Dynamic Competitive Analysis

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Spring 2001

Contents

CONTENTS — MANUSCRIPT

Chapter 1

Dynamic Programming

1.1 A Heuristic Approach

1.1.1 Neoclassical Growth Model

Consider the following optimization problem

$$\max_{\{c_t, k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \underbrace{U(c_t)}_{U:R_+ \to R}$$

subject to

$$c_t + k_{t+1} = \underbrace{F(k_t)}_{F:R_+ \to R_+} .$$

The above problem can be reformulated as

$$\max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(F(k_t) - k_{t+1}).$$

How can this problem be solved?

T-Period Problem

$$\max_{\{k_{t+1}\}_{t=1}^{T}} \sum_{t=1}^{T} \beta^{t-1} U(F(k_t) - k_{t+1}).$$

Period-T Problem

$$\begin{aligned} V^1(k_T) &\equiv \max_{k_{T+1}} \{U(F(k_T) - k_{T+1})\} && \text{P(1)} \\ &= U(F(k_T) - \underbrace{k^*_{T+1}}_{=0}), \end{aligned}$$

where k^*_{T+1} solves problem P(0).

$V^1(k_T)$ = value of entering last period with k_T units of capital and behaving optimally henceforth. The superscript refers to the number of periods remaining in the planning problem. The function $V^1(k_T)$ is called the value function, while k_T is known as the state variable.

Period-(T-1) Problem

$$\begin{aligned} V^2(k_{T-1}) &\equiv \max_{k_T} \{U(F(k_{T-1}) - k_T) + \beta V^1(k_T)\} && \text{P(2)} \\ &= U(F(k_{T-1}) - \underbrace{k^*_T}_{=G^2(k_{T-1})}) + \beta V^1(k^*_T), \end{aligned}$$

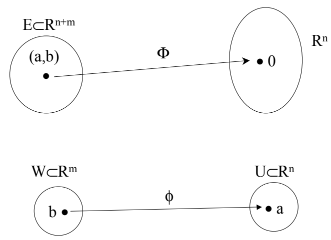

Figure 1.1: Implicit Function Theorem

where $k_T^* = G^2(k_{T-1})$ solves problem P(2). The function G^2 is called the decision rule or policy function. Here k_T^* solves the first-order condition

$$-U_1(F(k_{T-1}) - k_T^*) + \beta V_1^1(k_T^*) = 0. \tag{1.1}$$

Question: What can be said about the functions G^2 and V^2? Note that first-order condition (1.1) defines an implicit function determining k_T as a function of k_{T-1}. More generally in economics one often comes across equation systems of the form $\Phi(x, y) = 0$, where $x \in \mathcal{R}^n$, $y \in \mathcal{R}^m$, and $\Phi : \mathcal{R}^{n+m} \rightarrow \mathcal{R}^n$. Can a function ϕ be found that solves for x in terms of y so that $x = \phi(y)$?

Theorem 1 *Implicit Function. Let Φ be a $\mathcal{C}^q$ mapping from an open set $E \subset \mathcal{R}^{n+m}$ into $\mathcal{R}^n$ such that $\Phi(a, b) = 0$ for some point $(a, b) \in E$. Suppose that the Jacobian determinant $|J| = |\frac{\partial \Phi(a,b)}{\partial x}| \neq 0$. Then there exits a neighborhood $U \subset \mathcal{R}^n$ around a and a neighborhood $W \subset \mathcal{R}^m$ around b and a unique function $\phi : W \rightarrow U$ such that,* *1. $a = \phi(b)$,*

2. ϕ is class $\mathcal{C}^q$ on W,

3. for all $y \in W$, $(\phi(y), y) \in E$, and $\Phi(\phi(y), y) = 0$.

Now, applying the implicit function theorem to the first-order condition (1.1) it is apparent that under the standard conditions $k_T^* = G^2(k_{T-1})$ will be $\mathcal{C}^1$ function which implies that $V^2(k_{T-1})$ will be one too.

Period-t Problem

$$
\begin{aligned}
V^{T+1-t}(k_t) &\equiv \max_{k_{t+1}}\{U(F(k_t) - k_{t+1}) + \beta V^{T-t-1}(k_{t+1})\} \qquad \text{P(T+1-t)} \\
&= U(F(k_t) - k_{t+1}^*) + \beta V^{T-t-1}(k_{T+1}^*),
\end{aligned}
$$

where $k_{t+1}^* = G^{T+1-t}(k_t)$ solves problem P(T+1-t).

Observe that dynamic programming has effectively collapsed a single large problem involving $T+1-t$ choice variables into $T+1-t$ smaller problems, each involving one choice variable. To see this, solve out for $V^{T-t}(k_{t+1})$ in P(T+1-t) to get

$$
\begin{aligned}
V^{T+1-t}(k_t) &\equiv \max_{k_{t+1}}\{U(F(k_t) - k_{t+1}) + \beta \max_{k_{t+2}}\{U(F(k_{t+1}) - k_{t+2}) + \beta V^{T-t}(k_{t+2})\}\} \\
&= \max_{k_{t+1},k_{t+2}}\{U(F(k_t) - k_{t+1}) + \beta U(F(k_{t+1}) - k_{t+2}) + \beta^2 V^{T-t-1}(k_{t+2})\}.
\end{aligned}
$$

Solving out recursively for $V^{T-t}(k_{t+2})$, $V^{T-t-1}(k_{t+3})$, ..., yields

$$\max_{\{k_{t+j+1}\}_{j=0}^{T-t}} \sum_{j=0}^{T-t} \beta^j U(F(k_{t+j}) - k_{t+j+1}).$$

Infinite Horizon Problem

As $T \to \infty$ one might expect that

$$V^{T+1-t}(k_t) \to V(k_t)$$

and

$$G^{T+1-t}(k_t) \to G(k_t).$$

This is true but it takes some effort to show it. Thus, the problem for the infinite horizon will take the form:

$$\begin{aligned} V(k_t) &\equiv \max_{k_{t+1}} \{U(F(k_t) - k_{t+1}) + \beta V(k_{t+1})\} \qquad \mathrm{P}(\infty) \\ &= U(F(k_t) - k^*_{t+1}) + \beta V(k^*_{t+1})\}, \end{aligned}$$

where $k^*_{t+1} = G(k_t)$.

1.1.2 The Envelope Theorem

Assumption: V is continuously differentiable.

How is the solution to problem P(∞) characterized? The f.o.c. is

$$-U_1(F(k_t) - k_{t+1}) + \beta V_1(k_{t+1}) = 0 \tag{1.2}$$

or

$$U_1(F(k_t) - k_{t+1}) = \beta V_1(k_{t+1}). \tag{1.3}$$

Problem: This equation involves the unknown function V. What should be done?

