Econometrics 710
Final Exam, Spring 2013
Sample Answers

1. Linear IV

   (a) No. There is no exclusion restriction. There is only one instrument yet two coefficients. Thus the 2SLS estimator is not defined.

   (b) Yes. With two instruments we can define the 2SLS estimator. Both z_i and z_i^2 are valid instruments, since $E(z_i e_i) = 0$ and $E(z_i^2 e_i) = 0$ given $E(e_i | z_i) = 0$.

   (c) The excluded variable is z_i^2. The implicit exclusion restriction is that in the structural equation, z_i^2 has a true zero coefficient. That is, if we consider the augmented model

   $$y_i = x_i\beta_1 + z_i\beta_2 + z_i^2\beta_3 + e_i$$

   that the true value of $\beta_3 = 0$. This is what it means that z_i^2 is the excluded variable.

   (d) The reduced form for x_i is

   $$x_i = z_i\gamma_1 + z_i^2\gamma_2$$

   The excluded variable z_i^2 is relevant if $\gamma_2 \neq 0$. The implicit assumption is that $\gamma_2 \neq 0$, which means that the reduce form for x_i is quadratic in z_i

   (e) The use of z_i^2 as an instrument is valid when the reduced form for x_i is a non-trivial quadratic in z_i yet the equation for y_i is linear in z_i. Identification rests on this distinction. Linear structural equation with quadratic reduced form. This is generically arbitrary, and I would not be comfortable with these assumptions (especially the second) in a general application. The exception would be a case where a model specifically predicts that the effect of z_i on y_i is linear yet the effect of z_i on x_i is nonlinear.

2. Measurement error.

   (a) By substitution, we see that

   $$\begin{aligned} y_i &= x_i'\beta + e_i + u_i \\ &= x_i'\beta + v_i \end{aligned}$$

   where

   $$v_i = e_i + u_i.$$

   Note that

   $$\begin{aligned} E(v_i|x_i) &= E(e_i|x_i) + E(u_i|x_i) = 0 \\ E(v_i^2|x_i) &= E(e_i^2|x_i) + E(u_i^2|x_i) + 2E(e_iu_i|x_i) \\ &= \sigma^2 + \sigma_u^2(x_i) \end{aligned}$$

   Thus the equation error is a heteroskedastic CEF, with an error which has a larger variance than the case without measurement error.

   (b) The effect of this measurement error on OLS is

i. OLS remains consistent and asymptotically normal

ii. The asymptotic variance of $\hat{\beta}$ takes the heteroskedastic form

iii. The asymptotic variance of $\hat{\beta}$ is larger in the presence of measurement error than without measurement error. This means that the estimates are less precise.

(c) Standard errors should be calculated with the heteroskedasticity-consistent formula.

3. Just-identified 2SLS

(a)

$$\hat{\beta}_{2SLS} = \left(Z'X\right)^{-1}\left(Z'Y\right)^{-1} = \left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} z_i y_i\right)$$

(b) Note that using $x_i = \gamma z_i + u_i$ and $E\left(z_i u_i\right) = 0$

$$E\left(z_i x_i\right) = E\left(z_i\left(\gamma z_i + u_i\right)\right) = \gamma Q$$

Thus

$$\frac{1}{n}\sum_{i=1}^{n} z_i x_i \rightarrow_p E\left(z_i x_i\right) = \gamma Q$$

Hence

$$\begin{aligned}\sqrt{n}\left(\hat{\beta} - \beta\right) &= \left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i e_i\right) \\ &\rightarrow_d \left(\gamma Q\right)^{-1} N\left(0, \Omega\right) = N\left(0, \frac{\Omega}{\gamma^2 Q^2}\right)\end{aligned}$$

