ECON 714

Assignment 1 - due Nov. 11

The first three problems below are on the guess and verify technique. As we discussed in class this is pretty limited in scope and can be applied only in very special circumstances. Nevertheless that are pretty useful to guide intuition for some really simple dynamic programming problems. Also, in each of the three instances below, you will need to write down Bellman's functional equation before proceeding. When doing so, you need to be very clear on what the state variables are, what the control variables are etc. This comes with a lot of practice.

I) Habit Persistence: Consider the problem of choosing a consumption sequence to maximize

$$\sum_{t=0}^{\infty} \beta^t \left\{ \log\left(c_t\right) + \gamma \log(c_{t-1}) \right\}, 0 < \beta < 1, \gamma > 0$$

subject to

$$c_t + k_{t+1} \leq A k_t^{\alpha}, A > 0, 0 < \alpha < 1, \text{ and } k_0, c_{-1} \text{ given.}$$

Here c_t denotes consumption at time t, and k_t is the capital stock at the beginning of period t. Note that consumption purchases generate a stream of utility flow that extend for more than one period.

a) Formulate Bellman's functional equation.

b) Derive a closed-form solution for the value function and the policy function.

II) Hyperbolic Discounting: (This is a harder problem!) Suppose a planner chooses to maximize, by choice of $c_0, c_1, c_2, ...$, the following expression:

$$u(c_0) + \delta[\beta u(c_1) + \beta^2 u(c_2) + ...] \qquad (1)$$

where $u(c_t) = log(c_t)$ subject to

$$c_t = k_t^\alpha - k_{t+1}, 0 < \alpha < 1, c_t, k_{t+1} \geq 0, \;\; k_0 \text{ given}$$

where $0 < \delta < \beta < 1$. Note that when $\delta = 1$, this reduces to the problem commonly studied.

1. Write down the Bellman equation for the above problem.

2. Let $k_{t+1} = g_t(k_t)$ denote the policy rule that solves this problem, $t = 0, 1, ...$ Derive an explicit formula relating g_t to the parameters of the model, β, α, δ.

3. Derive a closed-form solution for the value function.

4. How does the saving rate from period $t = 1$ and on compare with the date 0 saving rate?

5. Is there a unique k^* with the property $k_t \to k^*$ as $t \to \infty$ for all k_0? Display a formula relating k^* to the parameters of the model.

III) A Vintner's Problem: Consider the following problem. A vintner has one unit of labor to use each day. (A vintner "One who makes wine") He can allocate that labor between the making of bread and the pressing of grapes for grape juice. The bread he makes today he can consume today. The grape juice he makes today will become tomorrow's wine (he doesn't care for grape juice!). The production technology is linear: it produces one unit of bread per unit of labor allocated to baking, one unit of juice per unit of labor allocated to grape pressing, and one unit of wine per unit of grape juice left to ferment. The transformation of juice into wine requires no labor input, only time. The vintner allocates his labor so as to maximize the utility of his own consumption. His utility function has the form

$$U = \sum_{t=0}^{\infty} \beta^t \sqrt{b_t w_t}$$

where b_t and w_t are the bread and wine consumption, respectively, in period t. The initial wine consumption w_0 is given. The discount factor is $\beta \in (0, 1)$.

(a) Write down this problem in recursive form.

(b) Derive a closed form solution for the value and policy functions.

IV) Problems on Metric Spaces

1. Let $X = C(a, b)$ be the set of continous functions $[a, b] \to R$, and for $x, y \in C(a, b)$ define $\rho(x, y)$ by

$$\rho(x, y) = \int_a^b \mid x(t) - y(t) \mid dt.$$

   (i) Show that (X, ρ) is a metric space.

   (ii) Is (X, ρ) complete? If so prove it, if not provide a counter-example.

2. Let $X = C(a, b)$ be the set of continous functions $[a, b] \to R$, and for $x, y \in C(a, b)$ define $\rho(x, y)$ by

$$\rho(x, y) = \left(\int_a^b [x(t) - y(t)]^2 \, dt \right)^{\frac{1}{2}} .$$

   (i) Show that (X, ρ) is a metric space.

   (ii) Is (X, ρ) complete? If so prove it, if not provide a counter-example.

3. (i) Show that R is a complete metric space with metric $\rho(x, y) = \mid x - y \mid$.

4. (i) Let $X = C(a, b)$ be the set of continous functions $[a, b] \to R$, and for $x, y \in C(a, b)$ define $\rho(x, y)$ by

$$\rho(x, y) = \max_{a \le t \le b} \mid x(t) - y(t) \mid, \; t \in [a, b].$$

   Let $x_1, x_2, x_3, \ldots$ be functions in X such that $x_1, x_2, x_3, \ldots \to x$. Show that for any $t \in [a, b]$, $x_1(t), x_2(t), x_3(t), \ldots \to x(t)$ in R.

   (ii) Let $X = C[0, 1]$ and let the metric be given as in part (i). Find an example of a sequence of functions in X such that for each $t \in [0, 1]$, $x_1(t), x_2(t), x_3(t), \ldots \to x(t)$; the function defined by collection of $x(t)s$ for $t \in [0, 1]$, however, *is not* in X. Hence, the converse of part (i) does not hold.