Answer: Differentiate both sides of P(∞) with respect to k_t to get

$$\begin{aligned}
V_1(k_t) &= U_1(F(k_t) - k_{t+1})F_1(k_t) - U_1(F(k_t - k_{t+1})\frac{\partial k_{t+1}}{\partial k_t} + V_1(k_{t+1})\frac{\partial k_{t+1}}{\partial k_t} \\
&= U_1(F(k_t) - k_{t+1})F_1(k_t) + [-U_1(F(k_t - k_{t+1}) + V_1(k_{t+1})]\frac{\partial k_{t+1}}{\partial k_t} \\
&= U_1(F(k_t) - k_{t+1})F_1(k_t),
\end{aligned}$$

since the term in brackets on the second line is zero by the first-order condition (1.2). Updating this expression from period t to period $t+1$ gives

$$V_1(k_{t+1}) = U_1(F(k_{t+1}) - k_{t+2})F_1(k_{t+1}).$$

This allows equation (1.3) to be rewritten as

$$U_1(F(k_t) - k_{t+1}) = \beta U_1(F(k_{t+1}) - k_{t+2})F_1(k_{t+1}).$$

1.2 A More Formal Analysis

1.2.1 Neoclassical Growth Model

Dynamic Programming Representation

$$V(k) \equiv \max_{k'}\{U(F(k) - k') + \beta V(k')\} \qquad \text{P(1)}$$

The problem at hand is to get answers to the following questions:

1. Will V exist?

2. Is V unique?

3. Is V continuous?

4. Is V continuously differentiable?

5. Is V increasing in k?

6. Is V concave in k?

1.2.2 Method of Successive Approximation

Goal: To approximate the value function V by a sequence of successively better guesses, denoted by V^j at stage j

Procedure:

- Stage 0. Make an initial guess for V. Call it V^0.

- Stage 1. Construct a revised guess for V, denoted by V^1.

$$V^1(k) \equiv \max_{k'}\{U(F(k) - k') + \beta V^0(k')\}$$

- Stage $n+1$. Compute V^{n+1} given V^n, as follows

$$V^{n+1}(k) \equiv \max_{k'}\{U(F(k)-k') + \beta V^n(k')\}. \tag{P(2)}$$

This procedure can be represented much more compactly using operator notation.

$$V^{n+1} = TV^n.$$

The operator T is shorthand notation for the list of operations, described by P(2) that are performed on the function V^n to transform it into the new one V^{n+1}. Often the operator T maps some set of functions, say $\mathcal{C}$, into itself. That is, $T : \mathcal{C} \to \mathcal{C}$. The hope is that as n gets large it transpires that $V^n \to V$, where $V = TV$.

1.2.3 Metric Space

Definition 2 *A metric space is a set S, together with a metric $\rho : S \times S \to \mathcal{R}_+$, such that for all $x, y, z \in S$ (see Figure 2):*

1. $\rho(x, y) \geq 0$, *with* $\rho(x, y) = 0$ *if and only if* $x = y$,
2. $\rho(x, y) = \rho(y, x)$,
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

Example 1

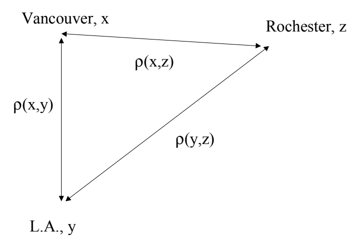

Figure 1.2: Distances Between Cities

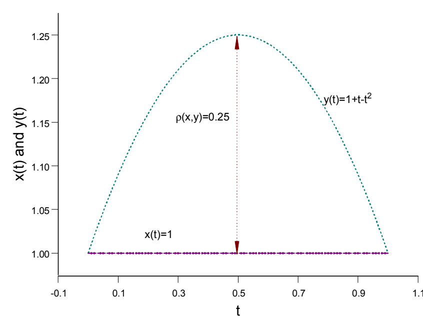

Figure 1.3: Uniform Metric

Space of continuous functions $\mathcal{C} : [a, b] \to \mathcal{R}_+$. See Figure 3.

$$\rho(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|.$$

Definition 3 *A sequence* $\{x_n\}_{n=0}^{\infty}$ *in* S *converges to* $x \in S$*, if for each* $\varepsilon > 0$ *there* *exists a N_ε such that*

$$\rho(x_n, x) < \varepsilon, \text{ for all } n \geq N_\varepsilon.$$

Definition 4 *A sequence $\{x_n\}_{n=0}^{\infty}$ in S is a Cauchy sequence if for each $\varepsilon > 0$ there exists a N_ε such that*

$$\rho(x_m, x_n) < \varepsilon, \text{ for all } m, n \geq N_\varepsilon.$$

Remark 5 *A Cauchy sequence in S may not converge to a point in S.*

Example 2

Let $S = (0, 1]$, $\rho(x, y) = |x - y|$, and $\{x_n\}_{n=0}^{\infty} = \{1/n\}_{n=0}^{\infty}$. Clearly, $x_n \to 0 \notin (0, 1]$. this sequence satisfies the Cauchy criteria, though, for

$$\rho(x_n, x_m) = |\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{m} + \frac{1}{n} < \varepsilon, \text{ if } m, n > \frac{2}{\varepsilon}.$$

Definition 6 *A metric space (S, ρ) is complete if every Cauchy sequence in S converges to a point in S.*

Theorem 7 *Let $X \subseteq \mathcal{R}^l$ and $\mathcal{C}(X)$ be the set of bounded continuous functions $V : X \to \mathcal{R}$ with the uniform metric $\rho(V, W) = \max_{x \in X} |V - W|$. Then $\mathcal{C}(X)$ is a complete metric space.*

Proof. : Let $\{V^n\}_n$ be any Cauchy sequence in $\mathcal{C}(X)$. Now, for each $x \in X$ the sequence $\{V^n(x)\}_n$ is Cauchy since

$$|V^n(x) - V^m(x)| \leq \sup_{y \in X} |V^n(y) - V^m(y)| = \rho(V^n, V^m).$$

By the completeness of the reals $V^n(x) \to V(x)$, as $n \to \infty$. Define the function V by $V(x)$ for each $x \in X$.

It will now be shown that $\rho(V^n, V) \to 0$ as $n \to \infty$. Choose an $\varepsilon > 0$. Now,

$$\begin{aligned} |V^n(x) - V(x)| &\leq |V^n(x) - V^m(x)| + |V^m(x) - V(x)| \\ &\leq \underbrace{\rho(V^n, V^m)}_{\leq \varepsilon/2} + \underbrace{|V^m(x) - V(x)|}_{\leq \varepsilon/2}. \end{aligned}$$

The first term on the left can be made smaller than $\varepsilon/2$ by the Cauchy criteria; that is, there exists a N_ε such that for all $n, m \geq N_\varepsilon$ it transpires that $\rho(V^n, V^m) \leq \varepsilon/2$. The second term can be made smaller than $\varepsilon/2$ by the pointwise convergence of V^m to V; that is, there exists a $M_\varepsilon(x)$ such that for all $m \geq M_\varepsilon(x)$ it follows that $|V^m(x) - V(x)| \leq \varepsilon/2$. Observe that while $M_\varepsilon(x)$ depends on x, N_ε does not. Also, note that for any value of x such a $M_\varepsilon(x)$ will always exist. Therefore, $|V^n(x) - V(x)| \leq \varepsilon$ for all $n \geq N_\varepsilon$ *independent* of the value of x. It follows that $\rho(V^n, V) \leq \varepsilon$, the desired result.

