Autocorrelation

- Recall the component decomposition

$$\mu_t = T_t + S_t + C_t$$

- We focusing on the cycle component C_t
- For the moment, we consider cycle-only models

$$\mu_t = C_t$$

Mean Stationary

- **Definition**: A time series Y_t has a constant mean, or is **mean stationary**, if

$$E(Y_t) = \mu$$

  is constant (stable) over time.
- Counter-example:
  – A trended time series is not mean stationary
- We assume the cyclical component C_t is mean stationary

Variance Stationarity

- **Definition**: A time series Y_t has a constant variance, or is **variance stationary**, if

$$\operatorname{var}(Y_t) = \sigma^2$$

  is constant (stable) over time.
- Counter-example:
  - A time-series with trended (increasing) variance is not variance stationary
- We assume the cyclical component C_t is variance stationary

Covariance

- The covariance of two random variables X and Z is

$$\text{cov}(X, Z) = E((X - EX)(Z - EZ))$$

- The covariance measures the linear dependence between X and Z.

Correlation

- The correlation normalizes the covariance

$$\operatorname{corr}(X,Z)=\frac{\operatorname{cov}(X,Z)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Z)}}$$

- Correlations lie between -1 and 1
- corr(X,Z)=0 means no linear association
- corr(X,Z)=1 means *X*=a+b*Z*
- corr(X,Z)=-1 means *X*=a-*bZ*

Lags

- The **first lag** of Y_t is its value in the preceding time period, Y_{t-1}
- The **second lag** of Y_t is its value in the two periods preceding, Y_{t-1}
- The **k'th lag** of Y_t is Y_{t-k}

U.S. Unemployment Rate
Lags 1 through 4

Y_t	Y_{t-1}	Y_{t-2}	Y_{t-3}	Y_{t-4}	t
3.4					1948m1
3.8	3.4				1948m2
4	3.8	3.4			1948m3
3.9	4	3.8	3.4		1948m4
3.5	3.9	4	3.8	3.4	1948m5
3.6	3.5	3.9	4	3.8	1948m6
3.6	3.6	3.5	3.9	4	1948m7
3.9	3.6	3.6	3.5	3.9	1948m8
3.8	3.9	3.6	3.6	3.5	1948m9
3.7	3.8	3.9	3.6	3.6	1948m10
3.8	3.7	3.8	3.9	3.6	1948m11
4	3.8	3.7	3.8	3.9	1948m12

Lag Operator

- The lag operator L is a useful way to manipulate lags
- It is defined by the relation

$$Ly_t = y_{t-1}$$

- Taking the lag operator to a power means that you apply it iteratively

$$L^2 y_t = LLy_t = Ly_{t-1} = y_{t-2}$$

- In general

$$L^k y_t = y_{t-k}$$

Lag Operator in STATA

- STATA uses the same notation
- **generate ur1=L.ur**
  - This creates a variable “ur1” which is the first lag of “ur”
- **generate ur5=L5.ur**
  - This creates a variable “ur5” which is the fifth lag
- **scatter ur L.ur**
  - This creates a scatter of “ur” and its first lag
- **regress ur L.ur**
  - This regresses “ur” on its first lag

Autocovariance

- The first **autocovariance** of a time series Y_t is the covariance of Y_t with its value in the preceding time period Y_{t-1}
- We call Y_{t-1} the **first lag** of Y_t
- We write the first autocovariance as

$$\begin{aligned}\gamma(1) &= \operatorname{cov}(Y_t, Y_{t-1}) \\ &= E\big((Y_t - \mu)(Y_{t-1} - \mu)\big)\end{aligned}$$

Autocorrelation

- The first **autocorrelation** of a time series Y_t is the correlation of Y_t with Y_{t-1}
- We write the first autocorrelation as

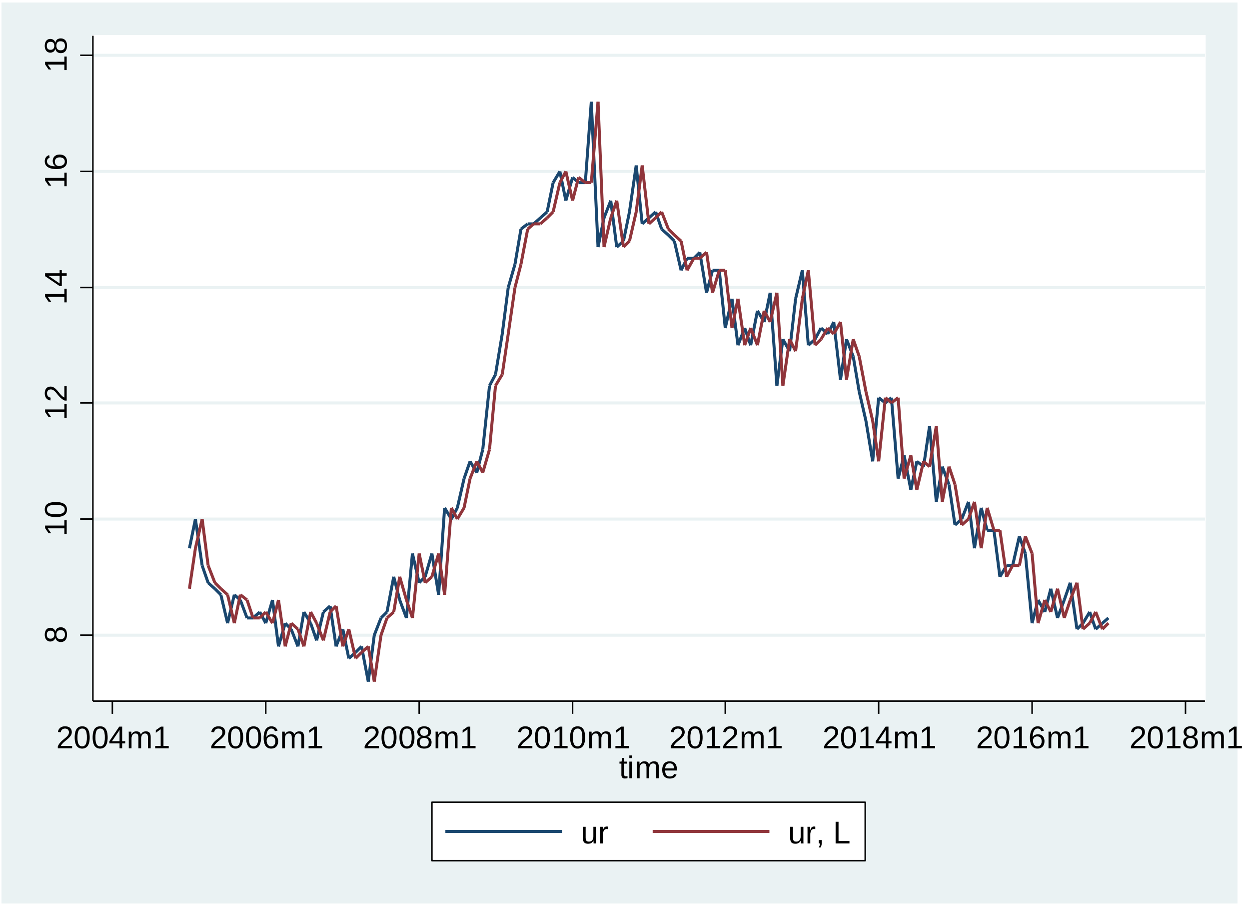
$$\begin{aligned}\rho(1) &= \operatorname{corr}(Y_t, Y_{t-1}) \\ &= \frac{\operatorname{cov}(Y_t, Y_{t-1})}{\sqrt{\operatorname{var}(Y_t)\operatorname{var}(Y_{t-1})}} \\ &= \frac{\operatorname{cov}(Y_t, Y_{t-1})}{\operatorname{var}(Y_t)}\end{aligned}$$

