Vector Autoregressions

- VAR: Vector AutoRegression
  – Nothing to do with VaR: Value at Risk (finance)
- Multivariate autoregression
- Multiple equation model for joint determination of two or more variables
- One of the most commonly used models for applied macroeconometric analysis and forecasting in central banks

Two-Variable VAR

- Two variables: y and x
- Example: output and interest rate
- Two-equation model for the two variables
- One-Step ahead model
- One equation for each variable
- Each equation is an autoregression plus distributed lag, with p lags of each variable

VAR(p) in 2 Variables

$$y_t = \mu_1 + \alpha_{11} y_{t-1} + \alpha_{12} y_{t-2} + \cdots + \alpha_{1p} y_{t-p}$$
$$+ \beta_{11} x_{t-1} + \beta_{12} x_{t-1} + \cdots + \beta_{1p} x_{t-p} + e_{1t}$$

$$x_t = \mu_2 + \alpha_{21} y_{t-1} + \alpha_{22} y_{t-2} + \cdots + \alpha_{2p} y_{t-p}$$
$$+ \beta_{21} x_{t-1} + \beta_{22} x_{t-1} + \cdots + \beta_{2p} x_{t-p} + e_{2t}$$

Multiple Equation System

- In general: k variables
- An equation for each variable
- Each equation includes p lags of y and p lags of x
- (In principle, the equations could have different # of lags, and different # of lags of each variable, but this is most common specification.)
- There is one error per equation.
  - The errors are (typically) correlated.

Unrestricted VAR

- An unrestricted VAR includes all variables in each equation
- A restricted VAR might include some variables in one equation, other variables in another equation
- Old-fashioned macroeconomic models (so-called simultaneous equations models of the 1950s, 1960s, 1970s) were essentially restricted VARs
  - The restrictions and specifications were derived from simplistic macro theory, e.g. Keynesian consumption functions, investment equations, etc.

VAR Revolution

- Christopher Sims
  - Princeton University
  - 2011 Nobel Prize in Economics
- "Macroeconomics and Reality" (1980)
  - Sims argued that conventional macro models were "incredible" – they were based on non-credible identifying assumptions

Sims and VARs

- Sims argued that the conventional models were restricted VARs, and the restrictions had no substantive justification
  - Based on incomplete and/or non-rigorous theory, or intuition
- Sims argued that economists should instead use unrestricted models, e.g. VARs
- He proposed a set of tools for use and evaluation of VARs in practice.

Estimation

- Each equation estimated by OLS

$$y_t = \mu_1 + \alpha_{11} y_{t-1} + \alpha_{12} y_{t-2} + \cdots + \alpha_{1p} y_{t-p} + \beta_{11} x_{t-1} + \beta_{12} x_{t-1} + \cdots + \beta_{1p} x_{t-p} + e_{1t}$$

$$x_t = \mu_2 + \alpha_{21} y_{t-1} + \alpha_{22} y_{t-2} + \cdots + \alpha_{2p} y_{t-p} + \beta_{21} x_{t-1} + \beta_{22} x_{t-1} + \cdots + \beta_{2p} x_{t-p} + e_{2t}$$

Estimation in Stata

- To estimate a VAR in the variables *y* & *x* with lags 1 through p included
  - .varbasic y x, lags(1/p)
- For example, using readgdpgrowth.dta and variables gdp and d.t1year with 3 lags
  - .gen rate=d.t1year
  - .varbasic rate gdp, lags(1/3)
- Could also use
  - .var rate gdp, lags(1/3)

Example: GDP and Interest Rate

```
. varbasic rate gdp, lags(1/3)

Vector autoregression

Sample:  1954q2 - 2016q4                        Number of obs     =        251
Log likelihood =  -882.6021                     AIC               =    7.14424
FPE            =    4.34278                     HQIC              =   7.223372
Det(Sigma_ml)  =   3.884258                     SBIC              =   7.340879

Equation           Parms      RMSE     R-sq      chi2     P>chi2
----------------------------------------------------------------
rate                  7     .640508   0.2133   68.04565   0.0000
gdp                   7     3.26728   0.1940   60.43203   0.0000
----------------------------------------------------------------
```

Interest Rate Equation

	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
rate						
rate						
L1.	.2403882	.0625725	3.84	0.000	.1177483	.3630282
L2.	-.3438512	.06209	-5.54	0.000	-.4655454	-.2221571
L3.	.2863313	.0644772	4.44	0.000	.1599584	.4127043
gdp						
L1.	.0430132	.0127419	3.38	0.001	.0180394	.0679869
L2.	.0241662	.0129016	1.87	0.061	-.0011205	.0494528
L3.	-.0137921	.0127449	-1.08	0.279	-.0387716	.0111874
_cons	-.1669861	.0655976	-2.55	0.011	-.295555	-.0384172

