

Econometrics 710
Midterm Exam, Spring 2003
Sample Answers

1. Substituting in $Y = X\beta + e$ and $\lambda = cn$ we see

$$\begin{aligned}\hat{\beta} &= (X'X + cnI_k)^{-1} (X'Y) \\ &= \left(\frac{1}{n}X'X + cI_k\right)^{-1} \left(\frac{1}{n}X'X\beta + \frac{1}{n}X'e\right) \\ &\xrightarrow{p} (Q + cI_k)^{-1} Q\beta \\ &= \beta - c(Q + cI_k)^{-1} \beta \neq \beta\end{aligned}$$

2. The moment conditions are

$$g(w_i, \beta, \Omega) = \begin{pmatrix} x_i(y_i - x_i'\beta) \\ (x_i x_i' (y_i - x_i'\beta)^2) - \Omega \end{pmatrix}.$$

The sample average is

$$\bar{g}_n(\beta, \Omega) = \frac{1}{n} \sum_{i=1}^n g(w_i, \beta, \Omega)$$

The MME $(\hat{\beta}, \hat{\Omega})$ sets $\bar{g}_n(\hat{\beta}, \hat{\Omega}) = 0$. Thus

$$\frac{1}{n} \sum_{i=1}^n x_i (y_i - x_i'\hat{\beta}) = 0 \tag{1}$$

$$\frac{1}{n} \sum_{i=1}^n \left((x_i x_i' (y_i - x_i'\hat{\beta})^2) - \hat{\Omega} \right) = 0 \tag{2}$$

From (1) we see $\hat{\beta} = (X'X)^{-1} (X'Y)$. From (2) we see

$$\hat{\Omega} = \frac{1}{n} \sum_{i=1}^n \left(x_i x_i' (y_i - x_i'\hat{\beta})^2 \right) = \frac{1}{n} \sum_{i=1}^n x_i x_i' \hat{e}_i^2$$

where $\hat{e}_i = y_i - x_i'\hat{\beta}$. In this model, these estimators are efficient in the sense of asymptotic semi-parametric efficiency – no other consistent estimator has a lower asymptotic variance. We know this is true because in just-identified moment models, where there is no information other than the stated moments, the MME is semi-parametrically efficient.

3. By the delta method, the asymptotic distribution of $\hat{\theta}$ is

$$\sqrt{n} (\hat{\theta} - \theta) \rightarrow^d N(0, V_\theta)$$

where

$$V_\theta = \left(\frac{d}{d\beta} (\beta^2) \right)^2 V = (2\beta)^2 V = 4\beta^2 V$$

Thus a valid asymptotic standard error for $\hat{\theta}$ is $s(\hat{\theta}) = 2|\hat{\beta}| \left(\hat{V}/n\right)^{1/2}$. Let $Z_{\alpha/2}$ denote the $\alpha/2$ quantile of the $N(0, 1)$ distribution. An asymptotic $(1 - \alpha)\%$ confidence interval for θ is

$$\hat{\theta} \pm Z_{\alpha/2}s(\hat{\theta}) = \hat{\beta}^2 \pm Z_{\alpha/2}2|\hat{\beta}| \left(\frac{\hat{V}}{n}\right)^{1/2}.$$

If $\beta = 0$, the asymptotic distribution for $\hat{\theta}$ is degenerate. Since $V_{\theta} = 0$, it follows that $\sqrt{n}(\hat{\theta} - \theta) \rightarrow^d N(0, 0) = 0$ almost surely. In consequence, the asymptotic confidence interval fails to have asymptotic coverage $(1 - \alpha)\%$.

Instead, when $\beta = 0$, you can observe that $\sqrt{n}\hat{\beta} \rightarrow^d N(0, V)$ implies

$$\frac{n\hat{\theta}}{V} = \frac{(\sqrt{n}\hat{\beta})^2}{V} \rightarrow^d \frac{N(0, V)^2}{V} = \chi_1^2.$$

The discontinuity in the asymptotic distribution at $\beta = 0$ is the source of the problem.

In this simple context, a better confidence interval for θ can be found by squaring the endpoints of the confidence interval for β :

$$\left(\hat{\beta} - Z_{\alpha/2} \left(\frac{\hat{V}}{n}\right)^{1/2}\right)^2, \quad \left(\hat{\beta} + Z_{\alpha/2} \left(\frac{\hat{V}}{n}\right)^{1/2}\right)^2$$

4. We know that $n^{1/2}(\hat{\beta} - \beta) \rightarrow^d N(0, V)$ where $V = Q^{-1}\Omega Q^{-1}$, $Q = E(x_i x_i')$, $\Omega = E(x_i x_i' e_i^2)$, so $\hat{\beta}$ is consistent. We also know that in the context of the regression model, $\hat{\beta}$ is generally inefficient. It is efficient (in the semi-parametric sense) under homoskedasticity: $E(e_i^2 | x_i) = \sigma^2$.

Now consider $\tilde{\beta}$. We see that if $E\left|\frac{e_i}{x_i}\right| < \infty$, then by the law of iterated expectations,

$$E\left(\frac{e_i}{x_i}\right) = E\left(E\left(\frac{e_i}{x_i} \mid x_i\right)\right) = E\left(\frac{1}{x_i} E(e_i \mid x_i)\right) = 0.$$

Thus by the WLLN,

$$\tilde{\beta} = \frac{1}{n} \sum_{i=1}^n \frac{y_i}{x_i} = \frac{1}{n} \sum_{i=1}^n \frac{x_i \beta + e_i}{x_i} = \beta + \frac{1}{n} \sum_{i=1}^n \frac{e_i}{x_i} \rightarrow_p \beta + E\left(\frac{e_i}{x_i}\right) = \beta$$

Hence $\tilde{\beta}$ is consistent.

Now observe that we can obtain the estimator $\tilde{\beta}$ by dividing $y_i = x_i \beta + e_i$ by x_i so that

$$\frac{y_i}{x_i} = \beta + \frac{e_i}{x_i}$$

and then estimating β by OLS (which is same as taking the average of y_i/x_i). Therefore $\tilde{\beta}$ is WLS using weights $w_i = x_i^{-1}$. WLS equals GLS when $w_i = \sigma_i^{-1}$. Since in the regression model GLS is efficient, $\tilde{\beta}$ is efficient when $\sigma_i^2 = x_i^2$.