- The third equality holds by variance stationarity

Autocorrelation

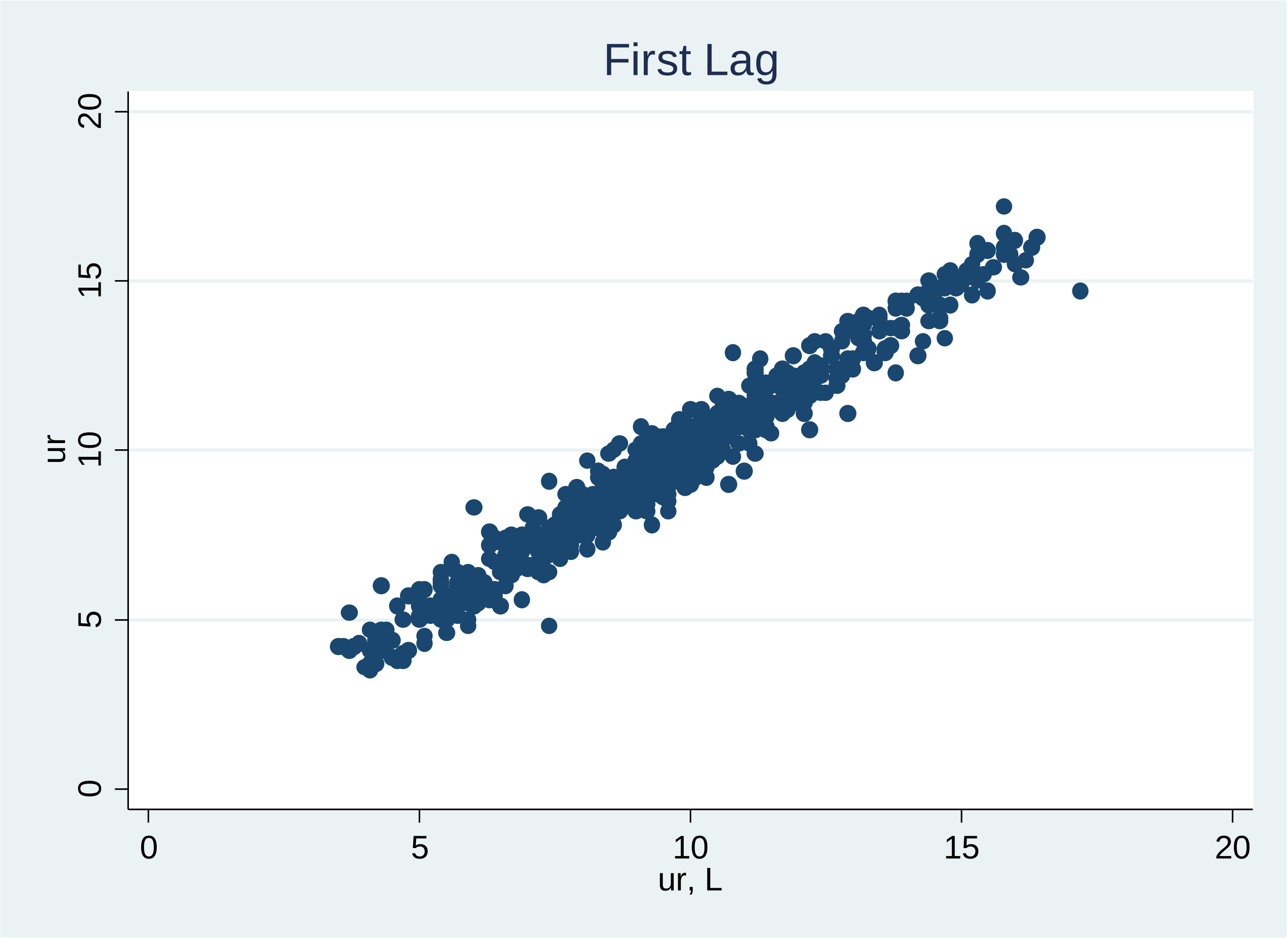
- The autocorrelation $\rho(1)$ lies between -1 and 1
- $\rho(1)$ is close to 1 for highly correlated series
- $\rho(1)$ is close to -1 if the correlation is negative – if there are movements back and forth
- $\rho(1)=0$ if the series is uncorrelated

Y_t and Y_{t-1} (Unemployment Rate)



Scatter Plot

Y_t and Y_{t-1}



First Autocorrelation Unemployment Rate

```
. correlate ur L.ur, covariance
(obs=828)

             |                   L.
             |        ur        ur
-------------+--------------------
          ur |
         --. |   6.75073
         L1. |    6.6074   6.76482


.
. correlate ur L.ur
(obs=828)

             |                   L.
             |        ur        ur
-------------+--------------------
          ur |
         --. |    1.0000
         L1. |    0.9777    1.0000
```

Autocovariances

- The *k*'th **autocovariance** of a time series Y_t is the covariance of Y_t with its lag Y_{t-k}
- It is written as

$$\begin{aligned}\gamma(k) &= \mathrm{cov}(Y_t, Y_{t-k}) \\ &= E\left((Y_t - \mu)(Y_{t-k} - \mu)\right)\end{aligned}$$

Autocorrelations

- The *k*'th **autocorrelation** of a time series Y_t is the correlation of Y_t with Y_{t-k}
- It is written as

$$\rho(k) = \frac{\operatorname{cov}(Y_t, Y_{t-k})}{\sqrt{\operatorname{var}(Y_t)\operatorname{var}(Y_{t-k})}}$$

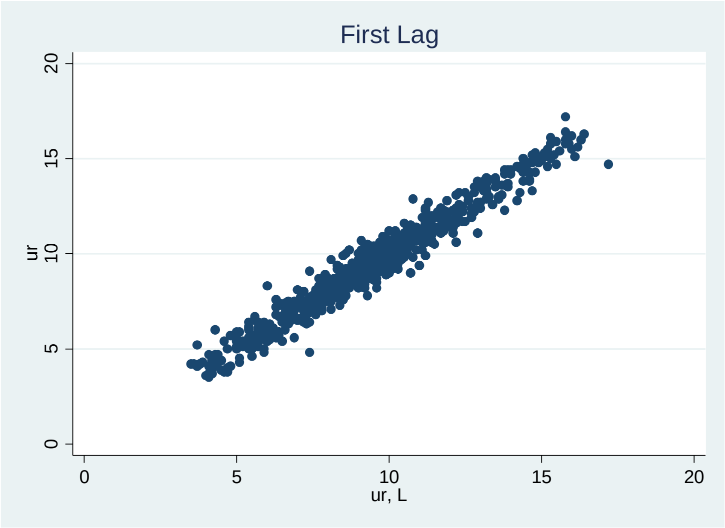
$$= \frac{\operatorname{cov}(Y_t, Y_{t-k})}{\operatorname{var}(Y_t)}$$