The last step is to show that V is a continuous function. To do this, pick an $\varepsilon > 0$. Does there exist a $\delta \geq 0$ such that $|V(x) - V(x_0)| \leq \varepsilon$ whenever $\rho(x, x_0) \leq \delta$?

Note that

$$|V(x)-V(x_0)| \leq \underbrace{|V(x)-V^n(x)|}_{\varepsilon/3} + \underbrace{|V^n(x)-V^n(x_0)|}_{\varepsilon/3} + \underbrace{|V(x_0)-V^n(x_0)|}_{\varepsilon/3}.$$

The first and third terms can be made arbitrarily small by the uniform convergence of V^n to V. The second term can be made to vanish by the fact that V^n is a continuous function; that is, by picking a δ small enough such that this term is less than $\varepsilon/3$. ■

Remark 8 *Pointwise convergence of a sequence of continuous functions does not imply that the limiting function is continuous.*

Example 3

Let $\{V^n\}_{n=1}^{\infty}$ in $\mathcal{C}[0,1]$ be defined by $V^n(t) = t^n$. As $n \to \infty$ it transpires that: (i) $V^n(t) \to 0$ for $t \in [0,1)$ and (ii), $V^n(t) \to 1$ for $t = 1$. Thus,

$$V(t) = \begin{cases} 0, & \text{for } t \in [0,1), \\ 1, & \text{for } t = 1. \end{cases}$$

Hence $V(t)$ is a discontinuous function. See Figure 4. Clearly, by the above theorem $\{V^n\}_{n=1}^{\infty}$ cannot describe a Cauchy sequence. This can be shown directly too, however. In particular, for given *any* N_ε it is always possible to pick a $m, n \geq N_\varepsilon$ and $t \in [0,1)$ so $|t^n - t^m| \geq 1/2$. To see this pick $n = N_\varepsilon$ and a $t < 1$ so that $t^n \geq 3/4$; i.e, choose

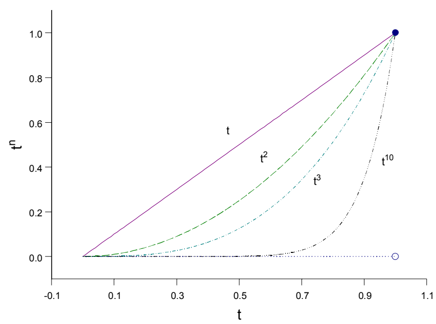

Figure 1.4: Pointwise Convergence to a Discontinuous Function

$t \geq (3/4)^{1/N_\varepsilon}$. Next, pick m large enough such that $t^m < 1/4$ or $m \geq (\ln 1/4)/(\ln t)$. The desired results obtains.

Example 4

Consider the space of continuous functions $\mathcal{C}[-1, 1]$ with metric

$$\rho(x, y) = \int_{-1}^{1} |x(t) - y(t)| dt.$$

Let $\{V^n\}_{n=1}^{\infty}$ in $\mathcal{C}[-1, 1]$ be defined by

$$V^n(t) = \begin{cases} 0, & \text{if } -1 \leq t \leq 0, \\ nt, & \text{if } 0 < t < 1/n, \\ 1, & \text{if } 1/n \leq t \leq 1. \end{cases}$$

Show that $\{V^n\}_{n=1}^{\infty}$ is a Cauchy sequence. Deduce that the space of continuous functions is not complete with this metric.

1.2.4 The Contraction Mapping Theorem

Definition 9 *Let (S, ρ) be a metric space and $T : S \to S$ be function mapping S into itself. T is a contraction mapping (with modulus β) if for $\beta \in (0, 1)$,*

$$\rho(Tx, Ty) \leq \beta\rho(x, y), \text{ for all } x, y \in S.$$

Theorem 10 *(Contraction Mapping Theorem or Banach Fixed Point Theorem). If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then*

1. *T has exactly one fixed point $V \in S$ such that $V = TV$,*
2. *for any $V^0 \in S$, $\rho(T^n V^0, V) < \beta^n \rho(V^0, V)$, $n = 0, 1, 2...$.*

Proof. Define the sequence $\{V^n\}_{n=0}^{\infty}$ by

$$V^n = TV^{n-1} = \underbrace{TT}_{T^2} V^{n-2} = T^n V^0.$$

Now, by the contraction property of T

$$\rho(V^2, V^1) = \rho(TV^1, TV^0) \leq \beta\rho(V^1, V^0).$$

Hence,

$$\rho(V^{n+1}, V^n) = \rho(TV^n, TV^{n-1}) \leq \beta\rho(V^n, V^{n-1}) \leq \beta^n \rho(V^1, V^0).$$

Therefore for any $m > n$

$$\begin{aligned}
\rho(V^m, V^n) &\leq \underbrace{\rho(V^m, V^{m-1}) + \rho(V^{m-1}, V^{m-2}) + ... + \rho(V^{n+1}, V^n)}_{\text{Triangle of Inequality}} \\
&\leq (\beta^{m-1} + \beta^{m-2} + ... + \beta^n)\rho(V^1, V^0) \\
&\leq \frac{\beta^n}{1-\beta}\rho(V^1, V^0).
\end{aligned}$$

Therefore $\{V^n\}_{n=0}^{\infty}$ is a Cauchy sequence, since $\frac{\beta^n}{1-\beta} \to 0$ as $n \to \infty$. Since S is complete $V^n \to V$.

To show that $V = TV$ note that for all $\varepsilon > 0$ and $V^0 \in S$

$$\begin{aligned}
\rho(V, TV) &\leq \rho(V, T^n V^0) + \rho(T^n V^0, TV) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2},
\end{aligned}$$

for large enough n since $\{V^n\}_{n=0}^{\infty}$ is a Cauchy sequence. Therefore, $V = TV$.

Finally suppose that another function $W \in S$ satisfies $W = TW$. Then,

$$\rho(V, W) = \rho(TV, TW) \leq \beta\rho(V, W),$$

a contradiction unless $V = W$.

$$\rho(T^n V^0, V) = \rho(T^n V^0, TV) \leq \beta\rho(T^{n-1}V^0, V) \leq \beta^n\rho(V^0, V).$$

■

Corollary 11 *Let (S, ρ) be a complete metric space and let $T : S \to S$ be a contraction mapping with fixed point $V \in S$. If S' is a closed subset of S and $T(S') \subseteq S'$ then $V \in S'$. If in addition $T(S') \subseteq S'' \subseteq S'$, then $V \in S''$.*

Proof. Choose $V^0 \in S'$ and note that $\{T^n V^0\}$ is a sequence in S' converging to V. Since S' is closed, it follows that $V \in S'$. If $T(S') \subseteq S''$, it then follows that $V = TV \in S''$.

