Econometrics 710
Final Exam, Spring 2010 Sample Answers

1. Reduced form equations:

   (a) $y_i = \mathbf{z}_i'\mathbf{\Pi}\boldsymbol{\beta} + v_i$
   (b) $v_i = \boldsymbol{u}_i'\boldsymbol{\beta} + e_i$
   (c) Let $\mathbf{w}_i = \mathbf{\Pi}'\mathbf{z}_i$ so that $y_i = \mathbf{w}_i'\boldsymbol{\beta} + v_i$. Since $E(\mathbf{w}_i v_i) = 0$ a simple answer is

$$\begin{aligned}\boldsymbol{\beta} &= \left(E\mathbf{w}_i\mathbf{w}_i'\right)^{-1}\left(E\left(\mathbf{w}_i y_i\right)\right) \\ &= \left(\mathbf{\Pi}'E\left(\mathbf{z}_i\mathbf{z}_i'\right)\mathbf{\Pi}\right)^{-1}\left(\mathbf{\Pi}'E\left(\mathbf{z}_i y_i\right)\right)\end{aligned}$$

   More generally, since $E(\mathbf{z}_i v_i) = 0$ the equation is overidentified. So for any weight matrix $\boldsymbol{W}$ we can also write the coefficient as

$$\begin{aligned}\boldsymbol{\beta} &= \left(E\left(\mathbf{w}_i\mathbf{z}_i'\right)\boldsymbol{W}E\left(\mathbf{z}_i\mathbf{w}_i'\right)\right)^{-1}\left(E\left(\mathbf{w}_i\mathbf{z}_i'\right)\boldsymbol{W}E\left(\mathbf{z}_i y_i\right)\right) \\ &= \left(\mathbf{\Pi}'E\left(\mathbf{z}_i\mathbf{z}_i'\right)\boldsymbol{W}E\left(\mathbf{z}_i\mathbf{z}_i'\right)\mathbf{\Pi}\right)^{-1}\left(\mathbf{\Pi}'E\left(\mathbf{z}_i\mathbf{z}_i'\right)\boldsymbol{W}E\left(\mathbf{z}_i y_i\right)\right)\end{aligned}$$

   (d) $\mathbf{\Pi} = E\left(\mathbf{z}_i\mathbf{z}_i'\right)^{-1}E\left(\mathbf{z}_i\mathbf{x}_i'\right)$
   (e) The identification condition is $\text{rank}(\mathbf{\Pi}) = k$

2. Estimation of $\boldsymbol{Q}$

   (a) $\widetilde{\boldsymbol{Q}} = \frac{1}{n}\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i'$
   (b) $\widehat{\boldsymbol{Q}} = \frac{1}{J}\sum_{j=1}^J \mathbf{z}_j\mathbf{z}_j'$
   (c) As $n \to \infty$, $\widetilde{\boldsymbol{Q}} \to_p E\left(\mathbf{z}_i\mathbf{z}_i'\right) = \boldsymbol{Q}$
   As $J \to \infty$, $\widetilde{\boldsymbol{Q}} \to_p E\left(\mathbf{z}_j\mathbf{z}_j'\right) = \boldsymbol{Q}$
   (d) Yes, these two limits are the same, because the distributions in the two samples are identical.
   (e) $\widetilde{\boldsymbol{Q}}$ is more efficient if $n > J$.
   $\widehat{\boldsymbol{Q}}$ is more efficient if $n < J$.
   They are equally efficient if $n = J$

3. Estimation of $\boldsymbol{\beta}$ given $\mathbf{\Pi}$

   (a) A simple estimator is

$$\widetilde{\boldsymbol{\beta}}_1 = \left(\mathbf{\Pi}'\left(\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i'\right)\mathbf{\Pi}\right)^{-1}\left(\mathbf{\Pi}'\left(\sum_{i=1}^n \mathbf{z}_i y_i\right)\right) \tag{1}$$

   .
   A GMM estimator is

$$\widetilde{\boldsymbol{\beta}} = \left(\mathbf{\Pi}'\left(\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i'\right)\boldsymbol{W}\left(\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i'\right)\mathbf{\Pi}\right)^{-1}\left(\mathbf{\Pi}'\left(\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i'\right)\boldsymbol{W}\left(\sum_{i=1}^n \mathbf{z}_i y_i\right)\right)$$

   The efficient GMM estimator sets $\boldsymbol{W} = \widehat{\mathbf{\Omega}}^{-1}$ where $\widehat{\mathbf{\Omega}}$ is an estimate of $\mathbf{\Omega} = E\left(\mathbf{z}_i\mathbf{z}_i'v_i^2\right)$. Notice that the error is v_i from the reduced form, not e_i from the structural form. This is because we are estimating $y_i = \mathbf{z}_i'\mathbf{\Pi}\boldsymbol{\beta} + v_i$ not $y_i = \mathbf{x}_i'\boldsymbol{\beta} + e_i$. Using the preliminary estimate (1) we construct $\hat{v}_i = y_i - \mathbf{z}_i'\mathbf{\Pi}\widetilde{\boldsymbol{\beta}}$ and

$$\widehat{\mathbf{\Omega}} = \frac{1}{n-k}\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i'\hat{v}_i^2$$

