Econometrics 710
Final Exam, Spring 2011
Sample Answers

1.
$$\hat{\beta} = \beta + \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i e_i\right) \rightarrow_p \beta + \boldsymbol{Q}_{xx}^{-1} E\left(\mathbf{x}_i e_i\right)$$

Because the equation is just-identified,

$$\begin{aligned}
\tilde{\beta} &= \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i\mathbf{x}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i y_i\right) \\
&= \beta + \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i\mathbf{x}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i e_i\right) \\
&\xrightarrow{p} \beta + \boldsymbol{Q}_{zx}^{-1} 0 = \beta
\end{aligned}$$

Thus

$$\begin{aligned}
\delta &= \operatorname*{plim}_{n\to\infty}\left(\widehat{\boldsymbol{\beta}} - \widetilde{\boldsymbol{\beta}}\right) \\
&= \beta + \beta + \boldsymbol{Q}_{xx}^{-1} E\left(\mathbf{x}_i e_i\right) - \beta \\
&= \boldsymbol{Q}_{xx}^{-1} E\left(\mathbf{x}_i e_i\right)
\end{aligned}$$

2. Equation (3) means that $\mathbf{x}_i$ is exogenous. Under this assumption, $\delta = 0$
3. Differencing the above equations,

$$\begin{aligned}
\hat{\beta} - \tilde{\beta} &= \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i e_i\right) - \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i\mathbf{x}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i e_i\right) \\
&= \left(\begin{array}{cc} \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i'\right)^{-1} & -\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i\mathbf{x}_i'\right)^{-1} \end{array}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} \mathbf{x}_i \\ \mathbf{z}_i \end{array}\right) e_i\right).
\end{aligned}$$

4. Since $E\left(\mathbf{x}_i e_i\right) = 0$, $E\left(\mathbf{z}_i e_i\right) = 0$ and

$$E\left(\left(\begin{array}{c} \mathbf{x}_i \\ \mathbf{z}_i \end{array}\right)\left(\begin{array}{cc} \mathbf{x}_i' & \mathbf{z}_i' \end{array}\right) e_i^2\right) = \left[\begin{array}{cc} E\left(\mathbf{x}_i\mathbf{x}_i' e_i^2\right) & E\left(\mathbf{x}_i\mathbf{z}_i' e_i^2\right) \\ E\left(\mathbf{z}_i\mathbf{x}_i' e_i^2\right) & E\left(\mathbf{z}_i\mathbf{z}_i' e_i^2\right) \end{array}\right] = \left[\begin{array}{cc} \Omega_{xx} & \Omega_{xz} \\ \Omega_{zx} & \Omega_{zz} \end{array}\right] = \Omega,$$

say, then by the CLT

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\begin{array}{c} \mathbf{x}_i \\ \mathbf{z}_i \end{array}\right) e_i \rightarrow_d N(0,\Omega)$$

5.
$$\begin{aligned}
\sqrt{n}\left(\hat{\beta} - \tilde{\beta}\right) &= \left(\begin{array}{cc} \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i'\right)^{-1} & \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i\mathbf{x}_i'\right)^{-1} \end{array}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\begin{array}{c} \mathbf{x}_i \\ \mathbf{z}_i \end{array}\right) e_i\right) \\
&\rightarrow_d \left(\begin{array}{cc} \boldsymbol{Q}_{xx}^{-1} & -\boldsymbol{Q}_{zx}^{-1} \end{array}\right) N(0,\Omega) \sim N(0,V)
\end{aligned}$$

where

$$\begin{aligned}
V &= \left(\begin{array}{cc} \boldsymbol{Q}_{xx}^{-1} & -\boldsymbol{Q}_{zx}^{-1} \end{array}\right)\left[\begin{array}{cc} \Omega_{xx} & \Omega_{xz} \\ \Omega_{zx} & \Omega_{zz} \end{array}\right]\left(\begin{array}{c} \boldsymbol{Q}_{xx}^{-1} \\ -\boldsymbol{Q}_{xz}^{-1} \end{array}\right) \\
&= \boldsymbol{Q}_{xx}^{-1}\Omega_{xx}\boldsymbol{Q}_{xx}^{-1} - \boldsymbol{Q}_{zx}^{-1}\Omega_{zx}\boldsymbol{Q}_{xx}^{-1} - \boldsymbol{Q}_{xx}^{-1}\Omega_{xz}\boldsymbol{Q}_{xz}^{-1} + \boldsymbol{Q}_{zx}^{-1}\Omega_{zz}\boldsymbol{Q}_{xz}^{-1}
\end{aligned}$$