GDP Equation

	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
gdp						
rate						
L1.	.1796421	.3191872	0.56	0.574	-.4459534	.8052375
L2.	-1.397838	.3167256	-4.41	0.000	-2.018609	-.7770669
L3.	.0143963	.3289028	0.04	0.965	-.6302413	.659034
gdp						
L1.	.2997937	.0649976	4.61	0.000	.1724007	.4271866
L2.	.1974569	.065812	3.00	0.003	.0684677	.3264462
L3.	-.0109821	.0650126	-0.17	0.866	-.1384043	.1164402
_cons	1.61183	.3346181	4.82	0.000	.9559905	2.267669

Order Selection

- A VAR(p) includes *p* lags of each variable in each equation
- In a two-variable system, the number of coefficients in each equation is $1+2p$
  - The total number is $2(1+2p)=2+4p$
- In a *k*-variable system, the number of coefficients in each equation is $1+kp$
  - The total number is $k(1+2p)=k+2kp$
- How should *p* be selected?
- Common approach:
  - Information criterion, primarily AIC

AIC and BIC for VAR Models

$$AIC = -2L + 2(k + 2kp)$$

$$BIC = -2L + (k + 2kp)\ln(T)$$

where *L* is log-likelihood from model

- Select model with smallest AIC (or BIC)

Stata Implementation

- varsoc command
- To calculate information criterion for a VAR in variables *x* and *y* up to a maximum lag of *pmax*:
  - .varsoc x y, maxlag(pmax)
- Produces a convenient table

Example: GDP and Interest Rate

```
. varsoc rate gdp, maxlag(8)

   Selection-order criteria
   Sample:  1955q3 - 2016q4                     Number of obs      =       246
```

lag	LL	LR	df	p	FPE	AIC	HQIC	SBIC
0	-912.301				5.79844	7.43334	7.44482	7.46184
1	-893.316	37.97	4	0.000	5.13337	7.31151	7.34594	7.39701
2	-876.194	34.244	4	0.000	4.61396	7.20483	7.2622	7.34732*
3	-866.1	20.189	4	0.000	4.39101*	7.15528*	7.23561*	7.35477
4	-863.74	4.7189	4	0.317	4.45012	7.16862	7.2719	7.42511
5	-859.652	8.1773	4	0.085	4.44715	7.1679	7.29413	7.48138
6	-858.708	1.8877	4	0.756	4.55939	7.19275	7.34192	7.56323
7	-853.8	9.816*	4	0.044	4.52634	7.18536	7.35749	7.61284
8	-851.802	3.9947	4	0.407	4.60127	7.20165	7.39672	7.68612

Result

- For this example
  - AIC selects *p=3*
  - BIC selects *p=3*
- Notice that the AIC value for p=3 in this table (AIC=7.155) is different from that obtained when we estimated the VAR(3) model (AIC=7.144).
  - This is because for the AIC comparison, all estimates are from a common sample, in this case excluding the first 8 observations since the maximum order is set to 8
- The varsoc command is correct

Double Check

```
. varbasic rate gdp if time>=tq(1955q3), lags(1/3)

Vector autoregression

Sample:  1955q3 - 2016q4                         Number of obs     =        246
Log likelihood =  -866.0997                      AIC               =   7.155282
FPE            =   4.391008                      HQIC              =   7.235608
Det(Sigma_ml)  =   3.918492                      SBIC              =   7.354773
```

- When we constrain the sample to exclude the first 8 observations, the reported AIC is 7.155, correctly.

Let’s look at the VAR(3) estimates again.

Example: GDP and Interest Rate

		Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
rate							
rate							
	L1.	.2399726	.0631826	3.80	0.000	.116137	.3638083
	L2.	-.3439793	.0626805	-5.49	0.000	-.4668308	-.2211279
	L3.	.286535	.0651217	4.40	0.000	.1588989	.4141712
gdp							
	L1.	.0430445	.0130613	3.30	0.001	.0174449	.0686441
	L2.	.0244459	.0131373	1.86	0.063	-.0013028	.0501945
	L3.	-.012818	.0130474	-0.98	0.326	-.0383905	.0127545
	_cons	-.1724748	.067853	-2.54	0.011	-.3054644	-.0394853
gdp							
rate							
	L1.	.180243	.3176448	0.57	0.570	-.4423294	.8028155
	L2.	-1.397335	.3151203	-4.43	0.000	-2.014959	-.7797104
	L3.	.004299	.3273933	0.01	0.990	-.6373801	.645978
gdp							
	L1.	.2816871	.0656643	4.29	0.000	.1529875	.4103867
	L2.	.1988894	.0660466	3.01	0.003	.0694405	.3283384
	L3.	.0124027	.0655948	0.19	0.850	-.1161608	.1409662
	_cons	1.532731	.3411249	4.49	0.000	.8641384	2.201324