■

Theorem 12 *(Blackwell's Sufficiency Condition) Let $X \subseteq \mathcal{R}^l$ and $B(X)$ be the space of bounded functions $V : X \to \mathcal{R}$ with the uniform metric. Let $T : B(X) \to B(X)$ be an operator satisfying*

1. *(Monotonicity) $V, W \in B(X)$. If $V \leq W$ [i.e., $V(x) \leq W(x)$ for all x] then $TV \leq TW$.*

2. *(Discounting) There exists some constant $\beta \in (0,1)$ such that $T(V+a) \leq TV + \beta a$, for all $V \in B(X)$ and $a \geq 0$. Then T is a contraction with modulus β.*

Proof. For every $V, W \in B(X)$, $V \leq W + \rho(V, W)$. Thus, (1) and (2) imply

$$TV \leq \underbrace{T(W + \rho(V, W))}_{\text{Monotonicity}} \leq \underbrace{TW + \beta\rho(V, W)}_{\text{Discounting}}.$$

Thus,

$$TV - TW \leq \beta\rho(V, W).$$

By permuting the functions it is easy to show that

$$TW - TV \leq \beta\rho(V, W).$$

Consequently,

$$|TV - TW| \leq \beta\rho(V, W),$$

so that

$$\rho(TV, TW) \leq \beta\rho(V, W).$$

Therefore T is a contraction. ■

1.2.5 Neoclassical Growth Model

Consider the mapping

$$(TV)(k) = \max_{k' \in \mathcal{K}} \{U(F(k) - k') + \beta V(k')\}, \qquad \text{P(3)}$$

where k, $k' \in \mathcal{K} = \{k_1, k_2, ..., k_n\}$. Is T a contraction?

1. Monotonicity. Suppose $V(k) \leq W(k)$ for all k. Need to show that $(TV)(k) \leq (TW)(k)$.

$$(TV)(k) = \{U(F(k) - k'^*) + \beta V(k'^*)\},$$

where k'^* maximizes P(3). Now, clearly

$$\begin{aligned}(TV)(k) &\leq \{U(F(k)-k'^*)+\beta W(k'^*)\}\\ &\leq \max_{k'\in\mathcal{K}}\{U(F(k)-k')+\beta W(k')\}\\ &= (TW)(k).\end{aligned}$$

2. Discounting.

$$\begin{aligned}T(V+a)(k) &= \max_{k'\in\mathcal{K}}\{U(F(k)-k')+\beta[V(k')+a]\}\\ &= \max_{k'\in\mathcal{K}}\{U(F(k)-k')+\beta V(k')\}+\beta a\\ &= (TV)(k)+\beta a.\end{aligned}$$

1.1 Consider the following dynamic programming problem

$$V(k_i,\varepsilon_r)=\max_{c,k'_j\in\mathcal{K}}\{U(c)+\beta\sum_{j=1}^{n}\pi_{rs}V(k'_j,\varepsilon_s)\},$$

subject to

$$c+i=F(k_i),$$

and

$$k'_j=(1-\delta)k_i+i\varepsilon_r. \tag{1.4}$$

Let U be a bounded, strictly increasing, strictly concave, continuous function. Suppose that $0 < \beta < 1$. The bounded positive random variable ε follows a m-point Markov process. In particular, ε is drawn from the discrete set $\mathcal{E} \equiv \{\varepsilon_1, \varepsilon_{2,...,} \varepsilon_m\}$ according to the probability distribution specified by $\pi_{rs} = \Pr\{\varepsilon' = \varepsilon_r | \varepsilon = \varepsilon_s\}$, where $0 \leq \pi_{rs} \leq 1$, and $\sum_{j=1}^{m} \pi_{rs} = 1$. Furthermore, suppose that $k \in \mathcal{K} = \{k_1, k_2, ..., k_n\}$.

1. Show that there exists a V that solves the above Bellman equation.

1.2.6 Characterizing the Value Function

What can be said about the function V?

1. Is V continuous in k?
2. Is V strictly increasing in k?
3. Is V strictly concave in k?
4. Is V differentiable in k?.

Definition 13 *A function $V : X \to \mathcal{R}$ is strictly increasing if $x > y$ implies $V(x) > V(y)$. A function $V : X \to \mathcal{R}$ is nondecreasing (or increasing) if $x > y$ implies $V(x) \geq V(y)$.*

Definition 14 *A function $V : X \to \mathcal{R}$ is strictly concave if*

$$V(\theta x + (1-\theta)y) > \theta V(x) + (1-\theta)V(y),$$

for all $x, y \in X$ such that $x \neq y$ and $\theta \in (0,1)$. A function $V : X \to \mathcal{R}$ is concave if $V(\theta x + (1-\theta)y) \geq \theta V(x) + (1-\theta)V(y)$,for all $x, y \in X$ such that $x \neq y$ and $\theta \in (0,1)$.

1.1 Show that the space of increasing functions with the uniform metric is complete.

1.1 Show that the space of concave function with the uniform metric is complete.

Assumption: Let U and F be strictly increasing functions.

Assumption: Let U and F be strictly concave functions.

Theorem 15 *The function V is both strictly increasing and strictly concave.*

Proof. Consider again the mapping given by

$$(TV)(k) = \max_{k'}\{U(F(k) - k') + \beta V(k')\}. \qquad \text{P(3)}$$

It will be shown that the operator T maps concave functions into *strictly* concave ones. It also maps increasing functions into strictly increasing ones. Let V be a concave function. Take two points $k_0 \neq k_1$ and let $k_\theta = \theta k_0 + (1-\theta)k_1$. Observe that $F(k_\theta) > \theta F(k_0) + (1-\theta)F(k_1)$, since F is strictly concave. It needs to be shown that

$$(TV)(k_\theta) > \theta(TV)(k_0) + (1-\theta)(TV)(k_1).$$

To this end, define $k_0'^*$ as the maximizer for $(TV)(k_0)$, $k_1'^*$ as the maximizer for $(TV)(k_1)$, and $k_\theta' = \theta k_0'^* + (1-\theta)k_1'^*$. Note that k_θ' is a feasible choice when $k = k_\theta$ since $k_0'^* \leq F(k_0)$ and $k_1'^* \leq F(k_1)$ while $\theta F(k_0) + (1-\theta)F(k_1) < F(k_\theta)$. Now,

$$
\begin{aligned}
(TV)(k_\theta) &\geq U(F(k_\theta) - k_\theta') + \beta V(k_\theta'), \text{ (k_θ' is nonoptimal)} \\
&> \theta[U(F(k_0) - k_0'^*) + \beta V(k_0'^*)] \\
&\quad +(1-\theta)[U(F(k_1) - k_1'^*) + \beta V(k_1'^*)], \text{ (by strict concavity)} \\
&> \theta(TV)(k_0) + (1-\theta)(TV)(k_1) \text{ (by definition).}
\end{aligned}
$$

■

Remark 16 *The space of strictly concave is not complete – see figure 4. Hence, to finish the argument an appeal to the corollary of the contraction mapping theorem can be made.*