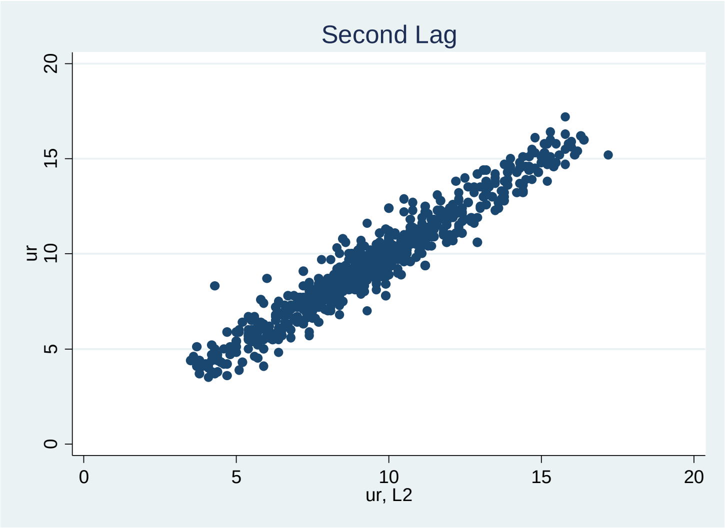
- Autocorrelations lie between -1 and 1

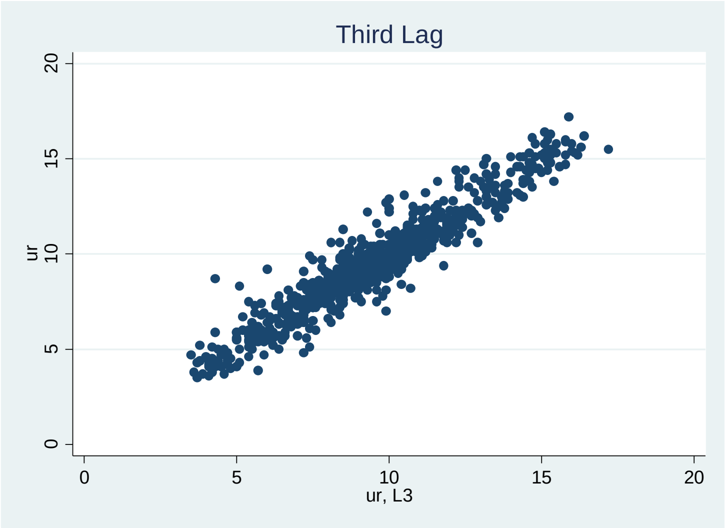
Covariance Stationarity

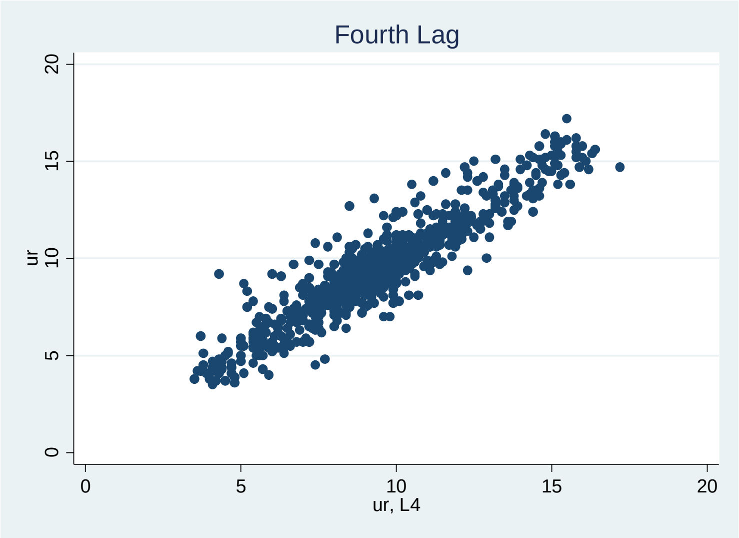
- **Definition**: A time series Y_t is **covariance stationary** if its mean EY_t, variance, and autocovariance function $\gamma(k)$ are constant (stable) over time
- Counter-example:
  - A time-series with changing correlations is not covariance stationary
- We assume the cyclical component C_t is covariance stationary

Scatters of Y_t with Y_{t-1}, Y_{t-2}, Y_{t-3} and Y_{t-4}









Autocorrelations 1 to 4

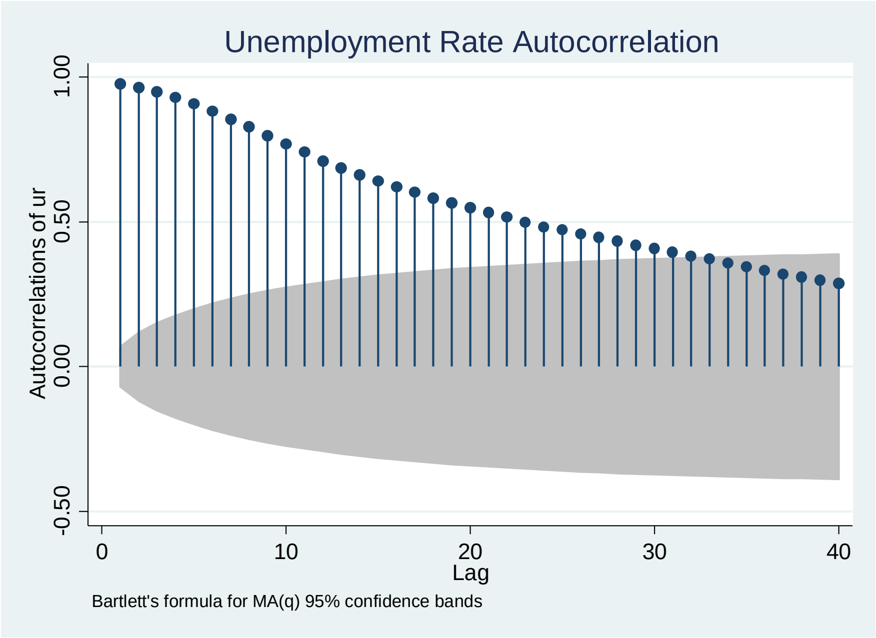
```
. correlate ur L.ur L2.ur L3.ur L4.ur
(obs=825)
```

	ur	L. ur	L2. ur	L3. ur	L4. ur
ur --.	1.0000				
L1.	0.9779	1.0000			
L2.	0.9658	0.9778	1.0000		
L3.	0.9500	0.9658	0.9777	1.0000	
L4.	0.9313	0.9500	0.9656	0.9777	1.0000

Autocorrelation Function

- The autocovariance $\gamma(k)$ and autocorrelation $\rho(k)$ are functions of the lag k.
- We call $\rho(k)$ the **autocorrelation function**.
- Plotted as a function of k it shows us how the dependence pattern alters with the lag.