Interpretation

- It is difficult to interpret the large number of coefficients in the VAR model
- Main tools for interpretation
  - Impulse responses

Impulse Response Analysis

- VAR(1) with no intercept

$$y_t = \alpha_{11} y_{t-1} + \beta_{11} x_{t-1} + e_{1t}$$
$$x_t = \alpha_{21} y_{t-1} + \beta_{21} x_{t-1} + e_{2t}$$

- The impulse responses are the time-paths of to y and x in response to shocks

Impulse Response Analysis

- The errors may be correlated.
- We “orthogonalize” them

$$
\begin{aligned}
e_{1t} &= u_{1t} \\
e_{2t} &= \rho e_{1t} + u_{2t} \\
&= \rho u_{1t} + u_{2t}
\end{aligned}
$$

Orthogonalized Model

$$y_t = \alpha_{11} y_{t-1} + \beta_{11} x_{t-1} + u_{1t}$$

$$x_t = \alpha_{21} y_{t-1} + \beta_{21} x_{t-1} + \rho u_{1t} + u_{2t}$$

- The shocks u_1 and u_2 are uncorrelated
- The ordering matters
  - The shock to y affects both y and x in period t
  - The shock to x affects only x in period t
- The impulse responses are the time-paths of to y and x in response to the shocks u_1 and u_2
- Imagine y=0 and x=0. Set u_1=1. Trace the history of y and x

Impulse Responses by Recursion

$$y_1 = \alpha_{11} 0 + \beta_{11} 0 + 1 = 1$$

$$x_1 = \alpha_{21} 0 + \beta_{21} 0 + \rho 1 = \rho$$

$$y_2 = \alpha_{11} y_1 + \beta_{11} x_1 = \alpha_{11} + \beta_{11}$$

$$x_2 = \alpha_{21} y_1 + \beta_{21} x_1 = \alpha_{21} + \beta_{21}\rho$$

$$y_3 = \alpha_{11} y_2 + \beta_{11} x_2 = \alpha_{11}(\alpha_{11} + \beta_{11}) + \beta_{11}(\alpha_{21} + \beta_{21}\rho)$$

$$x_3 = \alpha_{21} y_2 + \beta_{21} x_2 = \alpha_{21}(\alpha_{11} + \beta_{11}) + \beta_{21}(\alpha_{21} + \beta_{21}\rho)$$

Impulse Responses

- The impulse responses are these time-paths of y and x due to the shocks u_1 and u_2
- They are found by this recursion formula
- They are functions of the estimated VAR coefficients