Then the efficient estimator is

$$\widetilde{\boldsymbol{\beta}}_2 = \left(\boldsymbol{\Pi}'\left(\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\widehat{\boldsymbol{\Omega}}^{-1}\left(\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\boldsymbol{\Pi}\right)^{-1}\left(\boldsymbol{\Pi}'\left(\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\widehat{\boldsymbol{\Omega}}^{-1}\left(\sum_{i=1}^{n}\mathbf{z}_i y_i\right)\right)$$

(b) Sample 1

(c) As $n \to \infty$,

$$\widetilde{\boldsymbol{\beta}}_1 = \left(\boldsymbol{\Pi}'\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\boldsymbol{\Pi}\right)^{-1}\left(\boldsymbol{\Pi}'\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i y_i\right)\right) \xrightarrow{p} \left(\boldsymbol{\Pi}'\boldsymbol{Q}\boldsymbol{\Pi}\right)^{-1}\boldsymbol{\Pi}'E\left(\mathbf{z}_i y_i\right) = \boldsymbol{\beta}$$

as defined in 1(c). The asymptotic approximation is as n goes to infinity. Also,

$$\widetilde{\boldsymbol{\beta}}_2 = \left(\boldsymbol{\Pi}'\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\widehat{\boldsymbol{\Omega}}^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\boldsymbol{\Pi}\right)^{-1}\left(\boldsymbol{\Pi}'\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\widehat{\boldsymbol{\Omega}}^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i y_i\right)\right)$$

$$\xrightarrow{p} \left(\boldsymbol{\Pi}'\boldsymbol{Q}\boldsymbol{\Omega}^{-1}\boldsymbol{Q}\boldsymbol{\Pi}\right)^{-1}\boldsymbol{\Pi}'\boldsymbol{Q}\boldsymbol{\Omega}^{-1}E\left(\mathbf{z}_i y_i\right) = \boldsymbol{\beta}$$

4. Estimation of $\boldsymbol{\Pi}$

(a) $\widehat{\boldsymbol{\Pi}} = \left(\sum_{j=1}^{J}\mathbf{z}_j\mathbf{z}_j'\right)^{-1}\left(\sum_{j=1}^{J}\mathbf{z}_j\mathbf{x}_j'\right)$

(b) Sample 2

(c) As $J \to \infty$,

$$\widehat{\boldsymbol{\Pi}} = \left(\frac{1}{J}\sum_{j=1}^{J}\mathbf{z}_j\mathbf{z}_j'\right)^{-1}\left(\frac{1}{J}\sum_{j=1}^{J}\mathbf{z}_j\mathbf{x}_j'\right) \xrightarrow{p} E\left(\mathbf{z}_j\mathbf{z}_j'\right)^{-1}E\left(\mathbf{z}_j\mathbf{x}_j'\right) = \boldsymbol{\Pi}$$

as defined in 1(d). The asymptotics is as J goes to infinity.

5. Estimation of $\boldsymbol{\beta}$. when $\boldsymbol{\Pi}$ unknown

(a) $\widehat{\boldsymbol{\beta}_1} = \left(\widehat{\boldsymbol{\Pi}}'\left(\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\widehat{\boldsymbol{\Pi}}\right)^{-1}\left(\widehat{\boldsymbol{\Pi}}'\left(\sum_{i=1}^{n}\mathbf{z}_i y_i\right)\right)$ or

$$\widetilde{\boldsymbol{\beta}}_2 = \left(\widehat{\boldsymbol{\Pi}}'\left(\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\widehat{\boldsymbol{\Omega}}^{-1}\left(\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\widehat{\boldsymbol{\Pi}}\right)^{-1}\left(\widehat{\boldsymbol{\Pi}}'\left(\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\widehat{\boldsymbol{\Omega}}^{-1}\left(\sum_{i=1}^{n}\mathbf{z}_i y_i\right)\right)$$

This is just the answer in 3(a), replacing the known $\boldsymbol{\Pi}$ with the estimate $\widehat{\boldsymbol{\Pi}}$

(b) As $\min(n, J) \to \infty$

$$\widehat{\boldsymbol{\beta}}_1 = \left(\widehat{\boldsymbol{\Pi}}'\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\widehat{\boldsymbol{\Pi}}\right)^{-1}\left(\widehat{\boldsymbol{\Pi}}'\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i y_i\right)\right) \xrightarrow{p} \left(\boldsymbol{\Pi}'\boldsymbol{Q}\boldsymbol{\Pi}\right)^{-1}\boldsymbol{\Pi}'E\left(\mathbf{z}_i y_i\right) = \boldsymbol{\beta}$$

$$\widetilde{\boldsymbol{\beta}}_2 = \left(\widehat{\boldsymbol{\Pi}}'\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\widehat{\boldsymbol{\Omega}}^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\widehat{\boldsymbol{\Pi}}\right)^{-1}\left(\widehat{\boldsymbol{\Pi}}'\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\right)\widehat{\boldsymbol{\Omega}}^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i y_i\right)\right)$$

$$\xrightarrow{p} \left(\boldsymbol{\Pi}'\boldsymbol{Q}\boldsymbol{\Omega}^{-1}\boldsymbol{Q}\boldsymbol{\Pi}\right)^{-1}\boldsymbol{\Pi}'\boldsymbol{Q}\boldsymbol{\Omega}^{-1}E\left(\mathbf{z}_i y_i\right) = \boldsymbol{\beta}$$