Econometrics 710
Final Exam, Spring 2017
Sample Answers

1. IV Regression

(a) $\widehat{\beta} = (Z'X)^{-1} Z'Y = \left(\sum_{i=1}^n z_i x_i'\right)^{-1} \left(\sum_{i=1}^n z_i y_i\right)$

(b) Write

$$\widehat{\beta} - \beta = \left(\sum_{i=1}^n z_i x_i'\right)^{-1} \left(\sum_{i=1}^n z_i e_i\right)$$

Then

$$\begin{aligned}
E\left(\widehat{\beta} - \beta \mid Z, X\right) &= E\left(\left(\sum_{i=1}^n z_i x_i'\right)^{-1} \left(\sum_{i=1}^n z_i e_i\right) \mid Z, X\right) \\
&= \left(\sum_{i=1}^n z_i x_i'\right)^{-1} \left(\sum_{i=1}^n z_i E\left(e_i \mid Z, X\right)\right) \\
&= \left(\sum_{i=1}^n z_i x_i'\right)^{-1} \left(\sum_{i=1}^n z_i E\left(e_i \mid z_i, x_i\right)\right) \\
&= 0
\end{aligned}$$

Thus by the law of iterated expectations

$$E\left(\widehat{\beta}\right) = \beta$$

and $\widehat{\beta}$ is unbiased for β.

(c) Since $E\left(\widehat{\beta} \mid Z, X\right) = \beta$

$$\begin{aligned}
\mathrm{var}\left(\widehat{\beta} | X, Z\right) &= E\left(\left(\widehat{\beta} - \beta\right)\left(\widehat{\beta} - \beta\right)' \mid Z, X\right) \\
&= E\left(\left(\sum_{i=1}^n z_i x_i'\right)^{-1} \left(\sum_{i=1}^n z_i e_i\right) \left(\sum_{i=1}^n e_i z_i'\right) \left(\sum_{i=1}^n x_i z_i'\right)^{-1} \mid Z, X\right) \\
&= \left(\sum_{i=1}^n z_i x_i'\right)^{-1} \left(\sum_{i=1}^n \sum_{j=1}^n E\left(z_i e_i e_j z_j' \mid Z, X\right)\right) \left(\sum_{i=1}^n x_i z_i'\right)^{-1} \\
&= \left(\sum_{i=1}^n z_i x_i'\right)^{-1} \left(\sum_{i=1}^n E\left(z_i e_i e_i z_i' \mid z_i, x_i\right)\right) \left(\sum_{i=1}^n x_i z_i'\right)^{-1} \\
&= \left(\sum_{i=1}^n z_i x_i'\right)^{-1} \left(\sum_{i=1}^n z_i z_i' \sigma_i^2\right) \left(\sum_{i=1}^n x_i z_i'\right)^{-1}
\end{aligned}$$

where

$$\sigma_i^2 = E\left(e_i^2 \mid z_i, x_i\right)$$

Notice that this is the variance conditional on both z and x

2. Control function regression.

   (a) The reduced form equation for x_i is $x_i = \Gamma' z_i + u_i$ so

$$\begin{aligned} E(x_i \varepsilon_i) &= E((\Gamma' z_i + u_i)\varepsilon_i) \\ &= \Gamma' E(z_i \varepsilon_i) + E(u_i \varepsilon_i). \end{aligned}$$

   The definition for ε_i is from $e_i = u_i'\gamma + \varepsilon_i$ so $\varepsilon_i = e_i - u_i'\gamma$. Substituting into the first expression on the right side, we find

$$\begin{aligned} E(x_i \varepsilon_i) &= \Gamma' E(z_i (e_i - u_i'\gamma)) + E(u_i \varepsilon_i) \\ &= \Gamma' E(z_i e_i) - \Gamma' E(z_i u_i')\gamma + E(u_i \varepsilon_i) \end{aligned}$$

   The three components are each zero. First, $E(z_i e_i) = 0$ by the IV assumption. Second, $E(z_i u_i') = 0$ by the reduced form projection equation. Third, $E(u_i \varepsilon_i) = 0$ by the control function projection equation.