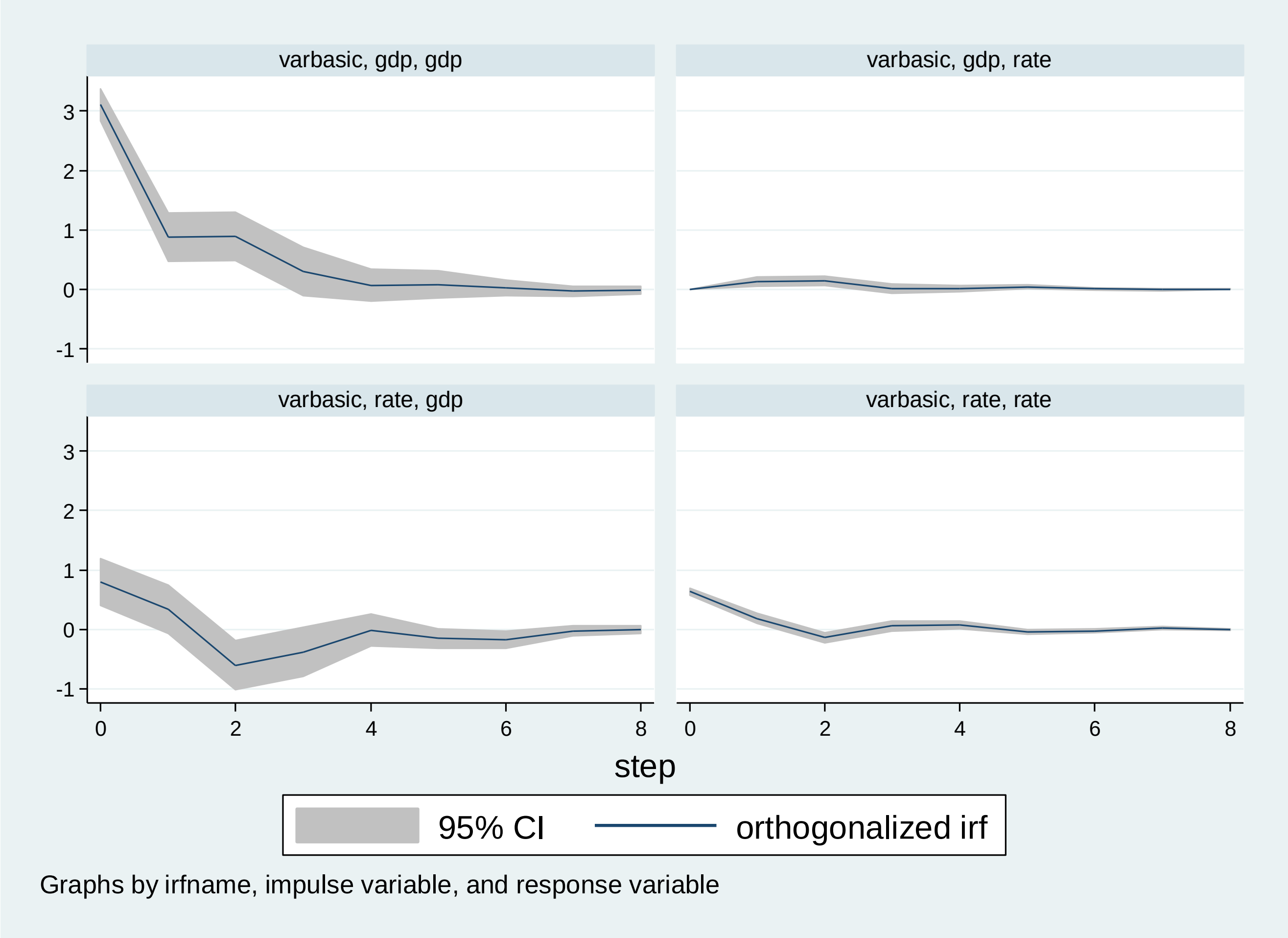
Impact of Shocks on Variables

- In a 2-variable system, there are 4 impulse response functions
  - The effect on y of a shock to y (u_1)
  - The effect on y of a shock to x (u_2)
  - The effect on x of a shock to y (u_1)
  - The effect on x of a shock to x (u_2)
- In a k-variable system, there are k^2 impulse response functions!

Stata Calculation

- Impulse response automatically calculated with varbasic command
- A kxk matrix of impulse response is created

GDP/Interest Rate Example



Interpretation

- Labeled “Graphs by irfname, impulse variable, and response variable”
  - “Impulse variable” means the source of the shock
  - “Response variable” means the variable being affected
- Upper left: “varbasic, gdp, gdp”
  - Impact of a gdp shock on the time-path of gdp
- Upper right: “varbasic, gdp, rate”
  - Impact of a gdp shock on the time-path of interest rates
- Lower left: “varbasic, rate, gdp,”
  - Impact of an interest rate shock on the time-path of gdp
- Lower fight: “varbasic, rate, rate”
  - Impact of an interest rate shock on the time-path of interest rates
- The impulse response is graphed as a function of forward time periods

Scale

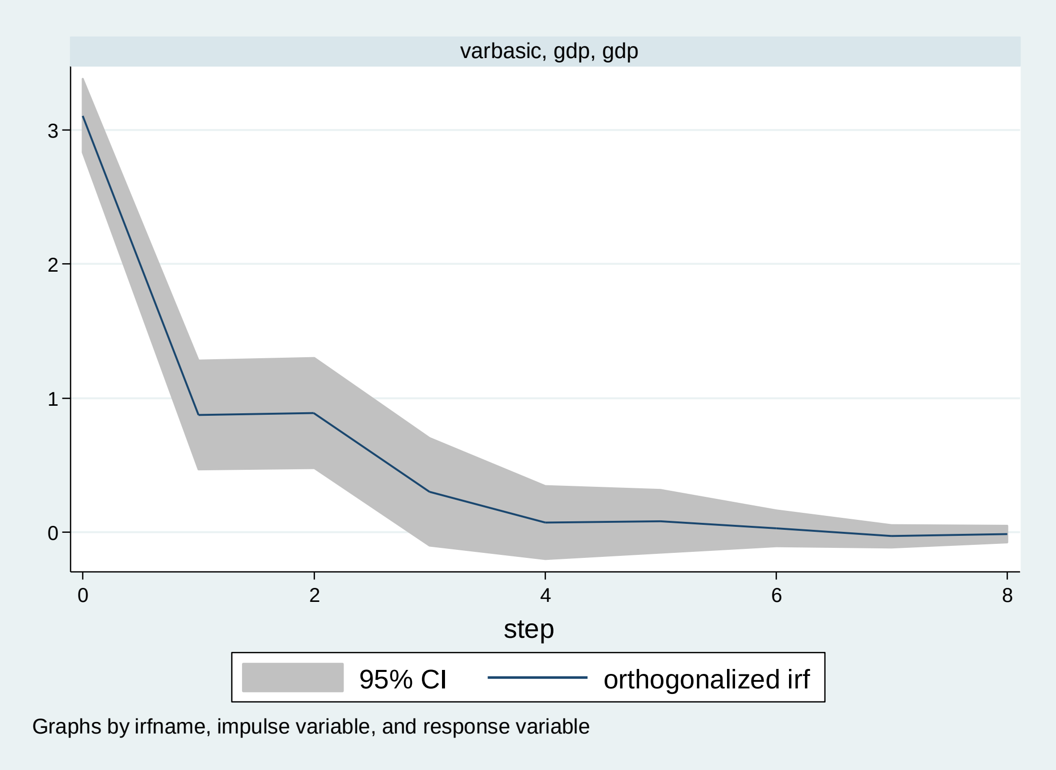
- The graphs are all created on the same scale, so difficult to read
- It may be better to create graphs separate for each impulse response

```
. irf graph oirf, impulse(gdp) response(rate)
```