4. Indirect Least Squares

(a) By substitution,

$$\begin{aligned}y_i &= \left(\gamma z_i + u_i\right)\beta + e_i \\ &= z_i\gamma\beta + u_i\beta + e_i \\ &= z_i\lambda + v_i\end{aligned}$$

with $\gamma\beta = \lambda$ and $v_i = u_i\beta + e_i$. Thus $\beta = \lambda/\gamma$ when $\gamma \neq 0$. Also, since $v_i = u_i\beta + e_i$

$$E\left(z_i v_i\right) = E\left(z_i u_i\right)\beta + E\left(z_i e_i\right) = 0$$

(b) From the standard OLS formula

$$\begin{aligned}\sqrt{n}\left(\hat{\lambda} - \lambda\right) &= \left(\frac{1}{n}\sum_{i=1}^{n} z_i^2\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i v_i\right) \\ \sqrt{n}\left(\hat{\gamma} - \gamma\right) &= \left(\frac{1}{n}\sum_{i=1}^{n} z_i^2\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i u_i\right).\end{aligned}$$

Stacking,

$$\sqrt{n}\left(\hat{\theta}-\theta\right)=\left(\frac{1}{n}\sum_{i=1}^{n}z_i^2\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}z_i\xi_i\right).$$

(c) Since $E(z_iv_i)=0$ and $E(z_iu_i)=0$, then

$$E(z_i\xi_i)=\begin{pmatrix} E(z_iv_i) \\ E(z_iu_i) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(d) By the CLT

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}z_i\xi_i \to_d N\left(0,\Omega_\xi\right)$$

and by the WLLN

$$\frac{1}{n}\sum_{i=1}^{n}z_i^2 \to_p Q=E\left(z_i^2\right).$$

Therefore

$$\sqrt{n}\left(\hat{\theta}-\theta\right)\to_d Q^{-1}N\left(0,\Omega_\xi\right)=N\left(0,Q^{-2}\Omega_\xi\right) \tag{1}$$

(e) Notice $\hat{\beta}=\hat{\lambda}/\hat{\gamma}=g(\hat{\theta})$ with

$$\frac{\partial}{\partial\theta}g(\theta)=\begin{pmatrix} \frac{\partial}{\partial\lambda}\left(\frac{\lambda}{\gamma}\right) \\ \frac{\partial}{\partial\gamma}\left(\frac{\lambda}{\gamma}\right) \end{pmatrix}=\begin{pmatrix} \frac{1}{\gamma} \\ -\frac{\lambda}{\gamma^2} \end{pmatrix}=\frac{1}{\gamma}\begin{pmatrix} 1 \\ -\beta \end{pmatrix}$$

The Delta method shows that the estimator $\hat{\beta}=\hat{\lambda}/\hat{\gamma}$ has the asymptotic distribution

$$\sqrt{n}\left(\hat{\beta}-\beta\right)\to_d N\left(0,V_\beta\right)$$

where

$$\begin{aligned} V_\beta &= \left(\frac{\partial}{\partial\theta}g(\theta)\right)' Q^{-2}\Omega_\xi\left(\frac{\partial}{\partial\theta}g(\theta)\right) \\ &= \frac{1}{\gamma^2Q^2}\begin{pmatrix} 1 & -\beta \end{pmatrix}\Omega_\xi\begin{pmatrix} 1 \\ -\beta \end{pmatrix} \end{aligned}$$

But observe that

$$\xi_i\begin{pmatrix} 1 \\ -\beta \end{pmatrix}=v_i-u_i\beta=e_i$$

so that

$$\begin{pmatrix} 1 & -\beta \end{pmatrix}\Omega_\xi\begin{pmatrix} 1 \\ -\beta \end{pmatrix}=E(z_i\xi_i\begin{pmatrix} 1 \\ -\beta \end{pmatrix})^2=E(z_ie_i)^2=\Omega$$

Thus $V_\beta=\Omega/\gamma^2Q^2$ and $\sqrt{n}\left(\hat{\beta}-\beta\right)\to_d N\left(0,\frac{\Omega}{\gamma^2Q^2}\right)$

(f) Yes, this is the same as the distribution in question 3. It should be, as the estimators are algebraically identical!