Autocorrelation Plot



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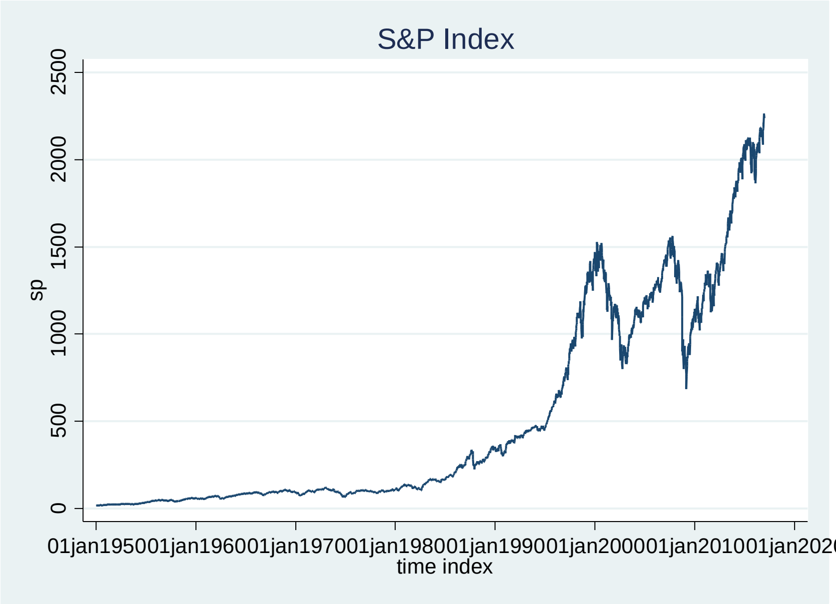
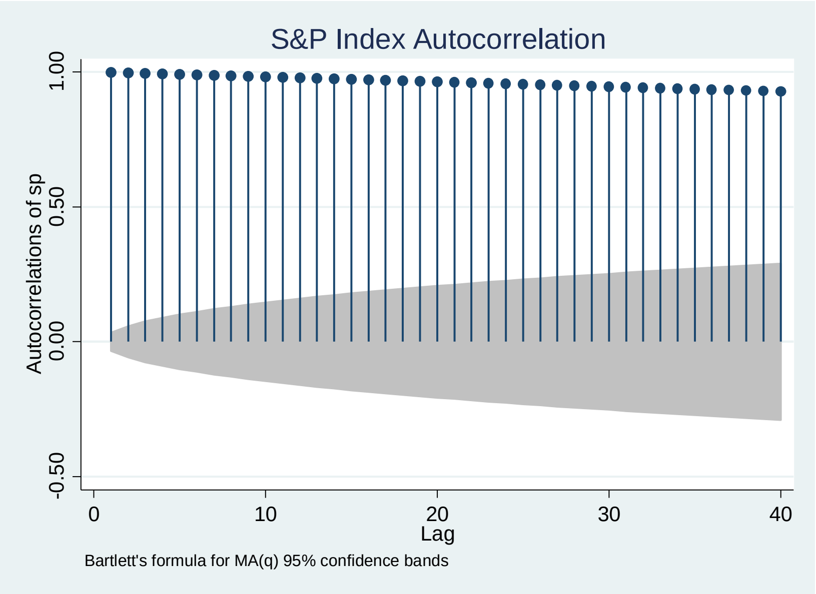
White Noise Autocorrelation

- **Definition**: A **white noise** process has zero autocorrelations: $\rho(k)=0$ for $k>0$
- Serially uncorrelated
- Linearly unforecastable
  - Level of Y_t does not help predict future values
- Common for asset returns, and some growth rates

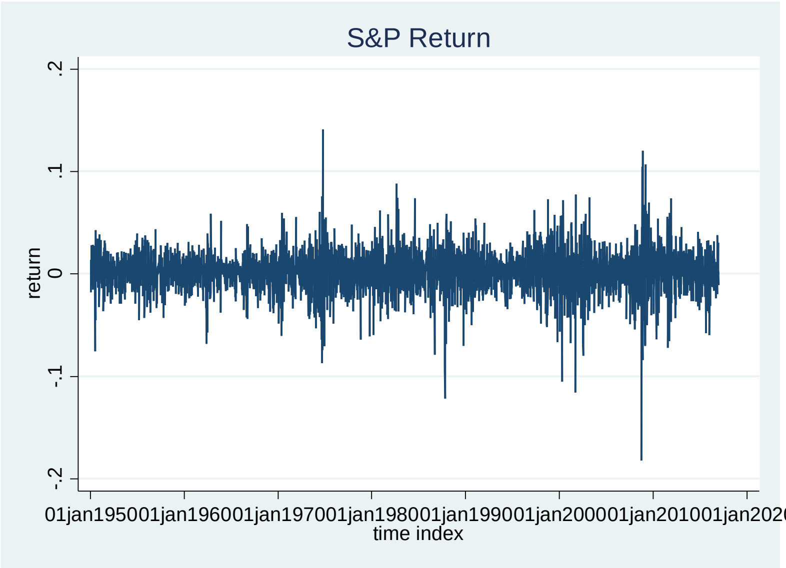
Example of High Autocorrelation: Stock Prices (S&P Index)