Theorem 17 *The function V is continuous in k.*

Proof. It will be shown that the operator described by P(3) maps strictly increasing, strictly concave C^2 functions into strictly increasing, strictly concave C^2 functions. Suppose that V^n is a continuous, strictly increasing, strictly concave C^2 function. The the decision rule for k' is determined from the first-order condition

$$U_1(F(k) - k') = \beta V_1^n(k').$$

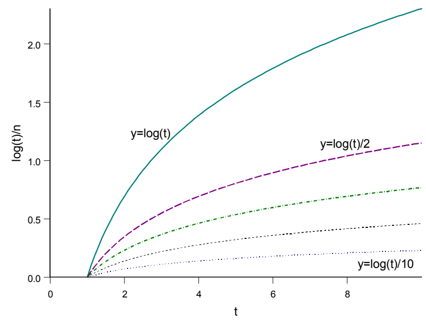

Figure 1.5: The space of strictly concave functions is not complete

This determines k' as a continuously differentiable function of k by the implicit function theorem. Note that $0 < dk'/dk < F_1(k)$. Therefore, $V^{n+1}(k)$ is a strictly increasing, strictly concave C^2 function too since $V_1^{n+1}(k) = U_1(F(k) - k')F_1(k)$. The limit of such a sequence must be a continuous function. (It is does *not* have to be a C^2 function) ■

Differentiability

Lemma 18 *Let $X \subseteq \mathcal{R}^l$ be a convex set, $V : X \to \mathcal{R}$ be a concave function. Pick an $x_0 \in intX$ and let D be a neighborhood of x_0. If there is a concave, differentiable function $W : D \to \mathcal{R}$ with $W(x_0) = V(x_0)$ and $W(x) \leq V(x)$ for all $x \in D$ then V is differentiable at x_0 and*

$$V_i(x_0) = W_i(x_0), \text{ for } i = 1, 2, ..., l.$$

Proof. See Figure 5

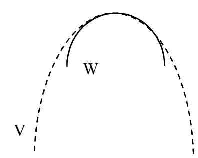

Figure 1.6: Differentiability of V

.

Theorem 19 *(Benveniste and Scheinkman) Suppose that K is a convex set and that U and F are strictly concave C^1 functions. Let $V : K \to \mathcal{R}$ in line with P(3) and denote the decision rule associated with this problem by $k' = G(k)$. Pick $k_0 \in intK$ and assume that $0 < G(k_0) < F(k_0)$. Then $V(k)$ is continuously differentiable at k_0 with its derivative given by*

$$V_1 = U_1(F(k_0) - G(k_0))F_1(k_0).$$

Proof. Clearly, there exists some neighborhood D of k_0 such that $0 < G(k_0) < F(k)$ for all $k \in D$. Define W on D by

$$W(k) = U(F(k) - G(k_0)) + \beta V(G(k_0)).$$

Now, W is concave and differentiable since U and F are. Furthermore, it follows that

$$W(k) \leq \max_{k'}\{U(F(k) - k') + \beta V(k')\} = V(k),$$

with this inequality holding strictly at $k = k_0$. The results then follows immediately from the above lemma. ■

1.2.7 Problems

1.1 Consider the problem described by

$$V(k) = \max_{x_1, x_2, \ldots, x_n} F(x_1, x_2, \ldots x_n; k).$$

Let F be a C^2 function. Presume that a maximum exists. Show that $V_{n+1}(k) = F_{n+1}(x_1, x_2, \ldots x_n; k)$.

1.2 (Aiyagari, 1994) Consider the following dynamic programming problem

$$V(a, z_i) = \max_{c, a'}\{U(c) + \beta \sum_{j=1}^{n} \pi_{ij} V(a', z_j)\},$$

subject to

$$c + a' = z_i + (1 + r)a,$$

and

$$a' \geq 0. \tag{1.5}$$

Let U be a bounded, strictly increasing, strictly concave, continuously differentiable function. Suppose that $0 < r < \beta < 1$. The bounded positive random variable z follows a n-point Markov process. In particular, z is drawn from the discrete set $\mathcal{Z} \equiv \{z_1, z_{2,}..., z_n\}$ according to the probability distribution specified by $\pi_{ij} = \Pr\{z' = z_j | z = z_i\}$, where $0 \leq \pi_{ij} \leq 1$, and $\sum_{j=1}^{n} \pi_{ij} = 1$.

1. Is the function $V(a, z)$ continuously differentiable in a, for all $a > 0$, whenever $a' > 0$?

2. Is the function $V(a, z)$ continuously differentiable in a, for all $a > 0$, when $a' = 0$. What is the issue here?

1.1 *Capacity Choice* (Harris, 1987): Here is a problem facing a monopolist. He faces a demand curve each period given by

$$q = (1 - p),$$

That is, if the price is p he can sell the quantity q. Production is costless but at each period in time the monopolist faces a capacity constraint,

$$q \leq c,$$

where c is the upper bound on his production. Capacity can be increased with a one period time delay according to the cost function

$$(c' - c)^2,$$

where $c' \geq c$ is the level of capacity that the monopolist chooses for next period. The monopolist faces the time-invariant gross interest rate r.

1. Formulate the monopolist dynamic programming problem.

2. *Prove* that the solution is given by

$$V(c) = \begin{cases} \alpha, & \text{for } c \geq 1/2, \\ \delta + \gamma c(1-c), & \text{for } c \leq 1/2, \end{cases}$$

where α and δ are some constants and

$$\gamma = \frac{2(1/r) - 1 + \sqrt{1 + 4(1/r)^2}}{2(1/r)},$$

and that the optimal policy is

$$c' = c + \begin{cases} 0, & \text{for } c \geq 1/2, \\ (1/r)\gamma(1/2 - c)/(1 + \gamma/r), & \text{for } c \leq 1/2. \end{cases}$$

Also solve for α and δ.

1.1 *The Replacement Problem*: Imagine a lot with an age-j building on it. Denote this amount of capital in this building by $k(j)$. The per period profit from the lot with an age-j building on it is $k(j)^\alpha$. Time flows continuously and the capital stock depreciates with age according to the law of motion $dk(j)/dj = \delta k(j)$. At any point in time the owner is free to tear down his existing building and replace it with a new one. The size of a new building is fixed at k. The interest rate is always r.

1. Write out the dynamic programming problem facing the owner.

2. Does a continuous value function solving this problem exist?

3. Is it concave?

4. Is it differentiable?

1.1 *Stochastic Goldmining* (Bellman, 1957): Consider the problem of an entrepreneur who owns two goldmines, Anaconda and Bonanza. Ananconda has x units of gold in its bowls and Bonanza has y units, both measured in dollars. The entrepreneur has a goldmining machine. The machine has the following properties: If it is used in Ananconda in a period it will reap the fraction r_a of the gold in the mine with probability p_a. With probability $1 - p_a$ the machine breaks down, mines no gold, and can never be used again. Getting another machine is impractical. Likewise, in any given period the machine can be used in Bonanza. There it may mine the fraction r_b of the gold with probability p_b and break down with probability $1 - p_b$. The entrepreneur's discount factor is β.