- This creates the impulse response for the impact of a gdp shock on the time-path of interest rates

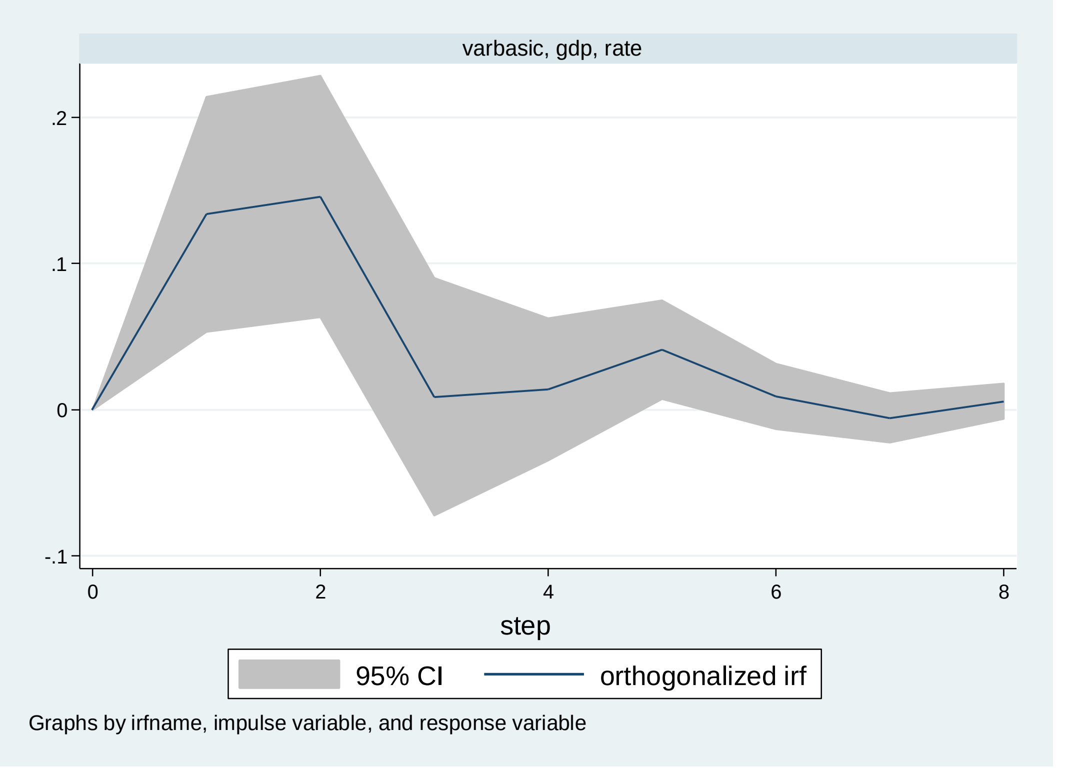
GDP on GDP

```
. irf graph oirf, impulse(gdp) response(gdp)
```