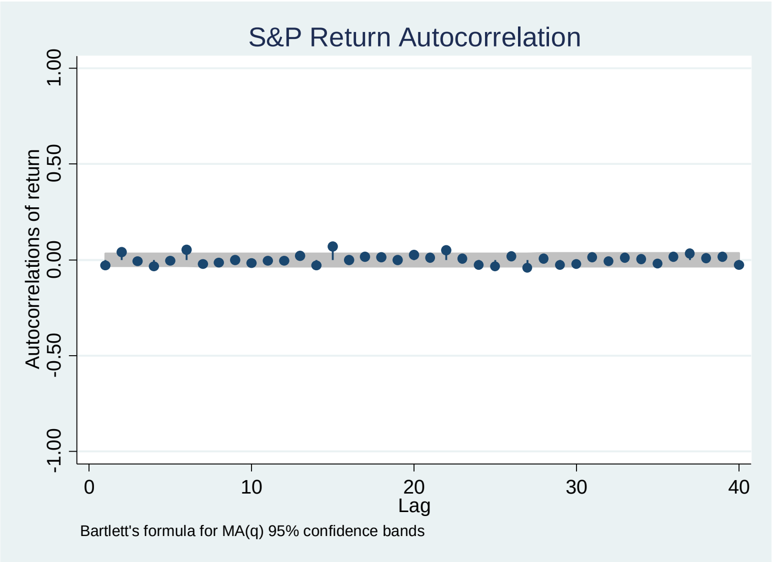




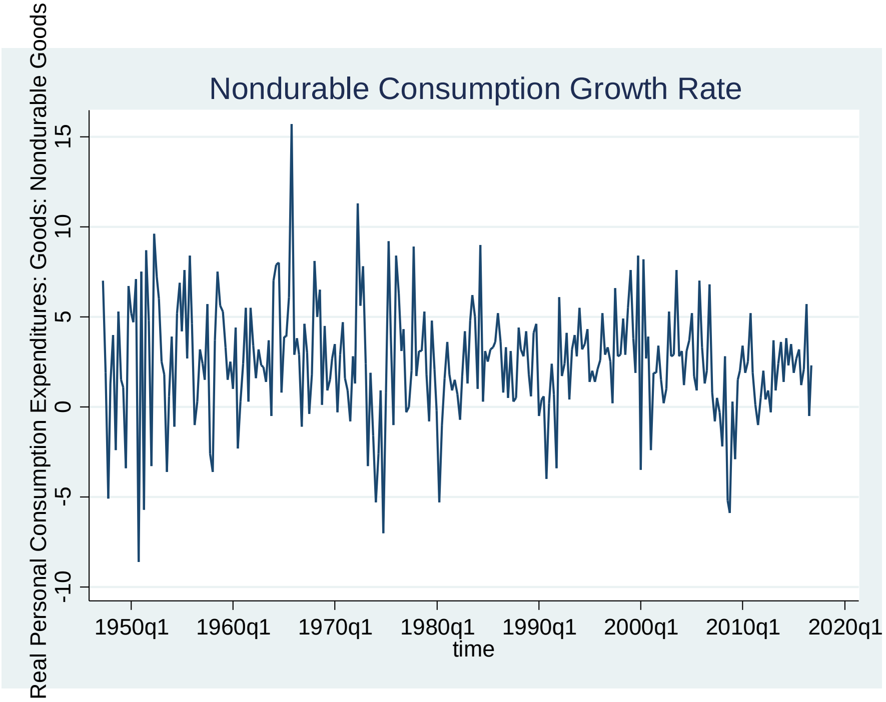
Example of White Noise: Stock Returns

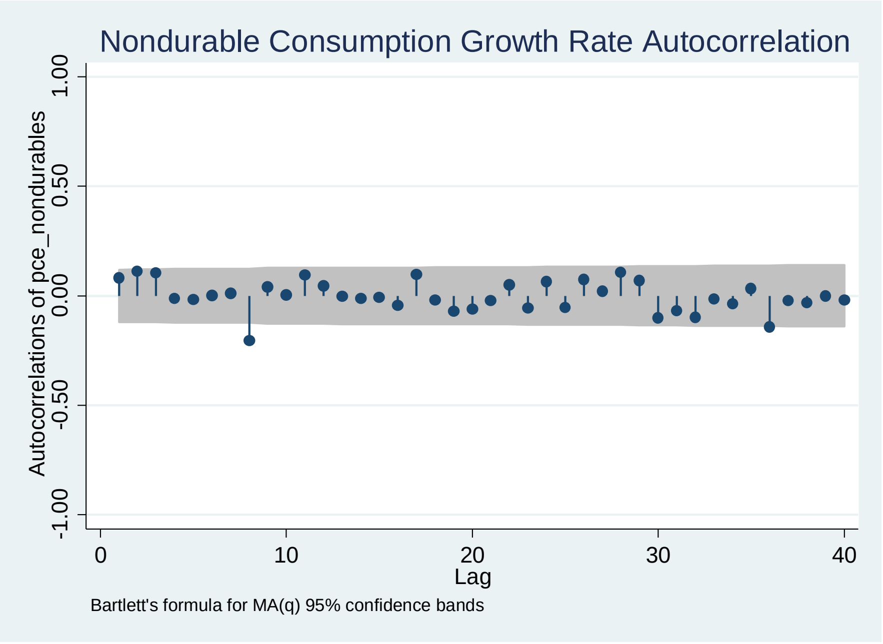
- return = % change in Price (+ dividends)
- gen return = (sp-L.sp)/L.sp





Another example of White Noise: NonDurable Consumption Growth Rate





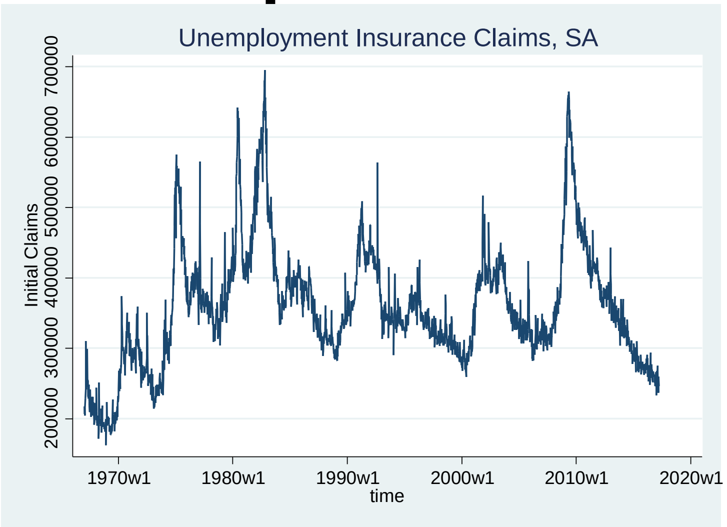
Positive Autocorrelation

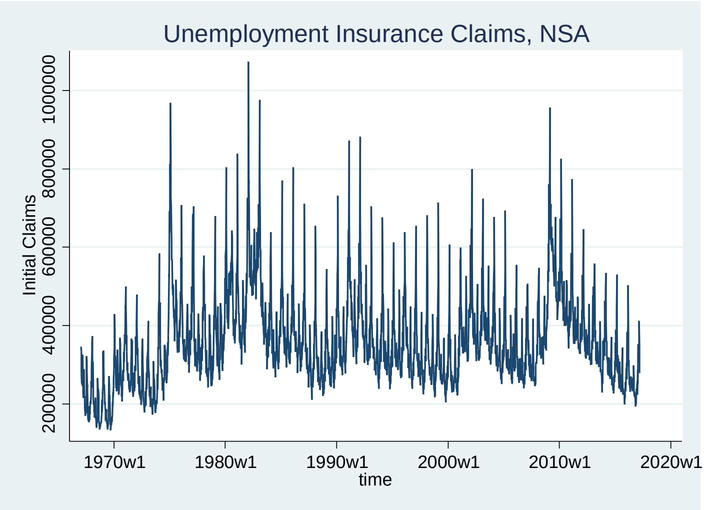
- $\rho(k)>0$
- Positive correlation
  - High values of Y_t predict future high values for Y_{t+h}
  - Low values of Y_t predict future low values for Y_{t+h}
- Commonly found
  - Most economic variables measured in levels