   (b) Derive the asymptotic distribution of $(\widehat{\beta}, \widehat{\gamma})$. First, write $w_i = (x_i', u_i')'$ so that we can write the estimators as

$$\begin{pmatrix} \widehat{\beta} \\ \widehat{\gamma} \end{pmatrix} = \left(\sum_{i=1}^n w_i w_i'\right)^{-1} \left(\sum_{i=1}^n w_i y_i\right).$$

   Centered and standardized, since $y_i = x_i'\beta + u_i'\gamma + \varepsilon_i$

$$\sqrt{n}\begin{pmatrix} \widehat{\beta} - \beta \\ \widehat{\gamma} - \gamma \end{pmatrix} = \left(\frac{1}{n}\sum_{i=1}^n w_i w_i'\right)^{-1} \left(\frac{1}{\sqrt{n}}\sum_{i=1}^n w_i \varepsilon_i\right).$$

   By the WLLN

$$\frac{1}{n}\sum_{i=1}^n w_i w_i' \to_p E(w_i w_i') = \begin{pmatrix} E(x_i x_i') & E(x_i u_i') \\ E(u_i x_i') & E(u_i u_i') \end{pmatrix} = Q$$

   say. The WLLN applies since w_i are iid, if x_i and u_i have finite second moments. By the CLT

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n w_i \varepsilon_i \to_d N(0, \Omega)$$

   where

$$\Omega = E(w_i w_i' \varepsilon_i^2) = \begin{pmatrix} E(x_i x_i' \varepsilon_i^2) & E(x_i u_i' \varepsilon_i^2) \\ E(u_i x_i' \varepsilon_i^2) & E(u_i u_i' \varepsilon_i^2) \end{pmatrix}$$

   The CLT applies if $w_i \varepsilon_i$ is iid, mean zero, and has finite second moments. The vector $w_i \varepsilon_i$ is iid because it is assumed to be a random sample. It is mean zero since $E(x_i \varepsilon_i) = 0$ by part (a) and $E(u_i \varepsilon_i) = 0$ by the control function projection. The variable $w_i \varepsilon_i$ has finite second moment if the observations have finite fourth moments.

Together, we obtain

$$\sqrt{n}\left(\begin{array}{c}\widehat{\beta}-\beta\\ \widehat{\gamma}-\gamma\end{array}\right)\rightarrow_d Q^{-1}N\left(0,\Omega\right)=N\left(0,Q^{-1}\Omega Q^{-1}\right).$$

In addition, this requires that Q^{-1} exists.

3. GMM criterion

(a) Evaluated at the true value β_0,

$$\overline{m}_n(\beta_0)=\frac{1}{n}\sum_{i=1}^{n}z_ie_i$$

which is the average of iid random vectors with mean zero (by assumption). Standardized, the CLT implies

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}z_ie_i\rightarrow_d Z\sim N\left(0,\Omega\right)$$

where

$$\Omega=E\left(z_iz_i'e_i^2\right)$$

It follows that

$$J_n(\beta_0)=\sqrt{n}\overline{m}_n(\beta_0)'W\sqrt{n}\overline{m}_n(\beta_0)\rightarrow_d Z'WZ$$

(b) If $W=\Omega^{-1}$ then the limit distribution is $Z'\Omega^{-1}Z\sim\chi^2_\ell$ where ℓ is the dimension of z_i.

(c) An estimator which takes advantage of H_0 is $\widetilde{\Omega}=\frac{1}{n}\sum_{i=1}^{n}z_iz_i'\widetilde{e}_i^2$ where $\widetilde{e}_i=y_i-x_i'\beta_0$ and $\widetilde{W}=\widetilde{\Omega}^{-1}$. The estimator $\widetilde{\Omega}$ is unbiased for Ω, and $\widetilde{W}$ is consistent for $W=\Omega^{-1}$ under H_0. This is different than the standard estimator $\widehat{\Omega}=\frac{1}{n}\sum_{i=1}^{n}z_iz_i'\widehat{e}_i^2$ and $\widehat{W}=\widehat{\Omega}^{-1}$ where $\widehat{e}_i=y_i-x_i'\widehat{\beta}$ and $\widehat{\beta}$ is a consistent estimator of β, for example $\widehat{\beta}=\left(X'Z\left(Z'Z\right)^{-1}Z'X\right)^{-1}\left(X'Z\left(Z'Z\right)^{-1}Z'Y\right)$. This does not take advantage of H_0.