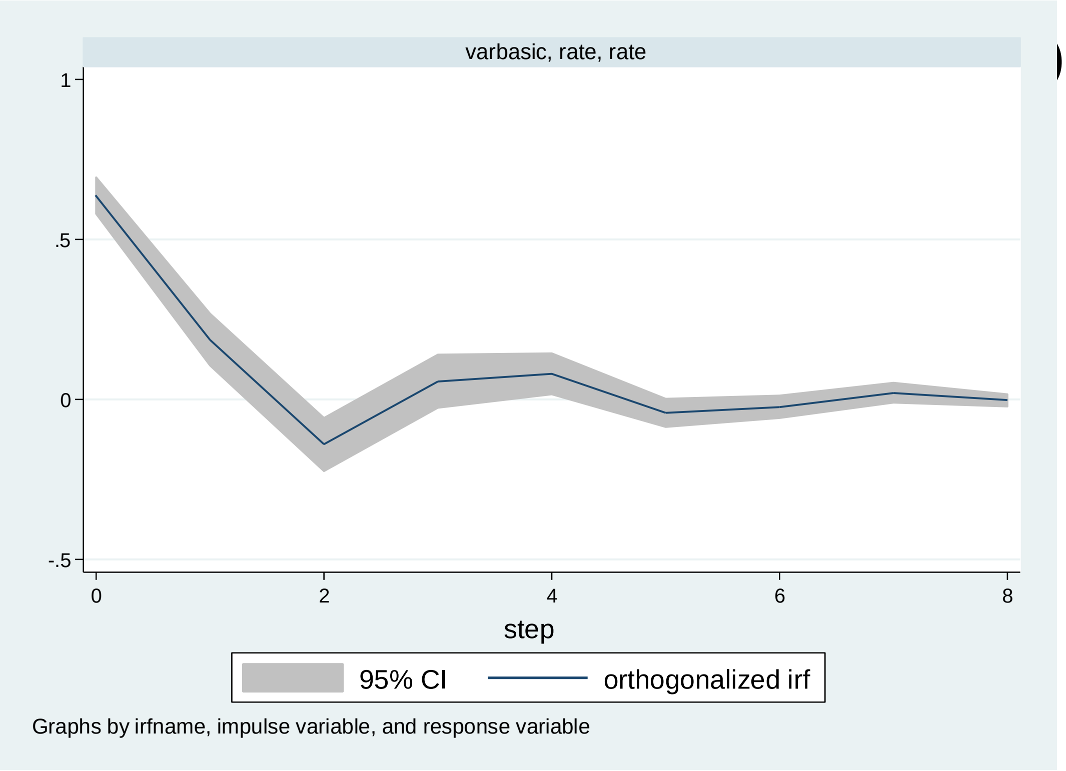
GDP on Interest Rates

```
. irf graph oirf, impulse(gdp) response(rate)
```