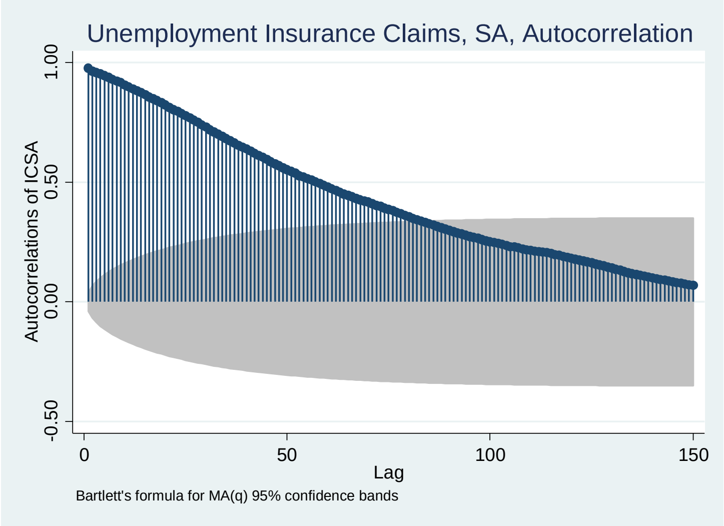
Ergodicity

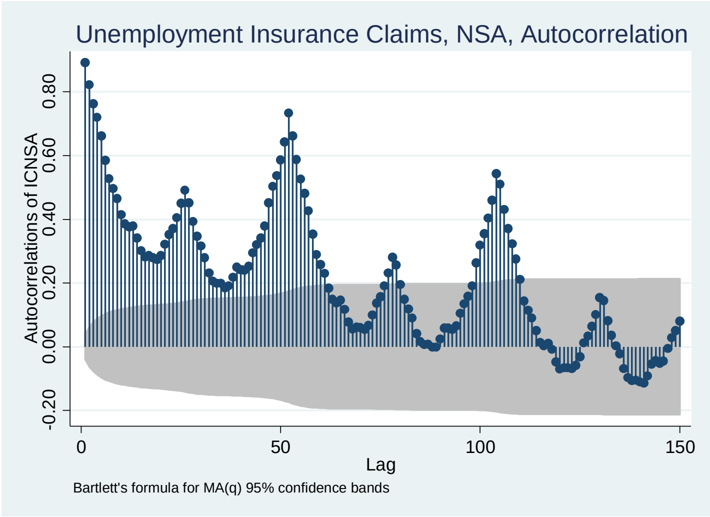
- The time series is **ergodic** if $\rho(k)$ declines to zero as k goes to infinity.
- If a time-series y_t is ergodic, then at long horizons (large h) the best forecast converges to the unconditional mean e.g. $\hat{y}_{T+h|T} \approx \mathrm{E}y_t$,
- Example:
  - Seasonal and trend components are not ergodic
  - NSA (not seasonally adjusted data) may not be ergodic

Example: Unemployment Claims





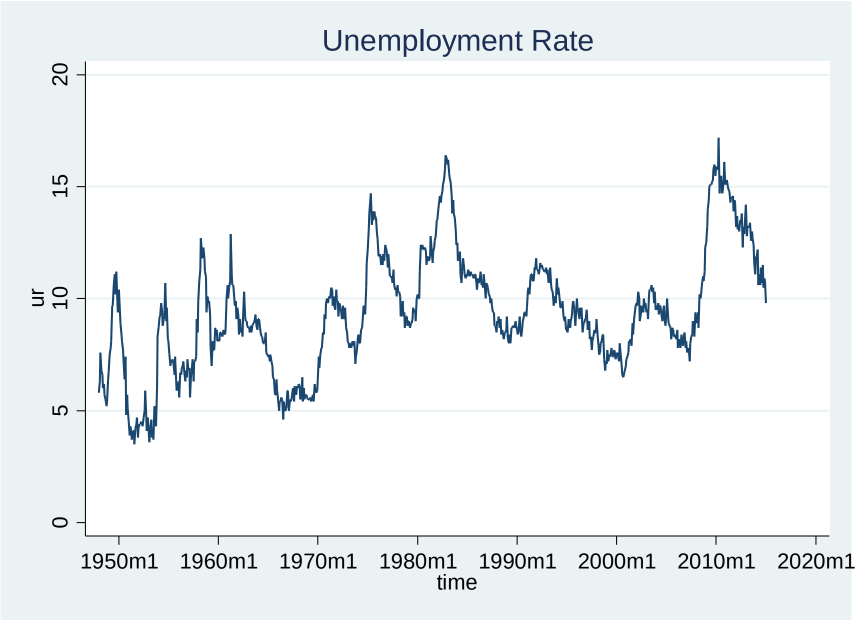


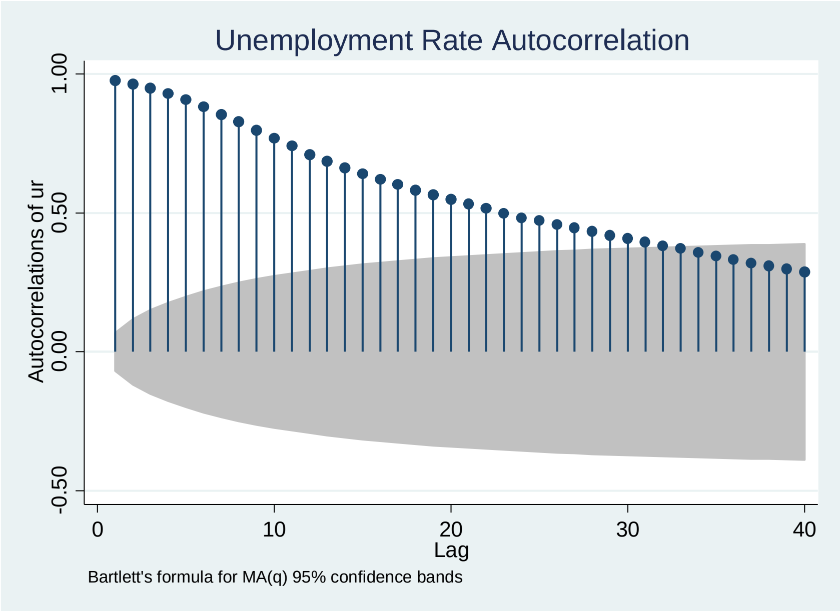


Autocorrelation with Geometric Decay

- Geometric Decay
  - $\rho(k) \approx \rho^k$ for some $\rho < 1$
- $\rho(k)$ decays smoothly to zero
  - ergodic
- Long-range forecasts are close to the unconditional mean
- Commonly found in economic variables measured in levels

