ECON 714

Assignment 3 - due December 9

1. The Prospect of Upward Mobility: One application of transition matrices lies in the study of income processes. It is well known that in many societies, even relatively poor people oppose high rates of redistributon in anticipation (either them or their children) of moving up the income ladder. (For instance, in the United States, even poor people oppose the estate tax (tax on bequests) which a tax levied on every dollar that you pass on to your kids in excess of a million dollars!) This is referred to as the prospect of upward mobility (POUM). Imagine that income of individual i at time t, y_t^i can take one of three values a_1, a_2 and a_3, $a_1 < a_2 < a_3$.

$$\Pi = \begin{bmatrix} 1-r & r & 0 \\ ps & 1-s & (1-p)s \\ 0 & q & 1-q \end{bmatrix}.$$

where $(p, q, r, s) \in (0, 1)$. Let the invariant distribution induced by Π be deoted by (π_1, π_2, π_3). We are in the business of seeking conditions on Π such that the majority oppose redistribution. When will that be the case. Well, the majority will oppose (future) redistribution if over half of the population is always poorer than average (positively skewed income distribution - which characterizes reality), BUT over half of the population always has expected incomes above average (negatively skewed *expected* income distribution - which, if true, will lead a strict majority to oppose redistribution). We seek restrictions on the parameters such that

(i) Next period's income y_{t+1}^i is stochastically increasing in current income y_t^i; (i.e. Π is a monotone transition matrix)

(ii) The median income level is $a_2 : \pi_1 < \pi_2 < \pi_1 + \pi_2$;

(iii) The median agent is poorer than the mean: $a_2 < \mu = \pi_1 a_1 + \pi_2 a2 + (1 - \pi_1 - \pi_2)a3$;

(iv) The median agent has expected income above the mean: $E[y_{t+1}^i | y_t^i = a2] > \mu$.

NOTE: $M = [m_{kl}]_{3\times3}$ is a monotone transition matrix if row $k+1$ stochastically dominates row k : $m_{11} \geq m_{21} \geq m_{31}$ and $m_{11} + m_{12} \geq m_{21} + m_{22} \geq m_{31} + m_{32}$.

Also note that conditions (b) and (c) together ensure that a strict majority of the population would vote for current redistribution, while (b) and (d) together imply that a strict majority will vote against future redistribution.

a) What are the (weakest) restrictions on the parameters needed for the above four properties to be satisfied?

b) From US data, we estimate that $p = .55$, $q = .6$, $r = .5$, and $s = .7$; and $(a_1, a_2, a_3) = (16000, 36000, 91000)$. Does this satisfy the parametric restriction that you found in part a? Can we then conclude that the POUM holds in the US?

2. Growth with Externality: Consider an economy with infinitely many agents each with preferences given by the utility function

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^\gamma}{\gamma} \qquad 0 < \beta < 1 \qquad \gamma < 1.$$

Each agent has one unit of time available in each period and can produce consumption goods according to

$$c_t = \alpha h_t^\theta \overline{h}_t^{1-\theta} u_t,$$

where $0 < \theta < 1$, $\alpha > 0$, u_t is the time spent in consumption goods production by the agent and $\overline{h}_t$ denotes the average human capital across all agents in the economy. The externality is such that each agent's productivity increases with the amnout of human capital that other agents have. An agent's human capital evolves according to

$$h_{t+1} = \delta h_t (1 - u_t),$$

where $\delta > 0$. Assume that each agent has the same initial stock of human capital.

a. Set up the social planner's problem as a dynamic program. Note that the social planner internalizes the externality and treats $\overline{h}_t = h_t$.

b. Solve for the optimal growth rates of human capital and consumption, and solve for u_t.

c. Set up the representative agent's dynamic programming problem and solve for the competitive equilibrium. Also, solve for a balanced growth path of the economy.