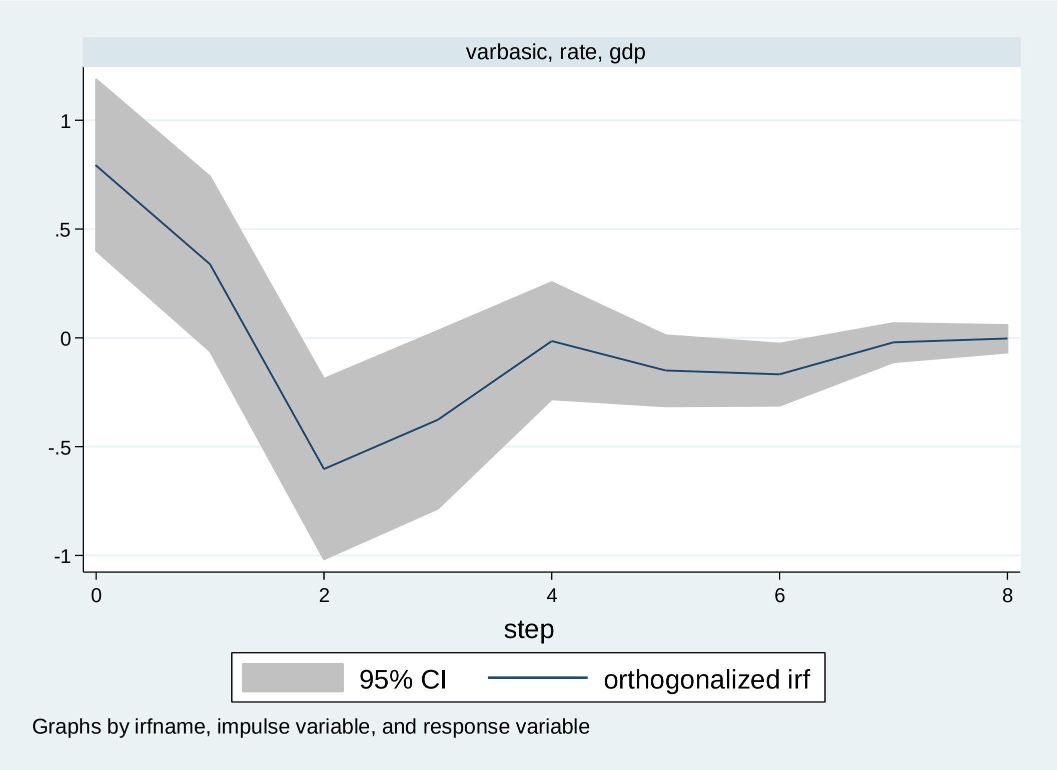
Interest Rates on Interest Rates

. irf grap **)**



Interest Rates on GDP

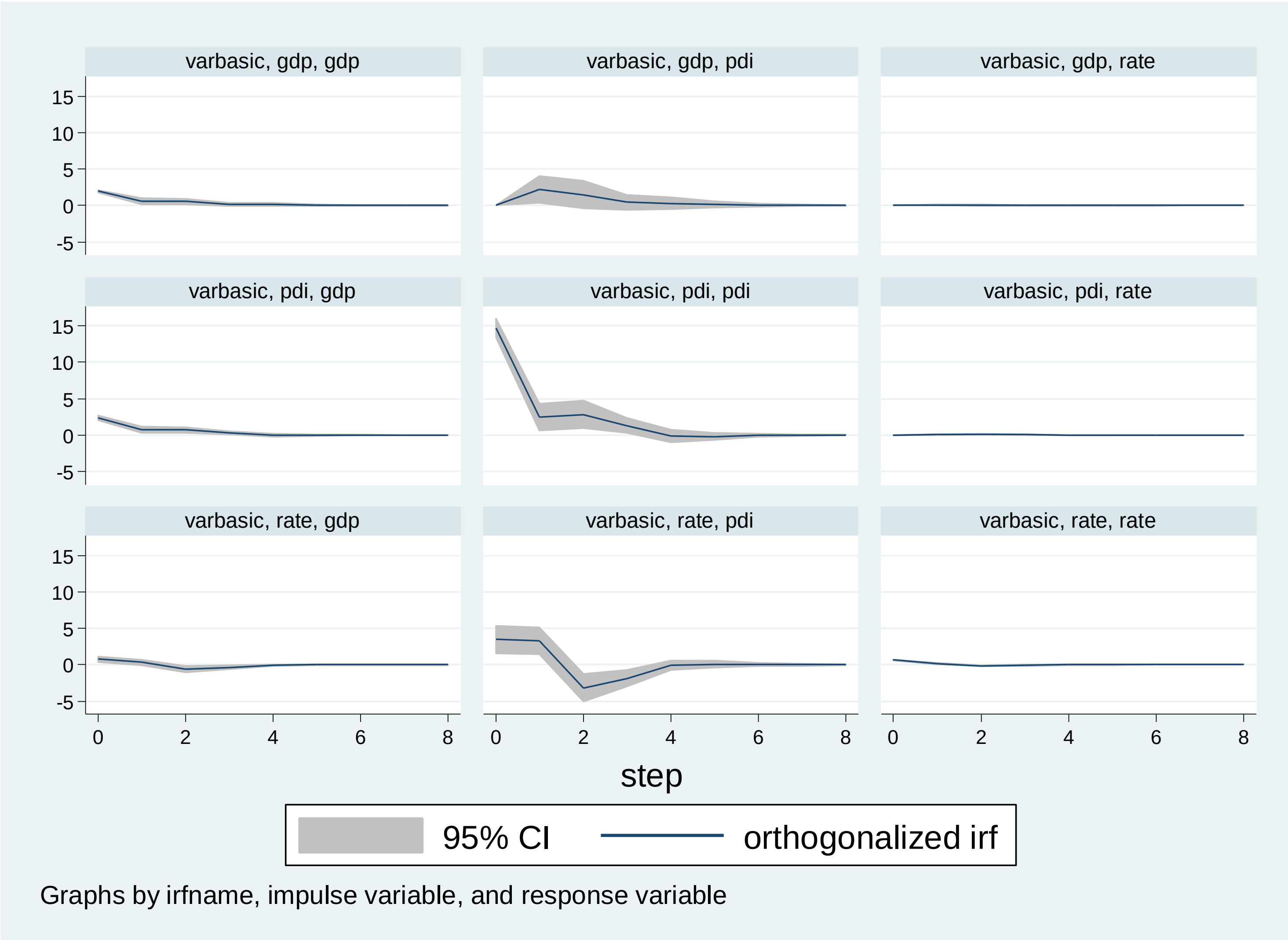
```
. irf graph oirf, impulse(rate) response(gdp)
```