Example of Positive Geometric Decay Unemployment Rate

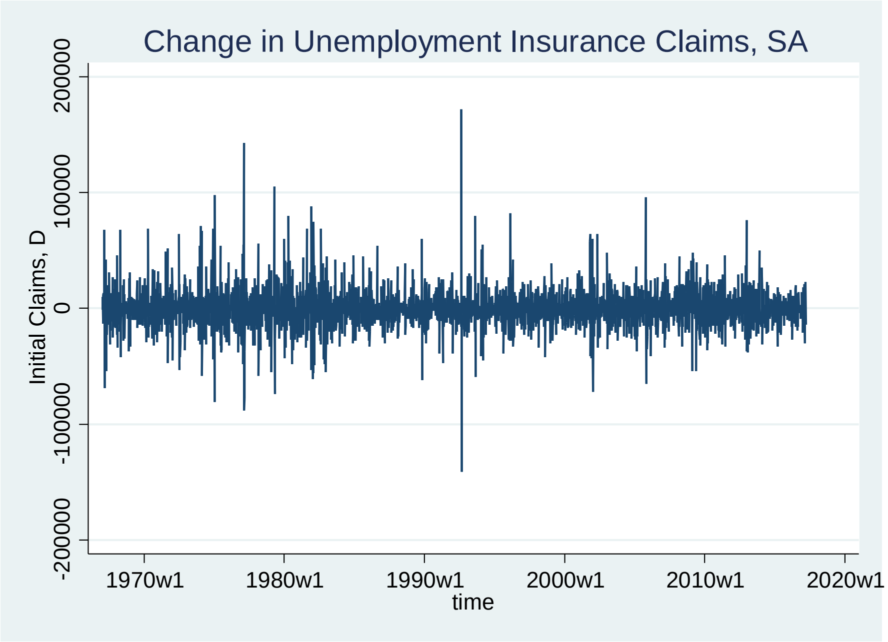


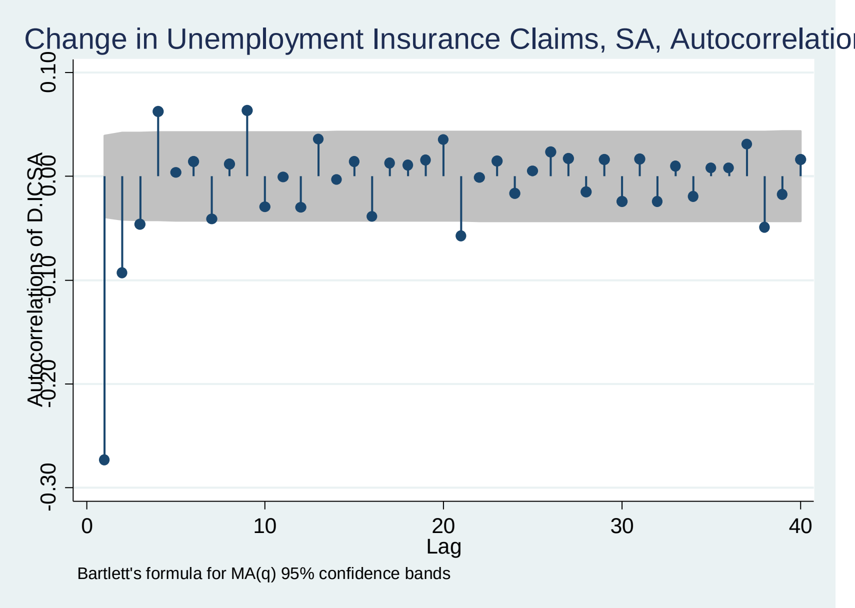


Negative Autocorrelation

- $\rho(1)<0$
- Y_t has immediate reversals in adjacent periods
- Occurs in some economic variables measured as changes (differences)
- Tends to alternate
  - $\rho(1)<0$
  - $\rho(k)>0$ for some k>1
  - Etc
- Forecasts can have opposite sign from current level
- Ergodic if $|\rho(k)|$ goes to zero as k goes to infinity

Example of Negative Autocorrelation Weekly Change in New Unemployment Insurance Claims



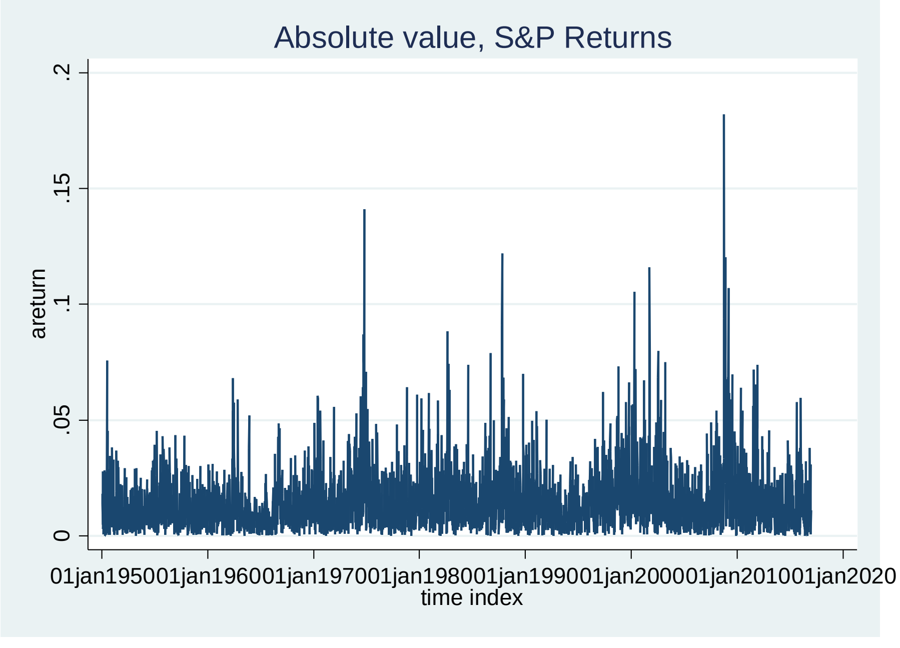


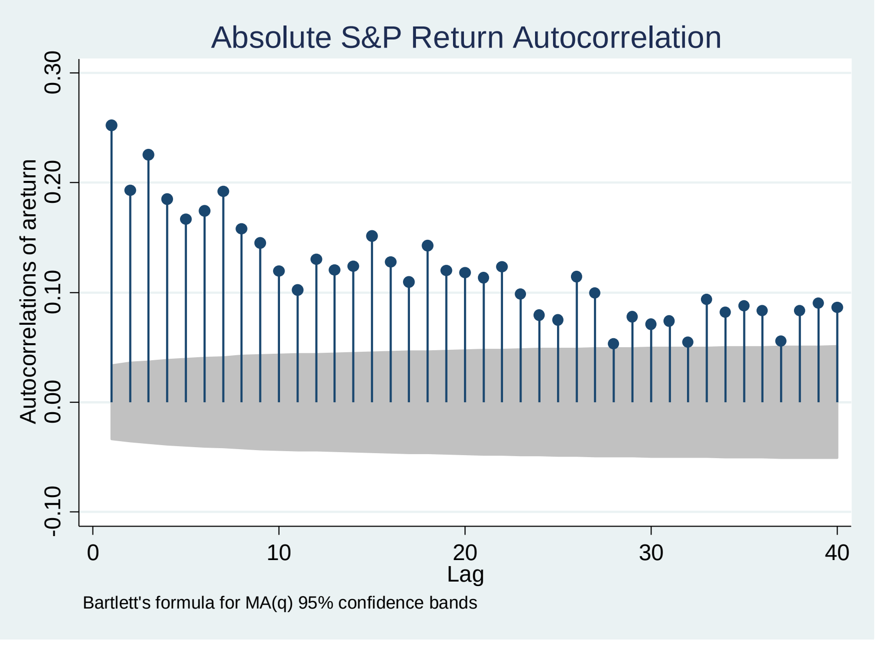
Autocorrelation with Slow Decay

- Slow Decay
  - $\rho(k)$ decays slowly to zero
  - Power law
    - $\rho(k) \approx k^{-d}$ for some $d>0$
  - ergodic
- Originally introduced in hydrology (patterns of the river Nile)
- Suggested for absolute stock returns
- Not common for economic variables

Example of Slow Decay: Absolute Stock Returns

- gen areturn=abs(return)





Estimation of Autocorrelations

- The autocorrelation is a function of moments

$$\rho(k) = \frac{\operatorname{cov}(Y_t, Y_{t-k})}{\operatorname{var}(Y_t)}$$

$$= \frac{\gamma(k)}{\gamma(0)}$$

$$\operatorname{cov}(Y_t, Y_{t-k}) = \gamma(k)$$

$$= E\big((Y_t - \mu)(Y_{t-k} - \mu)\big)$$

$$\mu = EY_t$$

- We estimate by replacing the population moments by sample moments

Estimation

- The population mean

$$\mu = EY_t$$

is estimated by the sample mean

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} Y_t$$

- The population covariance

$$\gamma(k) = E\big((Y_t - \mu)(Y_{t-k} - \mu)\big)$$

is estimated by the sample covariance

$$\hat{\gamma}(k) = \frac{1}{T}\sum_{t=k+1}^{T} (Y_t - \hat{\mu})(Y_{t-k} - \hat{\mu})$$