(d) An asymptotic test rejects H_0 in favor of H_1 at level α if $J_n(\beta_0)>c$ where c is the $1-\alpha$ quantile of the distribution of χ^2_ℓ. This test has asymptotic level of α since

$$P\left(J_n(\beta_0)>c|H_0\right)\rightarrow P\left(\chi^2_\ell>c\right)=\alpha$$

(e) A $1-\alpha$ confidence region for β is the set β for which the test does not reject. It is

$$C=\{\beta:J_n(\beta)\le c\}$$

where c is the $1-\alpha$ quantile of the distribution of χ^2_ℓ. When $\widetilde{W}(\beta)=\widetilde{\Omega}(\beta)^{-1}=\frac{1}{n}\sum_{i=1}^{n}z_iz_i'\left(y_i-x_i'\beta\right)^2$ then the weight matrix depends on β and the confidence region is

$$C=\left\{\beta:\left(Y'Z-\beta'X'Z\right)'\widetilde{\Omega}(\beta)^{-1}\left(Z'Y-Z'X\beta\right)\le nc\right\}$$

which is not an ellipse.

(f) For a bootstrap test, sample (y_i^*,x_i^*,z_i^*) iid from the data. If the estimator $\widetilde{\Omega}$ was used, then we set

$$
\begin{aligned}
J_n^* &= n\overline{m}_n^{*\prime}\widetilde{\Omega}^{*-1}\overline{m}_n^* \\
\overline{m}_n^* &= \frac{1}{n}\sum_{i=1}^{n} z_i^*\left(y_i^* - x_i^{*\prime}\widehat{\beta}\right) \\
\widetilde{\Omega}^* &= \frac{1}{n}\sum_{i=1}^{n} z_i^* z_i^{*\prime}\widetilde{e}_i^{*2} \\
\widetilde{e}_i^* &= y_i^* - x_i^{*\prime}\widehat{\beta}
\end{aligned}
$$

where $\widehat{\beta}$ is either 2SLS or GMM on the original sample. This is appropriate because $\widehat{\beta}$ is the analog of β_0 in the bootstrap distribution.
If the estimator $\widehat{\Omega}$ was used then we would alternatively set

$$
\begin{aligned}
J_n^* &= n\overline{m}_n^{*\prime}\widehat{\Omega}^{*-1}\overline{m}_n^* \\
\widehat{\Omega}^* &= \frac{1}{n}\sum_{i=1}^{n} z_i^* z_i^{*\prime}\widehat{e}_i^{*2} \\
\widehat{e}_i^* &= y_i^* - x_i^{*\prime}\widehat{\beta}^* \\
\widehat{\beta}^* &= \left(X^{*\prime}Z^*\left(Z^{*\prime}Z^*\right)^{-1}Z^{\prime *}X^*\right)^{-1}\left(X^{\prime *}Z^*\left(Z^{\prime *}Z^*\right)^{-1}Z^{\prime *}Y^*\right)
\end{aligned}
$$

That is, $\widehat{e}_i^*$ is calculated using the bootstrap estimate $\widehat{\beta}^*$
In either case, we obtain B replications of the statistic J_{nb}^* by simulation. The p-value for the test is then calculated as

$$
p_n^* = \frac{1}{B}\sum_{b=1}^{B} 1\left(J_{nb}^* > J_n(\beta_0)\right)
$$

The bootstrap test rejects H_0 at level α if $p_n^* < \alpha$, otherwise it does not reject H_0.