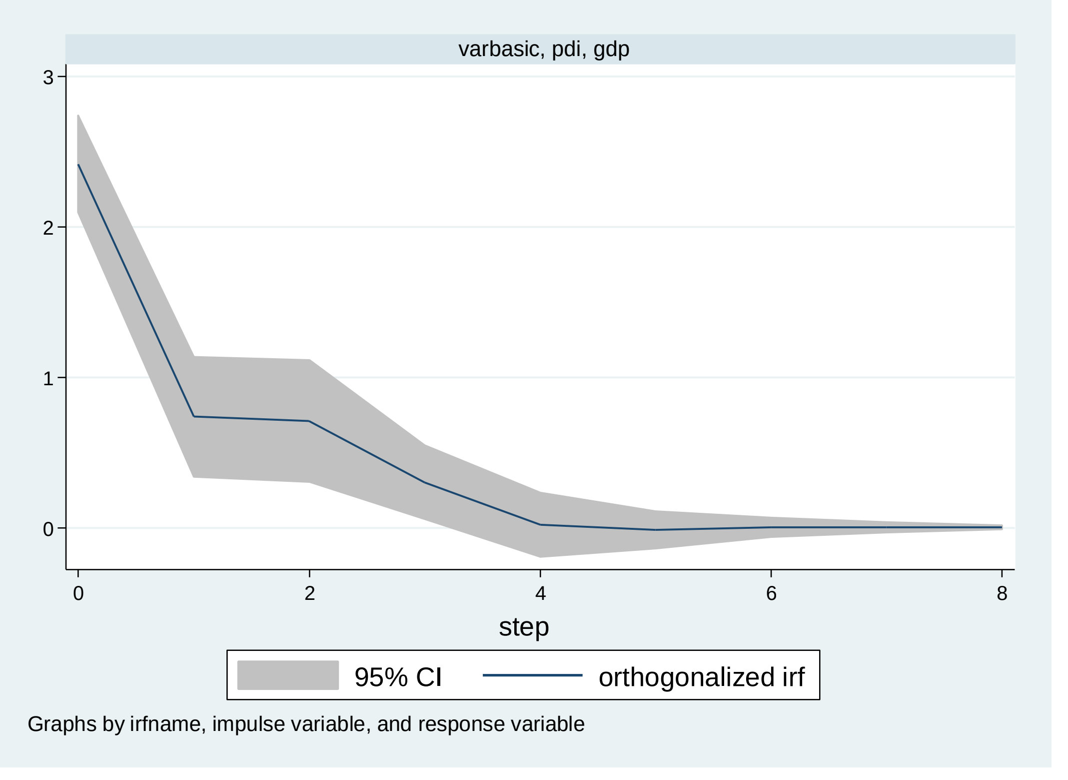
3-variable system

- Interest Rate Change (12-month T-Bill)
- Investment Growth Rate
- GDP Growth Rate

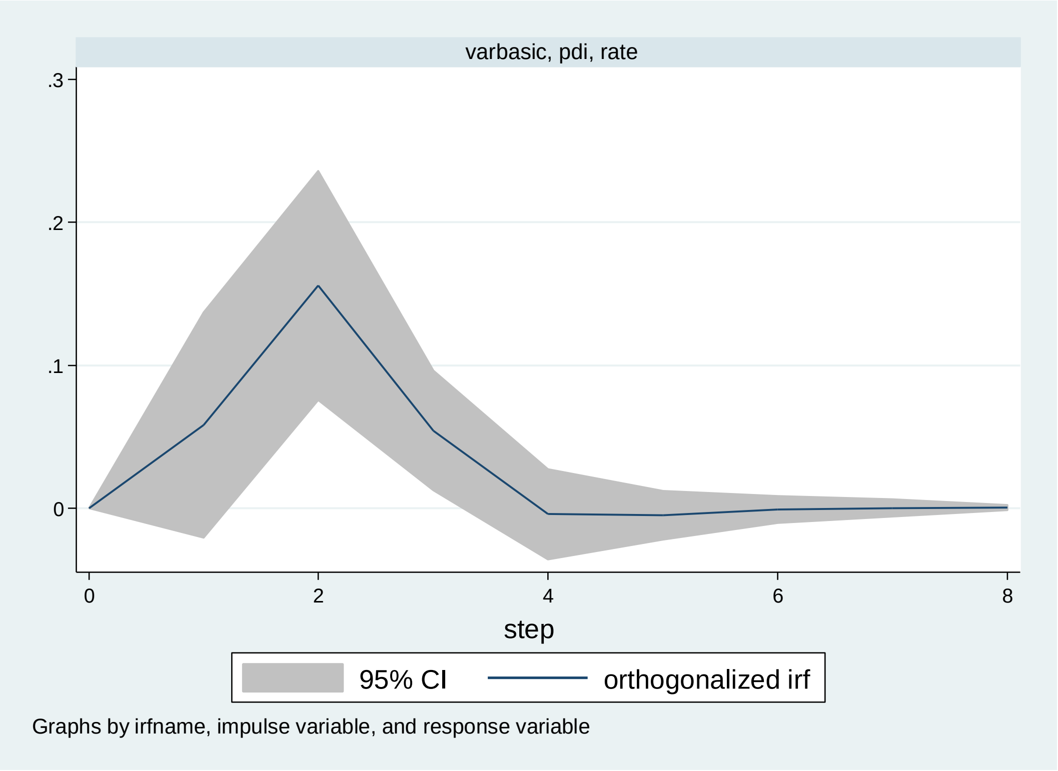
GDP/Investment/Interest Rate



Investment Shock on GDP



Investment Shock on Interest Rate



Interest Rate Shock on Investment