Estimation

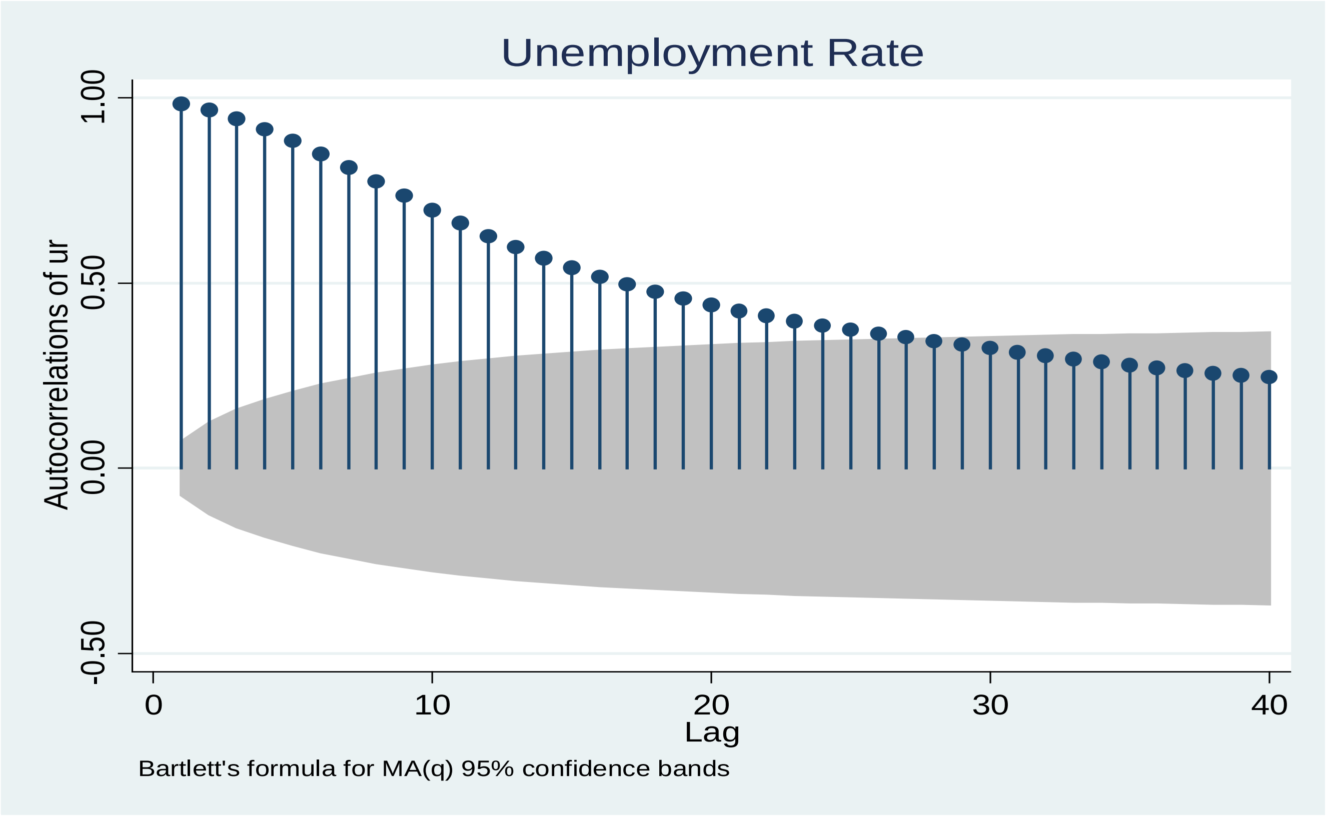
- The population autocorrelation

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)}$$

is estimated by the ratio of sample autocovariances

$$\hat{\rho}(k) = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)}$$

Autocorrelation Plot



Autocorrelations are calculated using the *ac* command

ac ur, title("Unemployment Rate")

Sampling Uncertainty

- The sample autocorrelations are estimates of the population autocorrelations, and are thus random.
- Just because the estimated autocorrelation is positive does not mean that the true autocorrelation is positive. The estimate contains sampling error.

Confidence Bands – White Noise Case

- If Y_t is independent white noise, then

$$\text{var}\left(\hat{\rho}(k)\right) \approx \frac{1}{T}$$

  which means that the standard error is $1/T^{1/2}$
- Thus the sample values will lie in the region $[-2/T^{1/2}, 2/T^{1/2}]$ with 95% probability.
- Consequently, a common measure of uncertainty for sample autocorrelations is to plot $\pm 2/T^{1/2}$ confidence bands about zero.
- The interpretation is that if the sample autocorrelation is within the bands, it is not statistically different from zero.

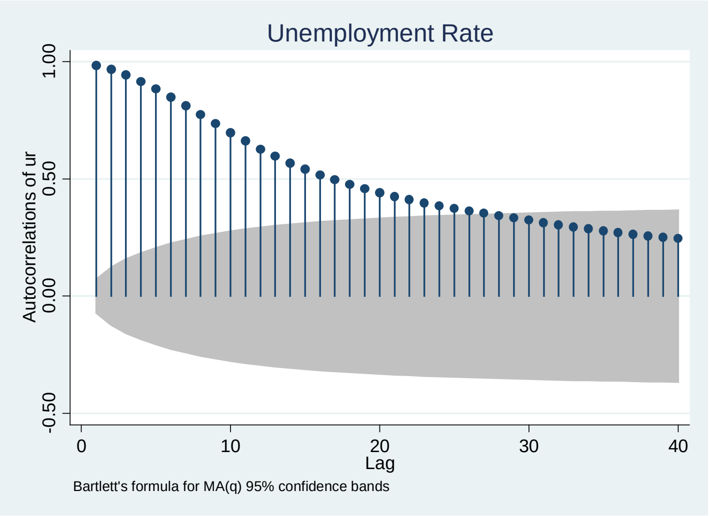
Confidence Bands – Correlated Case

- The $\pm 2/T^{1/2}$ confidence bands are only valid if the true process is white noise
- In general, the confidence bands depend upon the actual autocorrelation.
- Bartlett worked out an approximation based on a moving average model
  - Moving averages described in Diebold, ch. 8

Bartlett Confidence Bands

- Suppose that the autocorrelations up to order k-1 are the estimated values, but the remaining autocorrelations

  (order k and above) are zero.
- The Bartlett confidence band is the 95% sampling interval for the estimated k'th autocorrelation
- STATA displays the Bartlett bands as the shaded region
- The interpretation is that if the estimated autocorrelation falls outside the shaded region, it is statistically different than zero.



Assignments

- Read Diebold through Chapter 6
- Problem Set # 4
  - Due Tuesday (2/14)
- Read Chapter 4 from *The Signal and the Noise*
  - Reading Reflection
  - Due Thursday (2/16)