1. Let $V(x, y)$ be the value of the mine. Write out the entrepreneur's dynamic programming problem

2. Prove that V is increasing in x and y.

3. Derive the locus of x and y combinations that yield the same payoff. What does this say about how the value function can be written?

Chapter 2

Business Cycle Analysis

2.1 Introduction

2.1.1 Real Business Cycle Models — Kydland and Prescott (1982) and Long and Plosser (1983)

- Cycles are generated via exogenous contemporaneous productivity shocks, λ.
- Dynamic optimizing behavior on the part of agents in the economy implies that both consumption and investment react positively to supply shocks.
  - Output, $\lambda F(k,l)$, increases.
  - Consumption smoothing implies both current and future consumption should rise. Additionally, the marginal product of capital, $\lambda F_1(k,l)$, rises. This

should stimulate investment too.

- Labor productivity, $\lambda F_2(k,l)$, is directly affected. Results in employment and measures of labor productivity being procyclical.

- Capital accumulation provides a channel of persistence, even if the technology shocks are white noise. Note investment is reacting to a supply shock.

- *Conclusion*: Productivity shocks from a neoclassical perspective can generate the observable co-movements in macroeconomic variables and the persistence of economic fluctuations.

- *Criticism*: Do such productivity shocks occur in reality — oil shocks and harvest failures are two obvious examples but what is another one?

2.1.2 Keynesian Investment Multiplier Model

- "Animal spirits" cause investment fluctuations which generate the business cycle.

  - Marginal efficiency of investment shifts exogenously affecting investment demand and hence output, $y = c + i$, through the investment multiplier-accelerator mechanism.

- *Quintessential Case*: Change in the expected future marginal productivity of capital which does not affect the current production function. A positive shock in the neoclassical growth model will cause:

- Investment to increase[1].
- The real interest to rise to clear the goals market.
- Current consumption to fall and labor effort to rise (and hence leisure to fall.)
- The marginal product of labor to fall.

2.1.3 Current Analysis

- Adopts the Keynesian view that direct shocks to investment are important for business fluctuations
- Incorporates them into a neoclassical framework where the rate of capacity utilization is endogenous. Involves Keynes' (1936) notion of 'user cost' in production.

[1] If investment falls due to a strong income effect, then it is easy to show that consumption will rise but labor effort will fall.

2.2 The Economic Environment

Production Function

$$y = F(kh, l)$$

Here h represents the rate of capacity utilization, or the rate at which capital is utilized.

2.1 Let F be a constant-returns-to-scale function. Show that $F_{12} > 0$ and $F_{11}F_{22} - F_{12}^2 = 0$.

Law of Motion for Capital

$$k' = k[1 - \delta((h)] + i(1 + \varepsilon),$$

which implies

$$y' = F\left(\{k[1 - \delta(h)] + i(1 + \varepsilon)\}h', l'\right).$$

Observe that ε is a shock to the marginal efficiency of investment spending. A extra unit of investment spending today can purchase more units of new capital for tomorrow. Now, $1/(1+\varepsilon)$ can be thought of as the relative price of new capital in terms of forgone consumption. That is, it costs $1/(1 + \varepsilon)$ units of consumption to purchase an extra unit of capital. There is a cost of utilizing your capital today in terms of increased depreciation. Assume that δ_1, $\delta_2 > 0$.

Technology Shock

$$\varepsilon' \sim \Phi(\varepsilon' | \varepsilon).$$

Tastes

$$E[\sum_{t=0}^{\infty} \beta^t \underline{U}(c_t, l_t)],$$

with

$$\underline{U}(c, l) = U(c - G(l)).$$

Implication:

$$\left.\frac{\partial c}{\partial l}\right|_{\underline{U}} = \frac{\underline{U}_2(c, l)}{\underline{U}_1(c, l)} = G_1(l).$$

— Intertemporal substitution effect on labor supply is eliminated. Early critics of real business cycle theory complained that the models required implausible high elasticities of intertemporal substitution. This utility function implies that the supply of labor in the current period can be written solely as a function of the current wage. It is very convenient to use.

Resource Constraint

$$y = c + i.$$

2.2.1 The Representative Agent's Optimization Problem

$$V(k, \varepsilon) = \max_{c,k',h,l} [\underline{U}(c, l) + \beta \int_Q V(k', \varepsilon') d\Phi(\varepsilon'|\varepsilon) \tag{P(1)}$$

s.t.

$$c = F(kh, l) - \underbrace{\frac{k'}{(1+\varepsilon)} + \frac{k[1-\delta(h)]}{(1+\varepsilon)}}_{i/(1+\varepsilon)}.$$

The first-order conditions are:

$$\begin{aligned} U_1(c - G(l))/(1+\varepsilon) &= \beta \int_Q V_1(k', \varepsilon') d\Phi(\varepsilon'|\varepsilon) \\ &= \beta \int_Q U_1(c' - G(l'))[F_1(k'h', l')h' + \tfrac{(1-\delta(h'))}{1+\varepsilon'}] d\Phi(\varepsilon'|\varepsilon), \end{aligned}$$

$$F_1(kh, l) = \frac{\delta_1(h)}{(1+\varepsilon)}, \tag{2.1}$$

$$F_2(kh, l) = G_1(l). \tag{2.2}$$

Equation (2.1) is the efficiency condition regulating capacity utilization. The righthand side portrays Keynes' notion of the user cost of capital. Suppose capacity utilization is increased by a unit. This results in old capital depreciating by $\delta_1(h)$. But a unit of capital can be replaced at a cost of $1/(1+\varepsilon)$ in terms of current consumption. Keynes said (1936, pg. 69-70)

> "User cost constitutes the link between the present and the future. For in deciding the scale of his production an entrepreneur has to exercise a choice between using up his equipment now or preserving it to be used later on ...
> "

2.1 Suppose that Φ satisfies the Feller property: that is, for any continuous function $H(\cdot,\varepsilon')$ it transpires that the function $\int H(\cdot,\varepsilon')d\Phi(\varepsilon'|\varepsilon)$ is continuous too. Does the operator defined by P(1) map C^2 functions into C^2 functions?

2.1 Show that $O(k,l,\varepsilon) = \max_h\{F(kh,l) + k[1-\delta(h)]\}$ is concave in k and l. What does this imply about the value function?

2.2.2 Impact Effect of Investment Shocks

Capacity Utilization and Labor Effort

Consider the impact of a transitory shift in the current technology factor ε. Observe that (2.1) and (2.2) represent a system of two equations in two unknowns. Using Crammer's rule yields

$$\frac{dh}{d\varepsilon} > 0,$$

$$\frac{dl}{d\varepsilon} > 0.$$

2.1 Show that $dh/d\varepsilon > 0$ and $dl/d\varepsilon > 0$.

Interpretation

— An increase in ε reduces the cost of capacity utilization and induces a higher h.

