Econometrics 710
Final Exam, Spring 2014

1.

(a) $\widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\left(e_{1i}^2 - e_{2i}^2\right) = \widehat{\sigma}_1^2 - \widehat{\sigma}_2^2$ where $\widehat{\sigma}_1^2 = \frac{1}{n}\sum_{i=1}^{n} e_{1i}^2$ and $\widehat{\sigma}_2^2 = \frac{1}{n}\sum_{i=1}^{n} e_{2i}^2$

(b) Let $u_i = \left(e_{1i}^2 - e_{2i}^2\right) - \theta = \left(e_{1i}^2 - e_{2i}^2\right) - \left(\sigma_1^2 - \sigma_2^2\right)$. Note that $Eu_i^2 = 0$ and

$$\begin{aligned} var\left(u_i\right) &= E\left(\left(e_{1i}^2 - e_{2i}^2\right) - \left(\sigma_1^2 - \sigma_2^2\right)\right)^2 \\ &= E\left(\left(e_{1i}^2 - \sigma_1^2\right) - \left(e_{2i}^2 - \sigma_2^2\right)\right)^2 \\ &= E\left(e_{1i}^2 - \sigma_1^2\right)^2 + E\left(e_{2i}^2 - \sigma_2^2\right)^2 + 2E\left(\left(e_{1i}^2 - \sigma_1^2\right)\left(e_{2i}^2 - \sigma_2^2\right)\right) \\ &= E\left(e_{1i}^4\right) - \sigma_1^4 + E\left(e_{2i}^4\right) - \sigma_2^4 + 2E\left(e_{1i}^2 e_{2i}^2\right) - \sigma_1^2\sigma_2^2 \\ &= \sigma_v^2 \end{aligned}$$

say. Alternatively, you could write

$$var\left(u_i\right) = E\left(\left(e_{1i}^2 - e_{2i}^2\right)^2\right) - \left(\sigma_1^2 - \sigma_2^2\right)^2$$

By the CLT

$$\sqrt{n}\left(\widehat{\theta} - \theta\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} u_i \rightarrow_d N(0, \sigma_v^2)$$

(c) $\widehat{\sigma}_v^2 = \frac{1}{n}\sum_{i=1}^{n}\widehat{u}_i^2$ where $\widehat{u}_i = u_i = \left(e_{1i}^2 - e_{2i}^2\right) - \widehat{\theta}$. Equivalently,

$$\widehat{\sigma}_v^2 = \frac{1}{n}\sum_{i=1}^{n} e_{1i}^4 - \widehat{\sigma}_1^4 + \frac{1}{n}\sum_{i=1}^{n} e_{2i}^4 - \widehat{\sigma}_2^4 + 2\frac{1}{n}\sum_{i=1}^{n} e_{1i}^2 e_{2i}^2 - \widehat{\sigma}_1^2\widehat{\sigma}_2^2$$

or

$$\widehat{\sigma}_v^2 = \frac{1}{n}\sum_{i=1}^{n}\left(e_{1i}^2 - e_{2i}^2\right)^2 - \left(\widehat{\sigma}_1^2 - \widehat{\sigma}_2^2\right)^2$$

(d) A t-ratio is $T_n = \sqrt{n}\widehat{\theta}/\widehat{\sigma}_v$. A test of asymptotic size $\alpha$ rejects $H_0$ in favor of $H_1$ if $|T_n| > Z_{\alpha/2}$ where $Z_{\alpha/2}$ is the $\alpha/2$ quantile of the normal distribution, and accepts $H_0$ if $|T_n| \leq Z_{\alpha/2}$

(e) If the test accepts $H_0$, this means that the evidence cannot reject the hypothesis that the two models have equal fit. Based on the evidence, we cannot say if model (1) or model (2) is better than the other.

2.

(a) In matrix notation, $Y = X_1\beta_1 + X_2\beta_2 + e$ and

$$\widehat{\beta}_1 = \left(\left(X_1'Z\right)\left(Z'Z\right)^{-1}\left(Z'X_1\right)\right)^{-1}\left(X_1'Z\right)\left(Z'Z\right)^{-1}\left(Z'Y\right)$$

The estimate is overidentified since $\ell > k_1$ .

Substituting $Y = X_1\beta_1 + X_2\beta_2 + e$ for $Y$,

$$\begin{aligned}
\widehat{\beta}_1 &= \beta_1 + \left( (X_1'Z)(Z'Z)^{-1}(Z'X_1) \right)^{-1} (X_1'Z)(Z'Z)^{-1}(Z'X_2)\beta_2 \\
&\quad + \left( (X_1'Z)(Z'Z)^{-1}(Z'X_1) \right)^{-1} (X_1'Z)(Z'Z)^{-1}(Z'e) \\
&= \beta_1 + b_{1n} + r_{1n}
\end{aligned}$$

where

$$\begin{aligned}
b_{1n} &= \left( (X_1'Z)(Z'Z)^{-1}(Z'X_1) \right)^{-1} (X_1'Z)(Z'Z)^{-1}(Z'X_2)\beta_2 \\
r_{1n} &= \left( (X_1'Z)(Z'Z)^{-1}(Z'X_1) \right)^{-1} (X_1'Z)(Z'Z)^{-1}(Z'e)
\end{aligned}$$

