Assignment 2 - due November 25

I). **Capacity Choice**: Here is a problem facing a monopolist. He faces a demand curve each period given by $q = (1 - p)$, That is, if the price is p he can sell the quantity q. Production is costless but at each period in time the monopolist faces a capacity constraint, $q = c$, where c is the upper bound on his production. Capacity can be increased with a one period time delay according to the cost function $(c' - c)^2$, where $c' \geq c$ is the level of capacity that the monopolist chooses for next period. The monopolist faces the time-invariant gross interest rate r.

1. Formulate the monopolist dynamic programming problem.
2. Is the value function strictly increasing? If so, outline an argument.
3. Is the value function strictly concave? If so, outline an argument.
4. Is the value function differentiable? If so, outline an argument.
5. Using the guess and verify technique, provide a closed-form solution for the Value and Policy functions.

II) **Transition Functions**: Let

$$\Pi = \begin{bmatrix} 1-\gamma & \alpha\beta & \beta\gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

be a transition matrix associated with a 3-state Markov chain. Further, let α, β, γ ε $(0, 1)$ and $\alpha + \beta = 1$.

1. What is the stationary probability distribution associated with the above transition matrix?
2. How many ergodic sets are there? What are they?

III) **Neoclassical Growth Model with Depreciation**: Consider the following Bellman equation:

$$For\ all\ k > 0 : V(k) = \max\{\log[(1-\delta)k + Ak^{\alpha} - k'] + \beta V(k')\},$$

where $0 < \delta < 1, A > 0, 0 < \alpha < 1$ and $0 < \beta < 1$. Can $V(k)$ take the form $V(k) = E + Flog(k)$ for some constants E and F? If so, prove it. If not, show that $V(k)$ cannot take the above form.

IV) **Cauchy Sequences**

a) Let $C^1[a,b]$ be the set of all continuously differentiable functions on $[a,b] = X \subset R$, with the norm $\|f\| = \sup_{x\in X}\{|f(x)| + |f'(x)|\}$. Show that $C^1[a,b]$ is a complete metric space.

b) Show that this set of functions with the (standard) norm $\|f\| = \sup_{x\in X}\{|f(x)|\}$ is not complete. That is, give an example of a sequence of functions that is Cauchy in the given norm but doesn't converge to a function in the set (Basically differentiability will be lost in the limit!). Is this sequence Cauchy in the norm of part (a)?

V) **Ergodic Sets**: Let $S = \{s_1, s_2, ...\}$ and let

$$\Pi = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

What happens to the sequence Π^k as k becomes large? What are the ergodic sets? What is the invariant distribution?

VI) **Adjustment Costs**: Consider the following problem of a firm with convex investment costs. Specifically, the total cost of investment is a function $c : R \to R_+$ of gross investment that is strictly increasing, strictly convex and differentiable with $c(0) = 0$. The firm operates a technology $z = f(k)$, where physical capital $k \geq 0$ is the single input and $z \geq 0$ is output. Assume that f is continuously differentiable, strictly increasing and strictly concave and that

$$f(0) = 0, \lim_{k\to 0} f'(k) = +\infty, \lim_{k\to\infty} f'(k) = 0$$

Assume that p is the price of output, q the price of capital, and r the interest rate, with $\beta = (1+r)^{-1} \in (0,1)$. Assume that capital must be purchased one period in advance and depreciates at the rate $\delta \in (0,1)$. Then the firm's problem is

$$\sup \sum_{t=0}^{\infty} \beta^t \{pf(k_t) - c[k_{t+1} - (1-\delta)k_t]\}$$

where $c[.]$ denotes the cost of gross investment (note: $c[.]$ is a function, not a constant).

(a) Formulate the functional equation for this problem and show that there is a unique value function v^* that solves the functional equation. Also show that v^* is the supremum function for the sequence problem. Be careful to justify each step in your proof with reference to results discussed in class and to recall why these results are applicable.

(b) Show that v^* is strictly increasing and strictly concave.

(c) Show that v^* is differentiable. Then use the Euler equation to show that the policy function is strictly increasing in current capital.