— Since $F_{12} > 0$ labor's marginal product increases, resulting in a higher level of employment.

Now,

$$\frac{dw}{d\varepsilon} = \frac{dF_2(kh,l)}{d\varepsilon} = \frac{dF_2(kh/l,1)}{d\varepsilon} > 0,$$

if and only if

$$\frac{d(kh/l)}{d\varepsilon} > 0.$$

Hence,

$$\frac{dAP_L}{d\varepsilon} = \frac{d[F(kh,l)/l]}{d\varepsilon} = \frac{d[F(kh/l,1)]}{d\varepsilon} = F_1(kh/l,1)\frac{dkh/l}{d\varepsilon} > 0.$$

2.1 Show that $dw/d\varepsilon > 0$.

Capital accumulation

$$\frac{dk'}{d\varepsilon} = \underbrace{\frac{-U_1(\cdot)}{[U_{11}(\cdot) + (1+\varepsilon)^2\beta \int\limits_Q V_{11}(\cdot')d\Phi]}}_{\text{Substitution Effect}} + \underbrace{i\frac{U_{11}(\cdot)}{[U_{11}(\cdot) + (1+\varepsilon)^2\beta \int\limits_Q V_{11}(\cdot')d\Phi]}}_{\text{Income Effect}} > 0.$$

Interpretation

Substitution Effect — New capital is more productive, so invest more.

Income Effect — More resources available for capital accumulation and consumption, so invest more.

2.1 It was never established that V is continuously twice differentiable. Let $k' = K'(k, \varepsilon)$ represent the decision rule for investment. Is it continuous? Can you argue by induction that it must be nondecreasing in ε?

Alternative Specification of the Capital Evolution Equation (Capital Augmenting Technological Change).

Let

$$\begin{aligned} k' &= k[1-\delta(h)](1+\varepsilon) + i(1+\varepsilon) \\ &\Rightarrow \quad c = F(kh,l) - \frac{k'}{1+\varepsilon} + [1-\delta(h)] \end{aligned}$$

The efficiency condition governing the use of capital services now becomes

$$F_1(kh, l) = \delta_1(h).$$

2.2.3 Dynamic Effects of Investment Shocks

— Shock propagates into the next period via its effect on k'.

$$\frac{dh'}{dk'} < 0,$$

$$\frac{dl'}{dk'} > 0,$$

$$\frac{d(k'h')}{dk'} > 0,$$

$$\frac{dk''}{dk'} > 0.$$

2.3 Applied General Equilibrium Analysis

2.3.1 Sample Economy and Simulation Technique

Tastes and technology:

$$U(c,l) = \frac{1}{1-\gamma}[(c - \frac{l^{1+\theta}}{1+\theta})^{1-\gamma}]\,,$$

$$F(kh,l) = A(kh)^{\alpha} l^{1-\alpha},$$

$$\delta(h) = \frac{1}{\omega} h^{\omega}.$$

Technology shock

$$\varepsilon \in \mathcal{E} = \{e^{\xi_1} - 1, e^{\xi_2} - 1\},$$

with

$$\Pr[\varepsilon' = e^{\xi_s} - 1 \mid \varepsilon = e^{\xi_r} - 1] \equiv \pi_{rs} \quad \text{for } r,s = 1,2.$$

Figure 1 illustrates the situation.

The long-run (or unconditional) distribution function for technology shock

$$\Pr[\varepsilon = e^{\xi s} - 1] \equiv \phi_s^* = \frac{\pi_{rs}}{\pi_{12} + \pi_{21}} \quad \text{for } r,s = 1,2 \text{ and } r \neq s.$$

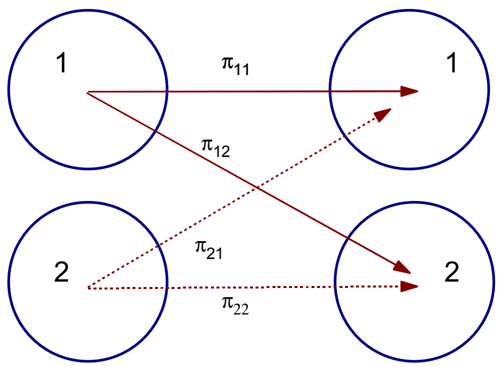

Figure 2.1: Two-Point Markov Chain

2.3.2 Discrete State Space Dynamic Programming Problem

Finally, the capital stock in each period is constrained to be an element of the finite time-invariant set, K, where

$$\mathcal{K} = \{k_1, ..., k_n\}.$$

Representative Agent's Dynamic Programming Problem

$$V(k_i, \xi_r) = \max_{c,k' \in \mathcal{K}} \{ \frac{1}{1-\gamma} (c - \frac{\hat{l}^{1+\theta}}{1+\theta})^{1-\gamma} + \beta \sum_{s=1}^{2} \pi_{rs} V(k', \xi_s) \},$$

subject to

$$c = A(k_i, \hat{h})^\alpha \hat{l}^{1-\alpha} - k' e^{-\xi_r} + k_i(1 - \frac{\hat{h}^\omega}{\omega}) e^{-\xi_r},$$

where

$$\hat{h}, \hat{l} = \arg\max[A(k_i, \hat{h})^\alpha \hat{l}^{1-\alpha} - k_i(1 - \frac{\hat{h}^\omega}{\omega}) e^{-\xi_r} - \frac{\hat{l}^{1+\theta}}{1+\theta}].$$

Observe that $V : \mathcal{K} \times \mathcal{E} \to \mathcal{R}$ is merely a list of $2n$ values, one for each $(k_i, \xi_r) \in \mathcal{K} \times \mathcal{E}$

Decision Rule for Capital

$$k' = K'(k_i, \xi_r) \in \mathcal{K}.$$

2.3.3 Construction of Markov Chain

$$\Pr[k' = k_j \mid k = k_i, \xi = \xi_r] = \begin{cases} 1, \text{ for some } j, \\ 0, \text{ for the rest.} \end{cases}$$

Trivially, then

$$\sum_{j=1}^{n} \Pr[k' = k_j \mid k = k_i, \xi = \xi_r] = 1 \text{ for all } (k, \xi) \in \mathcal{K} \times \mathcal{E}.$$

Transition Probabilities

$$p_{ir,js} = \Pr[k' = k_j, \xi' = \xi_s | k = k_i, \xi = \xi_r] = \Pr[k' = k_j \mid k = k_i, \xi = \xi_r] \pi_{rs}.$$

$$P = \underbrace{[p_{ir,js}]}_{2n\times 2n}.$$

— Given some initial probability distribution $\underbrace{\rho^0}_{1\times 2n}$ over the state.