(b) Since $\frac{1}{n}Z'e \xrightarrow{p} 0$,

$$\begin{aligned}
r_{1n} &= \left( \left(\frac{1}{n}X_1'Z\right) \left(\frac{1}{n}Z'Z\right)^{-1} \left(\frac{1}{n}Z'X_1\right) \right)^{-1} \left(\frac{1}{n}X_1'Z\right) \left(\frac{1}{n}Z'Z\right)^{-1} \left(\frac{1}{n}Z'e\right) \\
&\xrightarrow{p} \left(Q_{z1}'Q_{zz}^{-1}Q_{z1}\right)^{-1} Q_{z1}'Q_{zz}^{-1}0 \\
&= 0
\end{aligned}$$

where we write $Q_{z1} = Ez_ix_{1i}'$. Since $Q_{zx} = [Q_{z1}, Q_{z2}]$ has full rank, $Q_{z1}$ has full rank so the matrix $Q_{z1}'Q_{zz}^{-1}Q_{z1}$ is invertible under the assumptions.

(c)

$$\begin{aligned}
b_{1n} &= \left( \left(\frac{1}{n}X_1'Z\right) \left(\frac{1}{n}Z'Z\right)^{-1} \left(\frac{1}{n}Z'X_1\right) \right)^{-1} \left(\frac{1}{n}X_1'Z\right) \left(\frac{1}{n}Z'Z\right)^{-1} \left(\frac{1}{n}Z'X_2\right)\beta_2 \\
&\xrightarrow{p} \left(Q_{z1}'Q_{zz}^{-1}Q_{z1}\right)^{-1} Q_{z1}'Q_{zz}^{-1}Q_{z2}\beta_2
\end{aligned}$$

Since $\widehat{\beta}_1 = \beta_1 + b_{1n} + r_{1n}$, it follows that $\widehat{\beta}_1 \xrightarrow{p} \beta_1 + \left(Q_{z1}'Q_{zz}^{-1}Q_{z1}\right)^{-1} Q_{z1}'Q_{zz}^{-1}Q_{z2}\beta_2$

(d) The estimator $\widehat{\beta}_1$ suffers from omitted variables bias in the sense that it is inconsistent for $\beta_1$. The probability limit is $\beta_1 + \left(Q_{z1}'Q_{zz}^{-1}Q_{z1}\right)^{-1} Q_{z1}'Q_{zz}^{-1}Q_{z2}\beta_2$. This equals $\beta_1$ (and thus there is no omitted variables bias) in one of two cases. First, when $\beta_2 = 0$ (the variables $x_{2i}$ have true zero coefficients). Second, when $Q_{z1}'Q_{zz}^{-1}Q_{z2} = 0$. This second condition is a bit tricky to interpret. Notice that the assumptions include $Q_{zx} = [Q_{z1}, Q_{z2}]$ has full rank $k$, which implies that both $Q_{z1}$ and $Q_{z2}$ have full rank. Thus the answer "there is no asymptotic bias when $Q_{z2} = 0$" is in conflict with this assumption. However, it is still possible for $Q_{z1}'Q_{zz}^{-1}Q_{z2} = 0$. It occurs when the projections of $x_{1i}$ and $x_{2i}$ onto $z_i$ are uncorrelated. To see this, suppose without loss of generality that $Q_{zz} = I$ (or equivalently redefine the instrument vector by rotation). Then the plim of the estimator is $\beta_1 + \left(Q_{z1}'Q_{z1}\right)^{-1} Q_{z1}'Q_{z2}\beta_2$ This equals $\beta_1$ when $Q_{z1}'Q_{z2} = 0$. This can occur when both $Q_{z1}$ and $Q_{z2}$ are full rank but mutually orthogonal. You can also interpret $Q_{z1}$ as the population coefficients from a projection of $x_{1i}$ on $z_i$, and similarly $Q_{z2}$ as the coefficients from a projection of $x_{2i}$ on $z_i$. $Q_{z1}'Q_{zz}^{-1}Q_{z2} = 0$ occurs when these two regression coefficients are orthogonal.

(e)

$$
\begin{aligned}
\sqrt{n}\left(\widehat{\beta}_1 - \beta_1 - b_{1n}\right) &= \sqrt{n} r_{1n} \\
&= \left(\left(\frac{1}{n}X_1'Z\right)\left(\frac{1}{n}Z'Z\right)^{-1}\left(\frac{1}{n}Z'X_1\right)\right)^{-1}\left(\frac{1}{n}X_1'Z\right)\left(\frac{1}{n}Z'Z\right)^{-1}\left(\frac{1}{\sqrt{n}}Z'e\right) \\
&\xrightarrow{d} \left(Q_{z1}'Q_{zz}^{-1}Q_{z1}\right)^{-1}Q_{z1}'Q_{zz}^{-1}N(0,\Omega) \\
&= N(0,V)
\end{aligned}
$$

where $\Omega = Ez_iz_i'e_i^2$ and

$$
V = \left(Q_{z1}'Q_{zz}^{-1}Q_{z1}\right)^{-1}Q_{z1}'Q_{zz}^{-1}\Omega Q_{zz}^{-1}Q_{z1}\left(Q_{z1}'Q_{zz}^{-1}Q_{z1}\right)^{-1}
$$