3. Monopolist's problem: Consider this monopolistic version of the industry equilibrium model . A monopolist chooses output over time in order to maximize the present discounted value of its profits. Profits are defined as revenue minus cost, where the only cost that the monopolist incurs is the one associated with changing its level of production. Formally, the problem of the monopolist in sequence form is:

$$\max_{\{y_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t \Pi_t,$$

y_0 given where $0 < \beta < 1$ denotes the monopolist's discount factor. Profits, denoted by Π_t and are given by

$$\Pi_t = p_t y_t - \frac{1}{2}(y_{t+1} - y_t)^2.$$

The price p_t is given by the inverse demand function

$$p_t = a_0 - a_1 y_t,$$

where $a_0 > 0$ and $a_1 > 1$. Notice that the market price will exclusively depend on the production decision of the monopolist. The timing of the problem is as follows: at the beginning of period t the monopolist decides on the quantity y_{t+1} it wants to sell next period (at the price p_{t+1}). Revenue at time t is collected at the beginning of the period and is determined by the production choices made in period $t-1$.

a. Write down this problem as a dynamic programming problem.

b. Prove that there exists a unique value function V that satisfies the functional equation and that this value function corresponds to the supremum of present discounted utilities in the sequence problem. Be careful to specify the complete metric space within which you look for solutions to the functional equation.

c. Write down the first order condition for the monopolists' problem. Use it to show that the policy function $y\prime = g(y)$ is strictly increasing in y. [Remember that to be able to apply the Benveniste-Scheinkman theorem and take the derivative of the value function you need to show that it is concave].

4. **Linear Utility**: Consider the case of a planner or an economically self-sufficient household (Robinson Crusoe?). The agent faces the following problem:

$$\max \sum_{t=0}^{\infty} \left(\frac{1}{2}\right)^t c_t$$

where c_t represents consumption and $\frac{1}{2}$ is the discount factor. Output may be used either for consumption or capital formation. Output is produced with capital, according to $y_t = 4k_t - k_t^2$. Output installed as capital in period t incurs an adjustment cost equal to $\frac{1}{2}\left(k_{t+1} - k_t\right)^2$.

a. What is the Bellman equation associated with the above decision problem? Prove the differentiability of the value function. Outline a clear argument.

b. Derive a closed form solution for the value and policy functions.

c. Calculate the steady state value of capital.

5. Differentiability: Consider the following dynamic programming problem

$$V(a, z_i) = \max_{c,\, a'} \left\{ U(c) + \beta \sum_{j=1}^{n} \pi_{ij} V(a', z_j) \right\},$$

subject to

$$c + a' = z_i + (1 + r)a,$$

and

$$a' \geq 0.$$

Let U be a bounded, strictly increasing, strictly concave, continuously differentiable function. suppose that $0 < r < \beta < 1$. The bounded positive random variable z follows a n-point Markov process. In particular, z is drawn from the discrete set $\mathcal{Z} \equiv \{z_1, z_2, \ldots, z_n\}$ according to the probability distribution specified by $\pi_{ij} = prob\,\{z' = z_j \mid z = z_i\}$, where $0 \leq \pi_{ij} \leq 1$, and $\sum_{j=1}^{n} \pi_{ij} = 1$.

a) Is the value function $V(a, z)$ is continuously differentiable in a, for all $a > 0$, whenever $a' > 0$?

b) Is the value function $V(a, z)$ is continuously differentiable in a, for all $a > 0$, whenever $a' = 0$? What is the issue here?

6) Human Capital: Consider a model of human capital that we discussed in class. An adult forms his own household at age S. Until that point in time, he is attached to his parent and makes no decisions. Starting at age S, he begins making decisions. At age $B > S$, he gives birth to f identical children, who are attached to him until he is $B + S$ years old, at which time, the child moves out to form his own household. The adult works until his retirement age, R and dies at age $T > R$. Human capital evolves according to

$$h_{i+1} = (h_B^p)^{\gamma_1} (n_i h_i)^{\gamma_2} (e_i)^{\gamma_3} + (1 - \delta) h_i,$$

where h_B^p is the human capital of the parent when the individual in question was born, n_i and e_i represent the time and goods inputs into the production of human capital. An individual is altruistic in the sense that he cares about his children's consumption and the weight that he placed on his f identical children is $\alpha(f)$, which is increasing and concave in the choice of f. The parent derives utility from his own consumption and the welfare of his progeny. Formulate the Bellman equation at age S. Then proceed to derive the first order condition for the optimal choice of f and then assuming that we are in a steady state, eliminate the value function, V from that equation to present an equation governing the optimal choice of the number of kids, f.