— Next period's probability distribution is given by

$$\rho^1 = \rho^0 P,$$

or

$$(\rho^1_{11}, ..., \rho^1_{n2}) = (\rho^0_{11}, ..., \rho^0_{n2}) \begin{bmatrix} p_{11,11} & \cdots & p_{11,n2} \\ \vdots & & \vdots \\ p_{n2,11} & \cdots & p_{n2,n2} \end{bmatrix}.$$

Stationary Distribution

$$\rho^* = \rho^* P.$$

2.1 Is $(I - P)$ invertible? Suppose that the last equation of the system is replaced with $\rho^*_{n2} = 1 - \sum_{i\neq n, r\neq 2} \rho^*_{ir}$. Is there a direct way to compute ρ^*.

— Computation of Moments

$$E[y] = \sum_{r=1}^{2}\sum_{i=1}^{n} \rho^*_{ir} Y(k_i, \xi_r),$$

$$E[cy] = \sum_{r=1}^{2}\sum_{i=1}^{n} \rho^*_{ir} C(k_i, \xi_r) Y(k_i, \xi_r),$$

$$E[y'y] = \sum_{s=1}^{2}\sum_{j=1}^{n}\sum_{r=1}^{2}\sum_{i=1}^{n} p_{ir,js}\rho^*_{ir} Y(k'_i, \xi'_s) Y(k_i, \xi_r).$$

2.3.4 Calibration Procedure and Simulation Results

Experimental Design

- Using information from either the literature or US data, values were assigned for the various taste and technology parameters.

- Next by varying parameters governing the stochastic structure of the sample economy, the model is calibrated so that it yields the same standard elevation and first-order autocorrelation for output as is displayed by US data.

- Finally, the model is 'tested' so to speak, by comparing the standard deviations, correlations with output, and serial correlations of the other variables (consumption, investment, hours, and productivity) with the corresponding statistics in the US data.

Taste and Technology Parameters

$$\beta = 0.96, \text{ Kydland and Prescott(1982)}$$

$$\alpha = 0.29, \text{Capital's share of income}$$

$$1/\theta = 1.7, \text{ Intertemporal elasticity of labor supply}$$

— MaCurdy (1981) estimated this to be about .3 for adult males. Heckman and MaCurdy (1980,1982) found the corresponding value for females to be about 2.2.

$$\gamma = 1.0 \text{ and } 2.0, \text{Coefficient of relative risk aversion}$$

— A controversial parameter. The first estimate is close to what Hansen and Singleton (1983) found, the second is in accord with what Friend and Blume (1975) discovered.

$$\omega = 1.42, \text{ Elasticity of the depreciation function}$$

— This number was picked because it implies a steady-state depreciation rate of 10%.

Stochastic Process Parameters

Let

$$\pi_{11} = \pi_{22} \equiv \pi \text{ and } \xi_1 = -\xi_2 = \Xi. \tag{2.3}$$

Then

$$\begin{aligned} \sigma = &\quad \Xi \text{ (standard deviation)} \\ \lambda = &\quad 2\pi - 1 \text{ (autocorrelation coefficient)} \end{aligned} \tag{2.4}$$

— Picked so as make the model generate the same standard deviation and first order serial correlation for output as is observed in the data.

2.1 Show that (i) $\phi_s^* = \pi_{rs}/[\,\pi_{12}+\pi_{21}]$ and (ii), that given (2.3) the standard deviation and autocorrelation coefficient are given by (2.4).

2.3.5 Results

Cases: (Shocks Calibrated to Match US Output Fluctuations)

$$\text{Case 1:} \quad \gamma = 1 \quad \sigma = 0.047 \quad \lambda = 0.43$$

$$\text{Case 2:} \quad \gamma = 2 \quad \sigma = 0.051 \quad \lambda = 0.44$$

— Size of fluctuations in shocks relative to output seems about the same as Kydland and Prescott (1982) and Hansen (1985) $\frac{5.15}{3.50} = 1.5$.

— Required amount of persistence in shock much less than in Kydland and Prescott (1982) and Hansen (1985).

Stylized Facts

1. (Volatility) Investment much more volatile than output, consumption less. The model qualitatively mimics this behavior but quantitatively exaggerates it.

2. (Correlations) Hours has the highest correlation with output, but the other variables particularly consumption come fairly close. The procyclical behavior of consumption, however, is critically dependent on the value of γ. When $\gamma = 1$ the correlation of consumption with output is only 0.50. For $\gamma = 2$, this correlation

increases to 0.79 close to the 0.74 value with actual data. Also, increasing γ from 1 to 2 which corresponds to reducing the amount of intertemporal substitution] lowers the standard deviation of investment from 14.7% to 11.6%, closer to the actual data value of 10.5%. Higher values of γ could do even better. Overall, the best fit for the model corresponds to $\gamma = 2$.

3. (Persistence) In the data, consumption and productivity have the highest autocorrelations, and investment the lowest. In the sample economy consumption also has the highest autocorrelations productivity the second, and investment the lowest. the model displays a tendency though to overemphasize, though, the degree of persistence in investment spending.

U.S. Data				Model		
Var.	S.D.	Corr	Auto	S.D.	Corr.	Auto
Output	3.5	1.00	.66	3.5	1.00	.66
Cons.	2.5	.74	.72	2.2	.79	.94
Inv.	10.5	.68	.25	11.6	.90	.50
Hours	2.1	.81	.39	2.2	1.00	.66
Prod.	2.2	.82	.77	1.3	1.00	.66
Util.				5.6	0.61	.52

Capacity Utilization

So what role does capacity utilization play in the model's transmission mechanism? Figure 2 tells the story. Imagine that a positive investment-specific technology shock hits the economy. The productivity of new capital goods jumps up. Investment should rise (say from i to i'). But, at the old wage rate, w, this will cause the consumer/worker's budget constraint to drop down from $wl + rk - i$ to $wl + rk - i'$. Provided that consumption and leisure are normal goods this will cause an increase in work effort but a *fall* in consumption. Now, with capacity utilization the wage rate, w, increases to w'. This happens because: (i) the rate of capacity utilization increases and (ii), capital services and labor are Edgeworth-Pareto complements in the production function. The budget line rotates up from $wl + rk - i'$ to $w'l + rk - i'$, which permits consumption to rise.

2.4 Conclusions

- Addressed the macroeconomic effects of direct shocks to investment.

- A variable capacity utilization rate may be important for the understanding of business cycles. The modelling apparatus employed provides a mechanism through which investment shocks generate a higher utilization rate of the existing capital stock, and hence higher labor *demand*. This mechanism stands in contrast

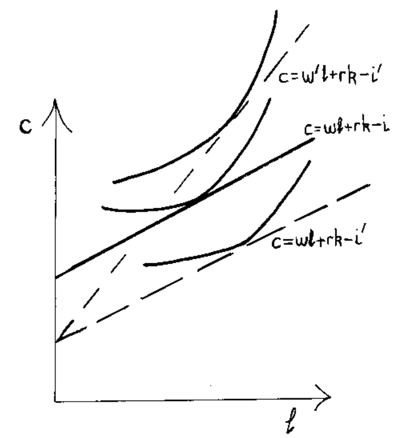

Figure 2.2: The Effect of Capacity Utilization on Consumption

to the intertemporal substitution effect which works on labor *supply.*

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