6. Under (4),

$$\left[\begin{array}{cc} \Omega_{xx} & \Omega_{xz} \\ \Omega_{zx} & \Omega_{zz} \end{array}\right] = \sigma^2 \left[\begin{array}{cc} \boldsymbol{Q}_{xx} & \boldsymbol{Q}_{xz} \\ \boldsymbol{Q}_{zx} & \boldsymbol{Q}_{zz} \end{array}\right]$$

so

$$\begin{aligned} V &= \boldsymbol{Q}_{xx}^{-1}\Omega_{xx}\boldsymbol{Q}_{xx}^{-1} - \boldsymbol{Q}_{zx}^{-1}\Omega_{zx}\boldsymbol{Q}_{xx}^{-1} - \boldsymbol{Q}_{xx}^{-1}\Omega_{xz}\boldsymbol{Q}_{xz}^{-1} + \boldsymbol{Q}_{zx}^{-1}\Omega_{zz}\boldsymbol{Q}_{xz}^{-1} \\ &= \sigma^2\left(\boldsymbol{Q}_{xx}^{-1} - \boldsymbol{Q}_{xx}^{-1} - \boldsymbol{Q}_{xx}^{-1} + \boldsymbol{Q}_{zx}^{-1}\boldsymbol{Q}_{zz}\boldsymbol{Q}_{xz}^{-1}\right) \\ &= \sigma^2\left(\boldsymbol{Q}_{zx}^{-1}\boldsymbol{Q}_{zz}\boldsymbol{Q}_{xz}^{-1} - \boldsymbol{Q}_{xx}^{-1}\right) \end{aligned}$$

7.

$$\hat{V} = \hat{\sigma}^2\left(\hat{\boldsymbol{Q}}_{zx}^{-1}\hat{\boldsymbol{Q}}_{zz}\hat{\boldsymbol{Q}}_{xz}^{-1} - \hat{\boldsymbol{Q}}_{xx}^{-1}\right)$$

where $\hat{\boldsymbol{Q}}_{xx} = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i'$, $\hat{\boldsymbol{Q}}_{xz} = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{z}_i'$, $\hat{\boldsymbol{Q}}_{zz} = \frac{1}{n}\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i'$, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n \hat{e}_i^2$, and $\hat{e}_i = y - \mathbf{x}_i'\hat{\boldsymbol{\beta}}$.

8. A test for H_0 is

$$W_n = n\left(\hat{\beta} - \tilde{\beta}\right)'\hat{V}^{-1}\left(\hat{\beta} - \tilde{\beta}\right)$$

Let the distribution in question 5 be $N \sim N(0, V)$. Then the asymptotic distribution of W_n is

$$W_n \to_d N'V^{-1}N \sim \chi_k^2$$

[Another feasible test would be a GMM overidentification test. But there are some pitfalls in taking this approach. It is important to base the test on all of the moment equations. Thus you need to set

$$g_i(\beta) = \left(\begin{array}{c} \mathbf{x}_i \\ \mathbf{z}_i \end{array}\right)\left(y_i - \mathbf{x}_i'\beta\right)$$

For example, if you set $g_i(\beta) = \mathbf{x}_i\left(y_i - \mathbf{x}_i'\beta\right)$ and use the LS estimator $\hat{\beta}$, then $\frac{1}{n}\sum_{i=1}^n g_i(\hat{\beta}) = 0$ identically so no test can be based on this moment. Or if you set $g_i(\beta) = \mathbf{z}_i\left(y_i - \mathbf{x}_i'\beta\right)$ and use the 2SLS estimator $\tilde{\beta}$ then $\frac{1}{n}\sum_{i=1}^n g_i(\tilde{\beta}) = 0$ identically.]

9. Under exogeneity, W_n is asymptotically χ_k^2. To test exogeneity, we compare W_n with the χ_k^2 distribution. If W_n is smaller than the 5% critical value, we do not reject the hypothesis of exogeneity. If W_n is larger than the critical value, we reject exogeneity in favor of endogeneity. The test works because under the alternative, $\hat{\beta} - \tilde{\beta} \to_p \beta^* - \beta = \boldsymbol{Q}_{xx}^{-1}\delta \neq 0$, so $W_n \to_p \infty$.

10. The asymptotic distribution implicity assumed $V = \sigma^2\left(\boldsymbol{Q}_{zx}^{-1}\boldsymbol{Q}_{zz}\boldsymbol{Q}_{xz}^{-1} - \boldsymbol{Q}_{xx}^{-1}\right) > 0$. This is true iff

$$\boldsymbol{Q}_{xx}^{-1} < \boldsymbol{Q}_{zx}^{-1}\boldsymbol{Q}_{zz}\boldsymbol{Q}_{xz}^{-1} >$$

or iff

$$\boldsymbol{Q}_{xx} > \left(\boldsymbol{Q}_{zx}^{-1}\boldsymbol{Q}_{zz}\boldsymbol{Q}_{xz}^{-1}\right)^{-1} = \boldsymbol{Q}_{xz}\boldsymbol{Q}_{zz}^{-1}\boldsymbol{Q}_{zx}$$

or iff

$$\boldsymbol{Q}_{xx} - \boldsymbol{Q}_{xz}\boldsymbol{Q}_{zz}^{-1}\boldsymbol{Q}_{zx} > 0$$

This holds when $V > 0$, but not generally.