_1, \hat{\beta}_2)$ be the OLS estimates of the coefficients, and let $\hat{V}$ denote the estimated asymptotic covariance matrix. The Wald statistic for this hypothesis is

$$W_n = \frac{n\left(2\hat{\beta}_1 + 80\hat{\beta}_2 - 1\right)^2}{R'\hat{V}R}$$

where

$$R = \begin{pmatrix} 2 \\ 80 \end{pmatrix}$$

It has an asymptotic χ_1^2 distribution under H_0. A 5% size test is to reject H_0 if W_n exceeds the 5% χ_1^2 critical value of 3.84. Otherwise, H_0 is not rejected.

Alternatively, the 10% or 1% level could be used, or a t-statistic used instead of the Wald statistic. Furthermore, since the model si a regression the FGLS estimator could be used instead of the OLS estimator.

3. We calculate that

$$\begin{aligned}
\tilde{\beta} &= \left(\sum_{i=1}^{n} w_i x_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} w_i x_i y_i\right) \\
&= \beta + \left(\sum_{i=1}^{n} w_i x_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} w_i x_i e_i\right) \\
&= \beta + \left(\frac{1}{n}\sum_{i=1}^{n} w_i x_i x_i'\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} w_i x_i e_i\right) \\
&\to_p \beta + \left(E\left(w_i x_i x_i'\right)\right)^{-1} E\left(w_i x_i e_i\right)
\end{aligned}$$

by the WLLN. This implicitly assumes that the $k \times k$ matrix $E\left(w_i x_i x_i'\right)$ is invertible. The probability limit in general is not β, thus $\tilde{\beta}$ is inconsistent for β.

The question asks to find a assumption under which $\tilde{\beta}$ is consistent for β. A sufficient condition is $E\left(w_i x_i e_i\right) = 0$, we need to find a reasonable assumption which implies this. One assumption is that the regression model $E\left(e_i \mid x_i\right) = 0$, for then $E\left(w_i x_i e_i\right) = E\left(w_i x_i E\left(e_i \mid x_i\right)\right) = 0$. Another assumption is that $w(x_i) = w$ is a constant, but that is not a very interesting assumption given the context of the question.

4. We know that $\hat{\beta} - \beta = \left(X'X\right)^{-1}\left(X'e\right)$ and $\tilde{\beta} - \beta = \left(X'D^{-1}X\right)^{-1}\left(X'D^{-1}e\right)$. Thus

$$\begin{aligned}
E\left(\left(\hat{\beta} - \beta\right)\left(\tilde{\beta} - \beta\right)' \mid X\right) &= E\left(\left(X'X\right)^{-1} X'ee'D^{-1}X\left(X'D^{-1}X\right)^{-1} \mid X\right) \\
&= \left(X'X\right)^{-1} X'E\left(ee' \mid X\right) D^{-1}X\left(X'D^{-1}X\right)^{-1} \\
&= \left(X'X\right)^{-1} X'DD^{-1}X\left(X'D^{-1}X\right)^{-1} \\
&= \left(X'X\right)^{-1} X'X\left(X'D^{-1}X\right)^{-1} \\
&= \left(X'D^{-1}X\right)^{-1}.
\end{aligned}$$

Furthermore, we know that

$$E\left(\left(\hat{\beta} - \beta\right)\left(\hat{\beta} - \beta\right)' \mid X\right) = \left(X'X\right)^{-1} X'DX\left(X'X\right)^{-1}$$

and

$$E\left(\left(\tilde{\beta} - \beta\right)\left(\tilde{\beta} - \beta\right)' \mid X\right) = \left(X'D^{-1}X\right)^{-1}$$

Thus

$$
\begin{aligned}
E\left(\left(\hat{\beta}-\tilde{\beta}\right)\left(\hat{\beta}-\tilde{\beta}\right)' \mid X\right) &= E\left(\left(\left(\hat{\beta}-\beta\right)-\left(\tilde{\beta}-\beta\right)\right)\left(\left(\hat{\beta}-\beta\right)-\left(\tilde{\beta}-\beta\right)\right)' \mid X\right) \\
&= E\left(\left(\hat{\beta}-\beta\right)\left(\hat{\beta}-\beta\right)' \mid X\right) \\
&\quad +E\left(\left(\tilde{\beta}-\beta\right)\left(\tilde{\beta}-\beta\right)' \mid X\right) \\
&\quad -E\left(\left(\hat{\beta}-\beta\right)\left(\tilde{\beta}-\beta\right)' \mid X\right) \\
&\quad -E\left(\left(\tilde{\beta}-\beta\right)\left(\hat{\beta}-\beta\right)' \mid X\right) \\
&= \left(X'X\right)^{-1} X'DX \left(X'X\right)^{-1} \\
&\quad + \left(X'D^{-1}X\right)^{-1} - \left(X'D^{-1}X\right)^{-1} - \left(X'D^{-1}X\right)^{-1} \\
&= \left(X'X\right)^{-1} X'DX \left(X'X\right)^{-1} - \left(X'D^{-1}X\right)^{-1} \\
&= Var(\hat{\beta}) - Var(\tilde{\beta})
\end{aligned}
$$